Classifying small index subfactors

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Invariants of subfactors

\[ A \subset B \]

\[ (P_+, P_-) \]

\[ (\Gamma_+, \Gamma_-) \]
** Definitions **

- A von Neumann algebra is a $\ast$-closed subalgebra $M \subseteq B(H)$ such that $M = M''$.
- A factor is a von Neumann algebra with trivial center $Z(M) = M' \cap M = \mathbb{C}1$.
- A $\text{II}_1$-factor $M$ is an infinite dimensional factor with a tracial state $\text{tr} : M \to \mathbb{C}$.
- A subfactor is a unital inclusion of factors.

Our subfactors will be $\text{II}_1$-subfactors.
**Definition**

Given a $\text{II}_1$-subfactor $A \subset B$, we say it has finite index if $B$ is a finitely generated projective $A$-module. The index $[B : A]$ is the trace of the corresponding idempotent in $K_0(A)^+$. 

**Theorem [Jon83]**

For a $\text{II}_1$-subfactor $A \subset B$,

$$[B : A] \in \left\{ 4 \cos^2 \left( \frac{\pi}{n} \right) \middle| n = 3, 4, \ldots \right\} \cup [4, \infty].$$
Where do subfactors come from?

Some examples include:

- Groups – from $G \rhd R$, we get $R^G \subset R$ and $R \subset R \rtimes_\alpha G$.
- finite dimensional unitary Hopf/Kac algebras
- Quantum groups
- Conformal field theory
- endomorphisms of Cuntz C*-algebras

However, there are certain possible infinite families without uniform constructions.

**Remark**

Just as von Neumann algebras come in pairs $(M, M')$, subfactors come in pairs $(A \subset B, B' \subset A')$. 
The standard invariant

**Definition**

The standard invariant of $A \subset B$ is the following unitary 2-category:

- **Objects**: $A$ and $B$

- **1-Morphisms**: bimodules generated by $A_B B$ and $B_A B$ (take tensor products and decompose into irreducible summands)

- **2-Morphisms**: bimodule intertwiners

- **Unitary structure**: Adjoint on $A, B$ is identity. Adjoint on bimodules is the contragredient, giving an involution on $A - A$ and $B - B$ bimodules, but swaps $A - B$ and $B - A$ bimodules. Adjoint on intertwiners is usual adjoint.

Choosing our favorite 1-morphism $A_B B$ gives a planar algebra.
Principal graphs

**Definition**

Given the 1-morphism \( A^B_B \), we define the principal graph of \( A \subset B \) as follows.

- **Even vertices:** isomorphism classes of simple \( A - A \) bimodules.
- **Odd vertices:** isomorphism classes of simple \( A - B \) bimodules.
- **Edges:** \( \dim(\text{Hom}_{A - B}(P \otimes_A B, Q)) \) unoriented edges from \( A^P_A \) to \( A^Q_B \).

- Get the dual principal graph by looking at \( B^B_A \) together with \( B - B \) and \( B - A \) bimodules.
- Can define the fusion graph with respect to any simple bimodule.
### Examples of principal graphs

- **index < 4**: $A_n, D_{2n}, E_6, E_8$. No $D_{odd}$ or $E_7$.
- **index = 4**: $A_{2n-1}^{(1)}, D_{n+2}^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, A_\infty, A_\infty^{(1)}, D_\infty$
- Graphs for $R \subset R \rtimes G$ obtained from $G$ and $\text{Rep}(G)$.  
  \[
  \begin{array}{c}
  \text{Haagerup 333} \\
  \text{extended Haagerup 733}
  \end{array}
  \]

  \[
  \begin{array}{c}
  \text{First graph is principal, second is dual principal.} \\
  \text{Leftmost vertex is the trivial bimodule } A_A, B_B \text{ resp.} \\
  \text{Red tags for duality of } A - A \text{ and } B - B \text{ vertices.} \\
  \text{Duality of } A - B \text{ to } B - A \text{ is by depth and height}
  \end{array}
  \]
Finite depth

**Definition**
If the principal graph is finite, then the subfactor and standard invariant/planar algebra are called finite depth.

**Example:** $R \subset R \rtimes G$ for finite $G$

For $G = S_3$:
- Principal graph: 
- Dual principal graph:
Supertransitivity

**Definition**

We say a principal graph is $n$-supertransitive if it begins with an initial segment consisting of the Coxeter-Dynkin diagram $A_{n+1}$, i.e., an initial segment with $n$ edges.

**Examples**

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  is 1-supertransitive

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  is 2-supertransitive

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  is 3-supertransitive
Recent classification to index 5 (contributed to parts 2 and 4) [MS12, MPPS12, IJMS12, PT12]

Map of known subfactors from Jones-Morrison-Snyder survey [JMS13], to appear Bulletin AMS.
Planar algebras [Jon99]

Definition

A shaded planar tangle has

- a finite number of inner boundary disks
- an outer boundary disk
- non-intersecting strings
- a marked interval $\star$ on each boundary disk
- a checkerboard shading
Composition of tangles

We can compose planar tangles by insertion of one into another if the number of strings matches up:

Definition

The shaded planar operad consists of all shaded planar tangles (up to isotopy) with the operation of composition.
Definition

A *planar algebra* is a family of vector spaces $P_{k,\pm}$, $k = 0, 1, 2, \ldots$ and an action of the shaded planar operad.

$$P_{2, -} \times P_{1, +} \times P_{1, +} \rightarrow P_{3, +}$$

$P_{2, -} \times P_{2, +} \times P_{1, +}$
Example: Temperley-Lieb

\( TL_{n, \pm}(\delta) \) is the complex span of non-crossing pairings of \( 2n \) points arranged around a circle, with formal addition and scalar multiplication.

\[
TL_{3, +}(\delta) = \text{Span}_\mathbb{C}\left\{ \begin{array}{c}
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Planar tangles act on \( TL \) by inserting diagrams into empty disks, smoothing strings, and trading closed loops for factors of \( \delta \).

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A planar algebra $P_\bullet$ is a subfactor planar algebra if it is:

- Finite dimensional: $\dim(P_k,\pm) < \infty$ for all $k$
- Evaluable: $P_{0,\pm} \cong \mathbb{C}$ by sending the empty diagram to $1_\mathbb{C}$
- Sphericality:

$$\begin{array}{c}
\hline
\text{X} \\
\hline
\end{array} = \begin{array}{c}
\hline
\text{X} \\
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\end{array}$$

- Positivity: each $P_{k,\pm}$ has an adjoint $\ast$ such that the sesquilinear form $\langle x, y \rangle := \text{Tr}(y^*x)$ is positive definite

From these properties, it follows that closed circles count for a multiplicative constant $\delta \in \{2 \cos(\pi/n) | n \geq 3\} \cup [2, \infty)$. 

David Penneys

Classifying small index subfactors
Planar algebras from tensor categories

Definition

Given a unitary fusion category and a choice of simple object $X$, we get a planar algebra by setting

$$P_{n,+} = \text{Hom}(1, (X \otimes X)^\otimes n) \quad \text{and} \quad P_{n,-} = \text{Hom}(1, (X \otimes X)^\otimes n)$$

- The strand is the identity 1-morphism: $\text{id}_X = \begin{array}{c}
\end{array}$ and $\text{id}_{X^*} = \begin{array}{c}
\end{array}$

- Caps are evaluation $\text{ev}_X = \begin{array}{c}
\end{array}$ and $\text{ev}_{X^*} = \begin{array}{c}
\end{array}$

- Cups are coevaluation $\text{coev}_X = \begin{array}{c}
\end{array}$ and $\text{coev}_{X^*} = \begin{array}{c}
\end{array}$

- Vertical join is composition $gf = \begin{array}{c}
\end{array}$

- Horizontal join is tensor product $f \otimes g = \begin{array}{c}
\end{array}$
If $f \in P_{n,+} = \text{Hom}(1, (X \otimes X) \otimes n)$, the tangle below is a composite map, read from bottom to top:

$$
\begin{align*}
\text{id}_{(X \otimes X)^{\otimes n}} \otimes \text{ev}_X \\
\text{id}_{(X \otimes X)^{\otimes n} \otimes X} \otimes \text{ev}_X \otimes \text{id}_{X} \\
\text{id}_{X \otimes X} \otimes f \otimes \text{id}_{X \otimes X} \\
\text{id}_X \otimes \text{coev}_X \otimes \text{id}_X \\
\text{coev}_X
\end{align*}
$$
Small index subfactor classification program

Focuses of the classification program:

- Enumerate graph pairs and apply obstructions.
- Construct examples when graphs survive.
- Place exotic examples into families.
The extended Haagerup subfactor

Bigelow-Morrison-Peters-Snyder, [BMPS12]

The extended Haagerup subfactor is the unique subfactor with principal graphs

(, )

- Last remaining possible graph in Haagerup’s classification to $3 + \sqrt{3}$ [Haa94] by work of Asaeda-Yasuda [AY09].
- Largest known supertransitivity outside the $A$ and $D$ series.
- Its planar algebra was constructed using Bigelow’s jellyfish algorithm.
The Haagerup and extended Haagerup subfactor planar algebras have a generator $S \in P_{n,+}$ where $n = 4, 8$ respectively satisfying:

- \[ S \star f(2n+2) = i \frac{\sqrt{[n][n+2]}}{[n+1]} \]
- \[ S \star f(2n+4) = \frac{[2][2n+4]}{[n+1][n+2]} \]

(Absorption) capping $S$ gives zero and $S^2 = f(n) \in TL_{n,+}$. 

**Bigelow-Morrison-Peters-Snyder, [BMPS12]**
We can evaluate all closed diagrams as follows:

1. First, pull all generators to the outside using the jellyfish relations.

2. Second, reduce the number of generators using the capping and absorption (multiplication) relations.
Consistency and positivity

Theorem [Jones-Penneys [JP11], Morrison-Walker]

Every subfactor planar algebra embeds in the graph planar algebra of its principal graph.

This serves two purposes:

1. To show the planar algebra is non-zero, give a representation.
2. Graph planar algebras are always finite dimensional, spherical, and positive. Only need to check evaluable.
Spoke graphs

Examples of spoke principal graphs

- $A_n, D_{2n}, E_6, E_8,
- E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$
- $A_\infty, A_\infty^{(1)}, D_\infty$
- Principal graphs for $R \subset R \rtimes G$, $G$ finite
- 2221
- Haagerup 333
- 3311
- 3333
- 4442
- extended Haagerup 733
Assume all generators of $P\bullet$ are at the same depth $n$.

**Theorem [Bigelow-Penneys [BP14]]**

- $P\bullet$ has 2-strand jellyfish relations $\iff$ one graph is a spoke.

- $P\bullet$ has 1-strand jellyfish relations $\iff$ both graphs are spokes.
Constructing spoke subfactors with jellyfish

**Theorem [Morrison-Penneys [MP13]]**

We automate finding 1-strand relations for these subfactors:

- **Izumi-Xu 2221** [Han11]
- **[GdlHJ89] 3311**
- **Izumi $3\mathbb{Z}/2 \times \mathbb{Z}/2$** (index $3 + \sqrt{5}$)
- **4442** (index $3 + \sqrt{5}$)

For the above, both principal graphs are the same spoke graph.
Theorem [Penneys-Peters [PP13]]

We give explicit 2-strand relations for the following subfactors:

- Haagerup $333$ $(\mathbb{Z}/3)$
  - $(\ldots,\ldots)$

- $3333$ $(\mathbb{Z}/2 \times \mathbb{Z}/2)$
  - $(\ldots,\ldots)$

- $3333$ $(\mathbb{Z}/4)$
  - $(\ldots,\ldots)$
Small index subfactor classification program

Focuses of the classification program:
- Enumerate graph pairs and apply obstructions.
- Construct examples when graphs survive.
- Place exotic examples into families.
Why do we care about index $3 + \sqrt{5}$?

- Standard invariants at index 4 are completely classified.
  - $\mathbb{Z}/2 \ast \mathbb{Z}/2 = D_\infty$ is amenable
- Standard invariants at index 6 are wild.
  - There is (at least) one standard invariant for every normal subgroup of the modular group $\mathbb{Z}/2 \ast \mathbb{Z}/3 = PSL(2, \mathbb{Z})$
  - There are unclassifiably many distinct hyperfinite subfactors with the same standard invariant [BNP07, BV13]
- $4 = 2 \times 2$ and $6 = 2 \times 3$ are composite indices, as is $3 + \sqrt{5} = 2\tau^2$ where $\tau = \frac{1 + \sqrt{5}}{2}$. 
Theorem [Liu13], partial proof by Izumi-Morrison-Penneys [IMP13]

There are exactly seven 1-supertransitive subfactor planar algebras with index $3 + \sqrt{5}$:

- (self-dual)
- (and its dual)
- (and its dual)
- (and its dual)

These are all the standard invariants of composed inclusions of $A_3$ and $A_4$ subfactors.

Open question

Are there infinitely many distinct hyperfinite subfactors with the same standard invariant at index $3 + \sqrt{5}$?

- $A_3 \ast A_4$ and $A_2 \ast T_2$ are not amenable [HI98].
1-supertransitive with index at most $6\frac{1}{5}$

**Theorem [Liu-Morrison-Penneys [LMP13]]**

An exactly 1-supertransitive subfactor planar algebra with index at most $6\frac{1}{5}$ either comes from a composed inclusion (and has index $3 + \sqrt{5}$ or 6), or is one of 3 self-dual planar algebras at index $3 + 2\sqrt{2}$:

- $(\quad, \quad)$
- $(\quad, \quad)$ two complex conjugate

- Can push classification results above index 6!
- Could hope that the only wildness at index 6 is “group-like”
Index $(5, 3 + \sqrt{5})$

**Conjecture [Morrison-Peters] [MP12]**

There are exactly two non Temperley-Lieb subfactor planar algebras in the index range $(5, 3 + \sqrt{5})$:

<table>
<thead>
<tr>
<th>name</th>
<th>Principal graphs</th>
<th>Index</th>
<th>Constructed</th>
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</thead>
<tbody>
<tr>
<td>$SU(2)_5$</td>
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<td>5.04892</td>
<td>[Wen90], [MP12]</td>
</tr>
<tr>
<td>$SU(3)_4$</td>
<td>(</td>
<td>5.04892</td>
<td>[Wen88], [MP12]</td>
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</table>

**Theorem [Morrison-Peters] [MP12]**

There is exactly one 1-supertransitive subfactor in the index range $(5, 3 + \sqrt{5})$.
### Conjecture [Morrison-Penneys]

At $3 + \sqrt{5}$, we have only the following subfactor planar algebras:

<table>
<thead>
<tr>
<th>name</th>
<th>Principal graphs</th>
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<tbody>
<tr>
<td>4442</td>
<td>(, )</td>
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<td>[MP13], Izumi</td>
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<tr>
<td>$3\mathbb{Z}/2 \times \mathbb{Z}/2$</td>
<td>(, )</td>
<td>2</td>
<td>Izumi, [MP13]</td>
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<tr>
<td>$3\mathbb{Z}/4$</td>
<td>(, )</td>
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<td>Izumi, [PP13]</td>
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<td>2D2</td>
<td>(, )</td>
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<td>Izumi, [MPP]</td>
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<td>$A_3 \otimes A_4$</td>
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<td>$\otimes$</td>
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<td>$A_3 \ast A_4$</td>
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<td>[BJ97]</td>
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<td>$A_\infty$</td>
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<td>[Pop93]</td>
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</tbody>
</table>

- The 1-supertransitive case is known by [Liu13, IMP13]
Subfactor planar algebras at index 6

Wildly optimistic conjecture [Penneys-Peters-Snyder]

At index 6, a $\geq 2$-supertransitive subfactor planar algebra is one of

<table>
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<tr>
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<tbody>
<tr>
<td>$A_5b$</td>
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<td>$S_5b$</td>
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<td>$A_5 \subset A_6$</td>
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Theorem [LMP13]

If a subfactor has principal graphs an extension of $( , )$, then it is a Bisch-Haagerup subfactor of the form $R\mathbb{Z}/2 \subset R \rtimes \mathbb{Z}/3$.

- We do not yet understand composed inclusions of $A_3$ and $A_5$.
- No subfactor with principal graph $\xrightarrow{\ }$ [EG12].
How can we prove these conjectures?

Biggest hurdle: need to eliminate certain weeds.

10 weeds:

*11 weeds:
Triple points

Fact

If the graph starts with a triple point at depth \( n - 1 \), e.g.

\[
\begin{align*}
\star \star S & = \omega_S \star \\
\end{align*}
\]

then the planar algebra has an uncappable rotational eigenvector at depth \( n \) with eigenvalue \( \omega_S \) where \( \omega_S^n = 1 \).

- If there is no merging two past the branch, we get a strong constraint in terms of the structure of the graph and \( \omega_S \).
New obstruction

Theorem [Pen13]

(1) If \((\Gamma_+), (\Gamma_-)\) is a translated extension of \((\bullet\rightarrow\bullet), (\bullet\rightarrow\bullet)\), then
\[
(r - 1)\frac{\tilde{r}}{r} + \left(\sigma_S + \sigma_S^{-1}\right)\frac{\sqrt{r}}{[n]} \frac{\sqrt{\tilde{r}}}{\tilde{r}} = \frac{r[n] - [n + 2]}{[n]}.
\]

(2) If \((\Gamma_+), (\Gamma_-)\) is a translated extension of \((\bullet\rightarrow\bullet\rightarrow\bullet), (\bullet\rightarrow\bullet\rightarrow\bullet)\), then
\[
(r - 1) + \left(\sigma_S + \sigma_S^{-1}\right)\frac{\sigma_S}{[n]} = \frac{[n + 2] - r[n]}{r[n]}.
\]

\(\sigma_S\) is the chirality (\(\sigma_S^2\) is rotational eigenvalue)

\(r, \tilde{r}\) are the branch factors (ratio of dimensions past branch)
Remarks on the new obstruction

- The key is the rotation $\star S \star$.
- The obstruction is far more general. Recovers results of Jones and Snyder.
- Key relation in the proof due to Liu, which is a variant of a relation due to Wenzl.
- This obstruction eliminates the $\ast 11$ weeds.
- Can obtain rotational eigenvalues for most small index subfactors.
- Gives easy non-existence result for $D_{odd}$ and $E_7$. 
Thank you for listening!

Recent articles:

- with Izumi and Morrison - 1-supertransitive at $3 + \sqrt{5}$ - Submitted - arXiv:1308.5723
- with Liu and Morrison - 1-supertransitive below $6\frac{1}{5}$ - Submitted - arXiv:1310.8566
- new obstruction - Submitted - arXiv:1307.5890


MR2314611 arXiv:math.OA/0604460
DOI:10.1142/S0129167X07004011.


Subfactors Planar algebras Jellyfish and spokes $3 + \sqrt{5}$

Why $3 + \sqrt{5}$? Conjectures A new obstruction


Zhengwei Liu, Scott Morrison, and David Penneys, *1-supertransitive subfactors with index at most $6\frac{1}{5}$*, 2013, arXiv:1310.8566.


Scott Morrison, David Penneys, and Emily Peters, *Equivariantizations and 3333 spoke subfactors at index $3 + \sqrt{5}$*, In preparation.


