Higher Categories and Quantum Structures

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II	**Higher categories**
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Bring a laptop with Chrome. If you don't have internet access in the room, pre-load *homotopy.io* in a few browser tabs beforehand.

Part I

Monoidal categories

I.1 Motivation for monoidal categories

Category theory describes systems and processes:

- physical systems, and physical processes governing them;
- data types, and algorithms manipulating them;
- algebraic structures, and structure-preserving functions;
- logical propositions, and implications between them.

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Monoidal category theory adds the idea of *parallelism*:

- independent physical systems evolve simultaneously;
- running computer algorithms in parallel;
- products or sums of algebraic or geometric structures;
- using separate proofs of P and Q to construct a proof of the conjunction (P and Q).

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- Maybe they should be *isomorphic* — but then what *equations* should these isomorphisms satisfy?
- How do we treat *trivial* systems?
- What should the relationship be between $A \otimes B$ and $B \otimes A$?

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Theorem 2 (Coherence for monoidal categories). *If the pentagon and triangle equations hold, then so does any well-typed equation built from α, λ, ρ and their inverses.*

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Exercise. Use the triangle and pentagon equations to prove $\lambda_I = \rho_I$.

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It is a categorical product, so it really arises as a *property*.

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Again, this tensor product arises from our understanding of physical reality. However, it is *not* a categorical product; it is extra *structure*.

We can similarly define $\mathbf{Vect}_k$ and $\mathbf{FVect}_k$, with vector spaces as objects.

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Theorem 6 (Interchange). *Any morphisms* $A \xrightarrow{f} B$*,* $B \xrightarrow{g} C$*,* $D \xrightarrow{h} E$ *and* $E \xrightarrow{j} F$ *in a monoidal category satisfy the interchange law:*

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Proof. This holds because of properties of the category $\mathbf{C} \times \mathbf{C}$, and from the fact that $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor:

$$\begin{aligned} (g \circ f) \otimes (j \circ h) &\equiv \otimes(g \circ f, j \circ h) \\ &= \otimes\big((g,j) \circ (f,h)\big) && \text{(composition in } \mathbf{C} \times \mathbf{C}\text{)} \\ &= \big(\otimes(g,j)\big) \circ \big(\otimes(f,h)\big) && \text{(functoriality of } \otimes\text{)} \\ &= (g \otimes j) \circ (f \otimes h) \end{aligned}$$

Remember the functoriality property: $F(g \circ f) = F(g) \circ F(f)$.

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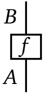
A morphism $f : A \to B$ can be depicted as a box, with an input wire of type A, and an output wire of type B.



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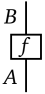
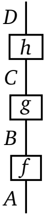


A composite chain of morphisms $f : A \to B$, $g : B \to C$, $h : C \to D$ can be depicted by stacking them vertically.



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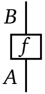

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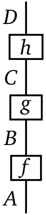

This representation is intrinsically 1-dimensional.

It is also nontrivial, because associativity and identity have been trivialized by the geometry of the line.

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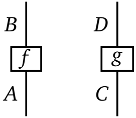
For morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, we draw their tensor product $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ like this:

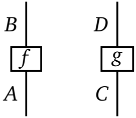


The idea is that f and g represent distinct processes taking place at the same time.

I.3 Graphical calculus

We can extend this graphical calculus to monoidal categories, in the following way.

For morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, we draw their tensor product $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ like this:



The idea is that f and g represent distinct processes taking place at the same time.

This representation is intrinsically 2-dimensional.

I.3 Graphical calculus

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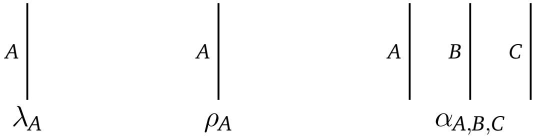

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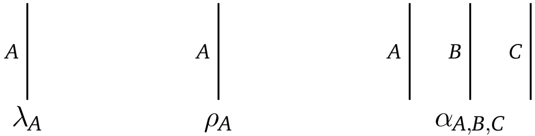

The coherence of α, λ and ρ is essential for the graphical calculus to function. Since there can only be a single morphism built from their components of any given type, it *doesn't matter* that their graphical calculus encodes no information.

I.3 Graphical calculus

Now let's look at the interchange law:

$$(g \circ f) \otimes (j \circ h) \quad = \quad (g \otimes j) \circ (f \otimes h)$$

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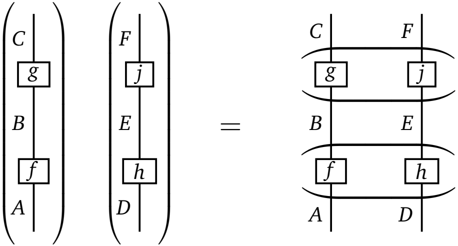

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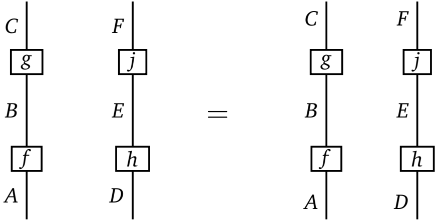

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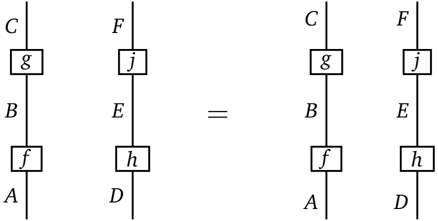

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The apparent complexity of the theory of monoidal categories—α, λ, ρ, coherence, interchange—was in fact complexity of the *geometry of the plane*. So when we use this notation, this complexity is absorbed, and becomes easy to handle.

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Two diagrams are *planar isotopic* when one can be deformed continuously into the other, such that:

- diagrams remain confined to a rectangular region of the plane;
- input and output wires terminate at the lower and upper boundaries of the rectangle;
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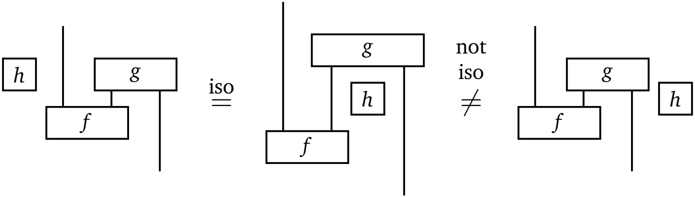
Here are examples of isotopic and non-isotopic diagrams:



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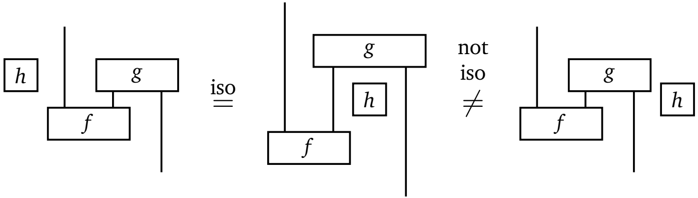

We will allow heights of the diagrams to change, and allow input and output wires to slide horizontally along the boundary, although they must never change order.

I.3 Graphical calculus

We can now state the correctness theorem.

Theorem 7 (Correctness of the graphical calculus for monoidal categories). *A well-formed equation between morphisms in a monoidal category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.*

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Let f and g be morphisms such that the equation $f = g$ is well-formed, and consider the following statements:

- $P(f, g)$ = 'under the axioms of a monoidal category, $f = g$'
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Completeness is the reverse assertion, that for all such f and g, $Q(f,g) \Rightarrow P(f,g)$. It is hard to prove; one must show that planar isotopy is generated by a finite set of moves, each being implied by the monoidal axioms.

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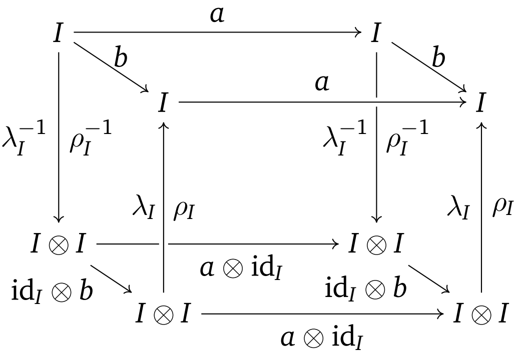

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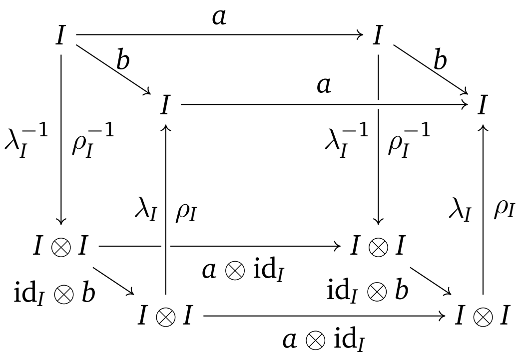

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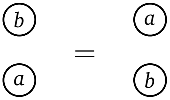
Here is an easier proof.

Proof (ii).



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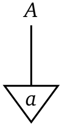
We draw a state $I \xrightarrow{a} A$ like this:



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- In **Hilb**, states of a Hilbert space H are linear functions $\mathbb{C} \to H$, which correspond to *elements* of H by considering the image of $1 \in \mathbb{C}$.

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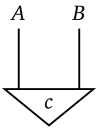
A morphism $I \xrightarrow{c} A \otimes B$ is a *joint state* of A and B. We depict it graphically in the following way.



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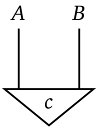
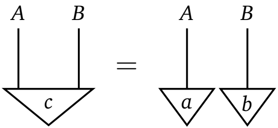


Definition 12. A joint state $I \xrightarrow{c} A \otimes B$ is a *product state* when it is of the form $I \xrightarrow{\lambda_I^{-1}} I \otimes I \xrightarrow{a \otimes b} A \otimes B$:



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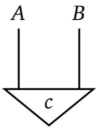

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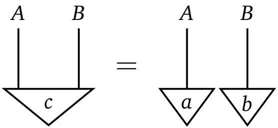

Definition 13. A joint state is *entangled* when it is not a product state.

I.4 States

Example 14. Let's investigate joint states, product states, and entangled states in our example categories.

- In **Set**:
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- In **Hilb**:
  - **joint states** of H and K are elements of $H \otimes K$;
  - **product states** are factorizable states;
  - **entangled states** are elements of $H \otimes K$ which cannot be factorized, i.e. entangled states in the quantum sense.

In this way, a central property of quantum reality can be seen in the behaviour of the monoidal structure.

I.5 Dagger structure

Our earlier definition of **Hilb** ignored the *inner products*.
These allow us to construct *adjoint* linear maps, as follows:

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We can *recover* the inner products from this functor:

$$(\mathbb{C} \xrightarrow{w} H \xrightarrow{v^\dagger} \mathbb{C}) \equiv v^\dagger(w(1)) = \langle 1 | v^\dagger(w(1)) \rangle = \langle v | w \rangle$$

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So † and $\langle - | - \rangle$ encode equivalent information.

This inspires the following abstract definition.

Definition 15. A *dagger structure* on a category **C** is an involutive contravariant functor $\dagger\colon \mathbf{C} \to \mathbf{C}$ that is the identity on objects. A *dagger category* is a category equipped with a dagger structure.

Definition 16. In a dagger category, a morphism $f : A \to B$ is *unitary* when $f \circ f^\dagger = \mathrm{id}_B$ and $f^\dagger \circ f = \mathrm{id}_A$.

Definition 17. A *monoidal dagger category* is a monoidal category with a dagger structure, such that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ and α, λ, ρ unitary.

I.6 Braiding and symmetry

In many settings, the systems $A \otimes B$ and $B \otimes A$ can be considered essentially equivalent. Developing this idea gives rise to *braided* and *symmetric* monoidal categories.

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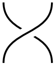
satisfying the following *hexagon equations*:

$$\begin{array}{ccc}
A \otimes (B \otimes C) & \xrightarrow{\sigma_{A,B\otimes C}} & (B \otimes C) \otimes A \\
\downarrow \alpha^{-1}_{A,B,C} & & \uparrow \alpha^{-1}_{B,C,A} \\
(A \otimes B) \otimes C & & B \otimes (C \otimes A) \\
\downarrow \sigma_{A,B} \otimes \mathrm{id}_C & & \uparrow \mathrm{id}_B \otimes \sigma_{A,C} \\
(B \otimes A) \otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B \otimes (A \otimes C)
\end{array}
\qquad
\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{\sigma_{A\otimes B,C}} & C \otimes (A \otimes B) \\
\downarrow \alpha_{A,B,C} & & \uparrow \alpha_{C,A,B} \\
A \otimes (B \otimes C) & & (C \otimes A) \otimes B \\
\downarrow \mathrm{id}_A \otimes \sigma_{B,C} & & \uparrow \sigma_{A,C} \otimes \mathrm{id}_B \\
A \otimes (C \otimes B) & \xrightarrow[\alpha^{-1}_{A,C,B}]{} & (A \otimes C) \otimes B
\end{array}$$

I.6 Braiding and symmetry

We include the braiding in our graphical notation like this:



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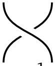

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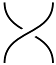

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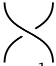

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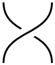

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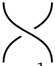

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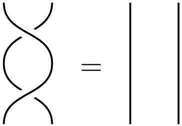
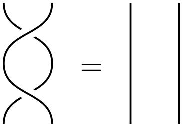
Invertibility takes the following graphical form:





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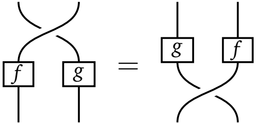
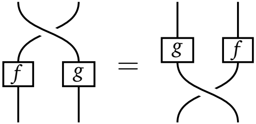
Naturality has the following graphical representation:





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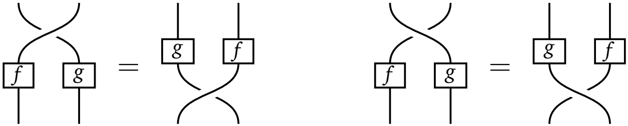
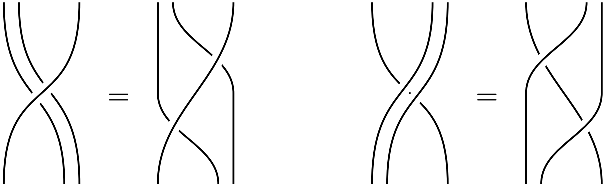


The hexagon equations look like this:



So braiding with a tensor product of two objects is the same as braiding with one then the other separately.

I.6 Braiding and symmetry

Braided monoidal categories have a sound and complete graphical calculus, as established by the following theorem.

Theorem 19 (Correctness of graphical calculus for braided monoidal categories). *A well-formed equation between morphisms in a braided monoidal category follows from the axioms if and only if it holds in the graphical language up to 3-dimensional isotopy.*

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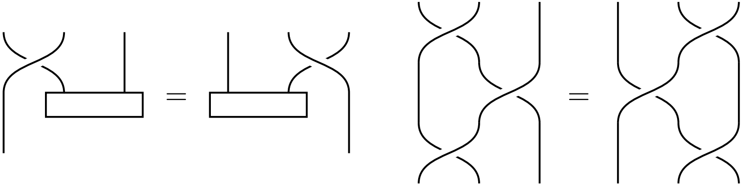
The coherence theorem is very powerful. For example, the following equations hold:



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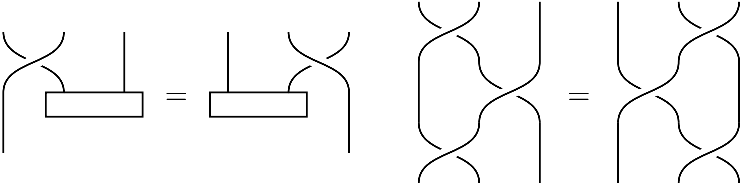

The second equation is called the *Yang–Baxter equation*, which plays an important role in the mathematical theory of knots.

I.6 Braiding and symmetry

Let’s consider this structure for our example categories.

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Definition 20. The monoidal categories **Hilb** and **Set** can all be equipped with a canonical braiding.

- In **Hilb**, $H \otimes K \xrightarrow{\sigma_{H,K}} K \otimes H$ is the unique linear map extending $a \otimes b \mapsto b \otimes a$ for all $a \in H$ and $b \in K$.

I.6 Braiding and symmetry

Let's consider this structure for our example categories.

Definition 20. The monoidal categories **Hilb** and **Set** can all be equipped with a canonical braiding.

- In **Hilb**, $H \otimes K \xrightarrow{\sigma_{H,K}} K \otimes H$ is the unique linear map extending $a \otimes b \mapsto b \otimes a$ for all $a \in H$ and $b \in K$.
- In **Set**, $A \times B \xrightarrow{\sigma_{A,B}} B \times A$ is defined by $(a, b) \mapsto (b, a)$ for all $a \in A$ and $b \in B$.

I.6 Braiding and symmetry

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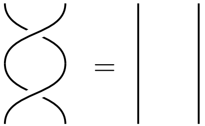

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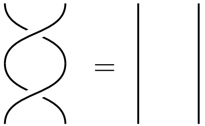

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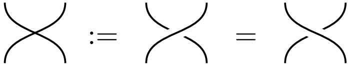
Lemma 22. In a symmetric monoidal category $\sigma_{A,B} = \sigma_{B,A}^{-1}$, with the following graphical representation:



Part II

Higher categories

II.1 Introduction to 2-categories

Definition 23. A *2-category* **C** consists of the following data:

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- a collection $\mathrm{Ob}(\mathbf{C})$ of *objects*;

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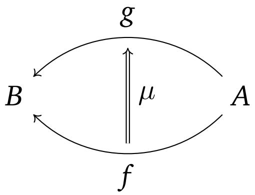
Definition 23. A *2-category* **C** consists of the following data:

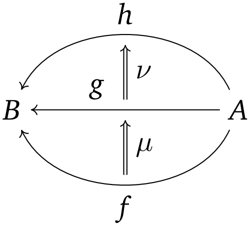
- a collection $\mathrm{Ob}(\mathbf{C})$ of *objects*;

- for any two objects A, B, a category $\mathbf{C}(A, B)$, with objects called *1-morphisms* drawn as $A \xrightarrow{f} B$, and morphisms μ called *2-morphisms* drawn as $f \overset{\mu}{\Longrightarrow} g$, or in full form as follows:



II.1 Introduction to 2-categories

- for 2-morphisms $f \overset{\mu}{\Longrightarrow} g$ and $g \overset{\nu}{\Longrightarrow} h$, an operation called *vertical composition* given by their composite as morphisms in $\mathbf{C}(A,B)$:



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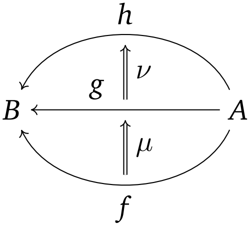
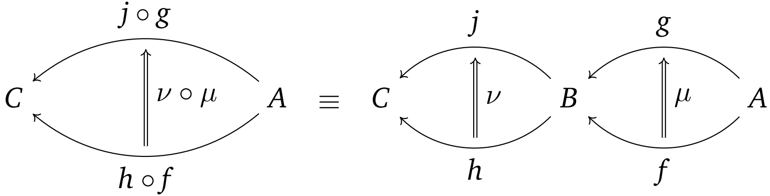


- for any triple of objects A, B, C a *horizontal composition* functor:

$$\circ : \mathbf{C}(A, B) \times \mathbf{C}(B, C) \to \mathbf{C}(A, C)$$



II.1 Introduction to 2-categories

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Definition 24. A 2-category is *strict* just when every λ_f, ρ_f, $\alpha_{h,g,f}$ is an identity morphism.

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Unit object	Identity 1-morphism

The transformations α, λ and ρ are the same for both structures.

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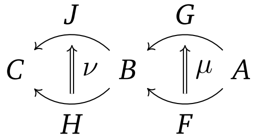
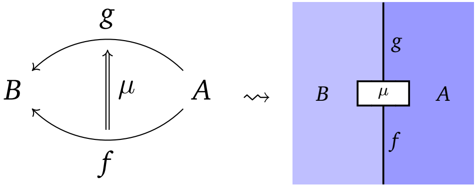
- **horizontal composition** of 

  is $(\nu \circ \mu)_A := \nu_{G(A)} \circ H(\mu_A) = J(\mu_A) \circ \nu_{F(A)}$.

II.2 Graphical calculus for 2-categories

We can extend the graphical calculus to 2-categories.

Objects are represented by regions, 1-morphisms by vertically-oriented lines, and 2-morphisms by vertices:



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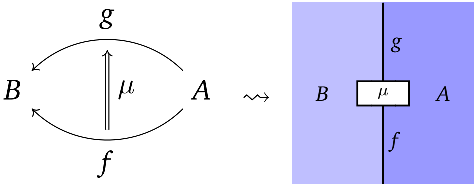
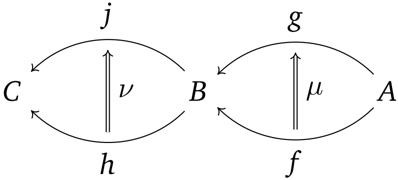


The graphical calculus is the *dual* of the traditional pasting diagram notation given on the left.

II.2 Graphical calculus for 2-categories

Horizontal and vertical composition is represented like this:



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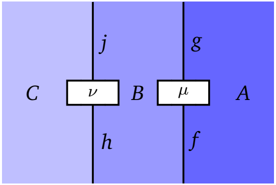
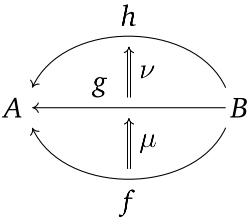


$= \nu \circ \mu$



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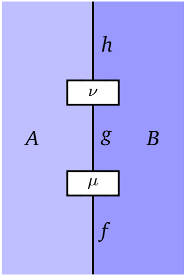

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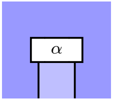
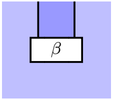
If we have only a single object A, we may as well denote this by a region coloured white. Then the graphical calculus is identical to that of a monoidal category.

II.3 Equivalence and duality

We can use the graphical calculus to define a notion of equivalence.

Definition 25. In a 2-category, an *equivalence* is a pair of 1-morphisms $A \xrightarrow{F} B$ and $B \xrightarrow{G} A$, and 2-morphisms $G \circ F \overset{\alpha}{\Longrightarrow} \mathrm{id}_A$ and $\mathrm{id}_B \overset{\beta}{\Longrightarrow} F \circ G$:





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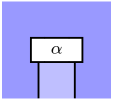
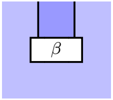
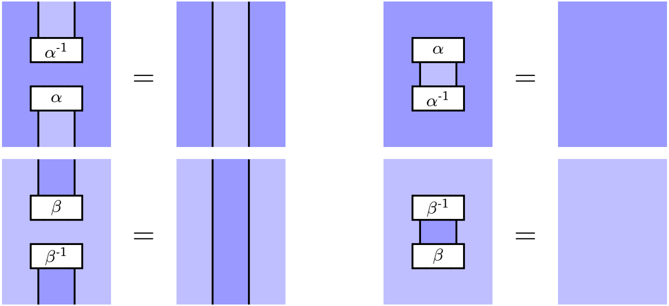




They must satisfy the following equations:



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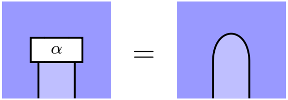
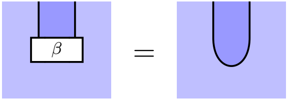
Definition 26. In a 2-category, a 1-morphism $A \xrightarrow{L} B$ has a *right dual* $B \xrightarrow{R} A$ when there are 2-morphisms $G \circ F \xRightarrow{\alpha} \mathrm{id}_A$ and $\mathrm{id}_B \xRightarrow{\beta} F \circ G$





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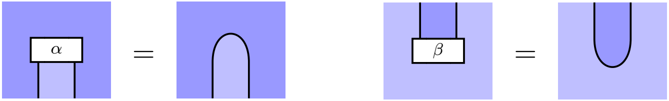

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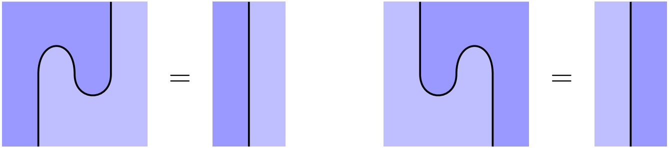

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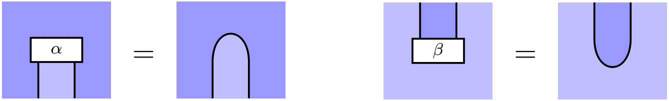

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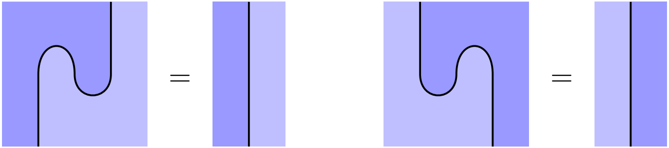

Theorem 27. *In* **Cat***, a duality $F \dashv G$ is exactly an adjunction $F \dashv G$ between F and G as functors.*

II.3 Equivalence and duality

We now prove a nontrivial theorem relating equivalences and duals.

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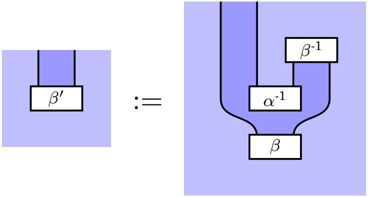
Proof. Suppose we have an equivalence in a 2-category, witnessed by invertible 2-morphisms α and β. Then we will build a new equivalence witnessed by α and β', with β' defined like this:



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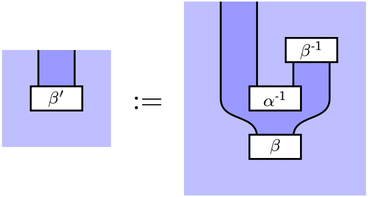
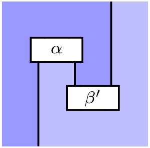


Since α' is composed from invertible 2-morphisms it must itself be invertible, and so it is clear that α' and β still yield an equivalence.

II.3 Equivalence and duality

We now demonstrate that the adjunction equations are satisfied.

The first adjunction equation takes following form:



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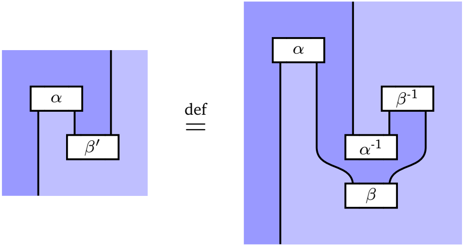

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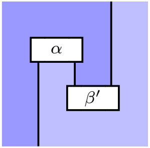
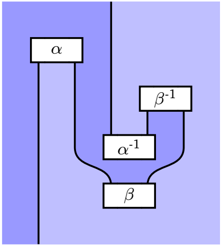


$\overset{\text{def}}{=}$



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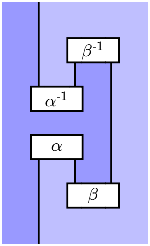

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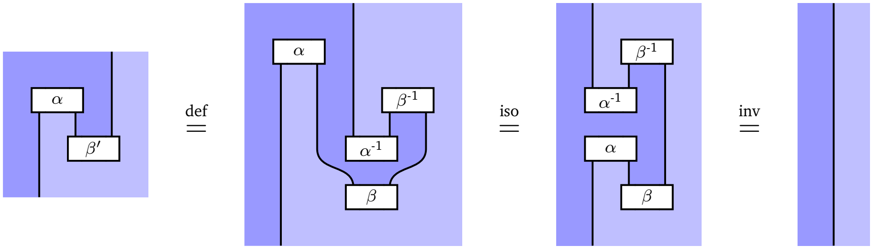

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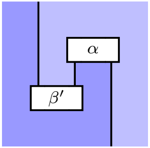

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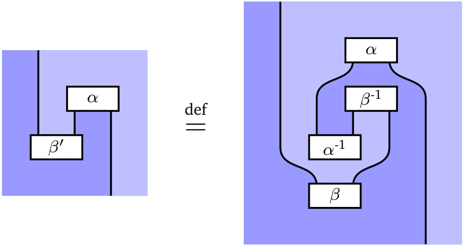

□

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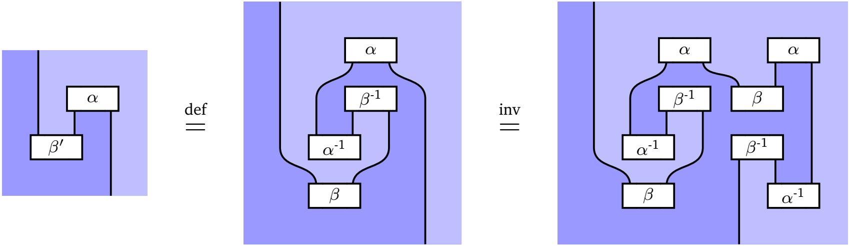

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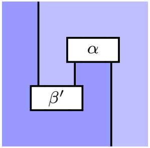
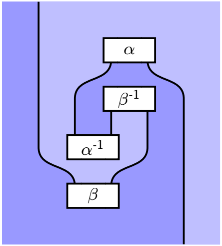
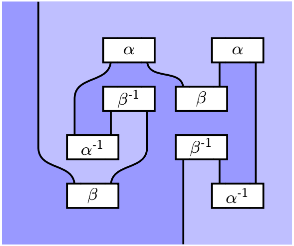


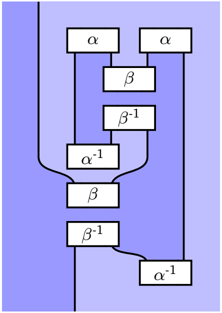
$\overset{\text{def}}{=}$



$\overset{\text{inv}}{=}$



$\overset{\text{iso}}{=}$



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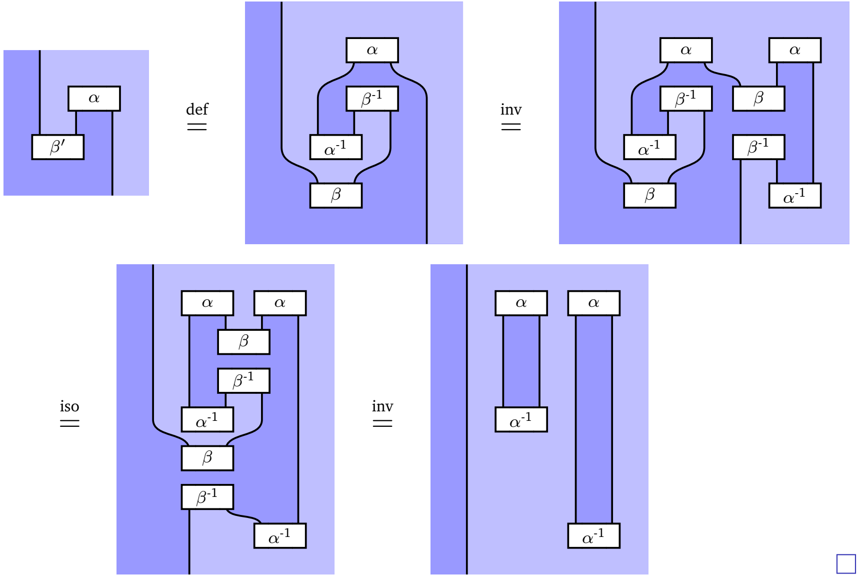

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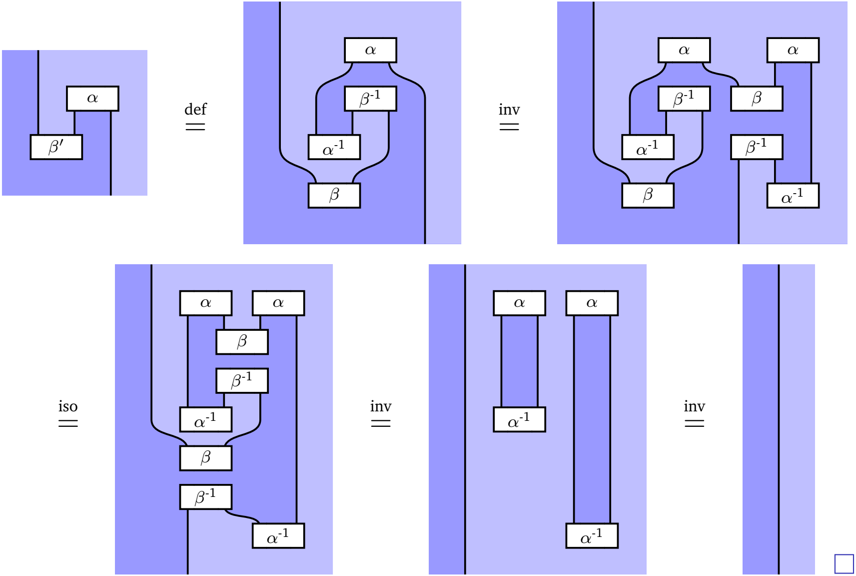

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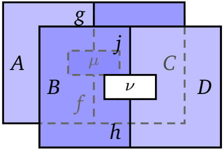
Tensor product. Given 2-morphisms $f \overset{\mu}{\Longrightarrow} g$ and $h \overset{\nu}{\Longrightarrow} j$, the their *tensor product* 2-morphism $\mu \boxtimes \nu$ is depicted like this:



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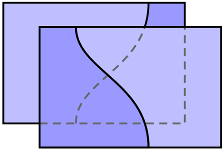
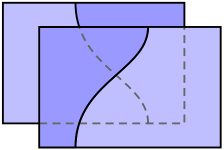
Interchange. Components can move freely in their separate layers, with the order of 1-morphisms in separate sheets being *interchanged*:





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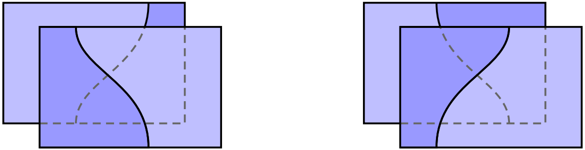

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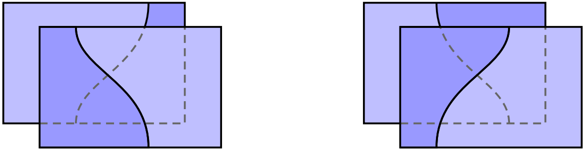

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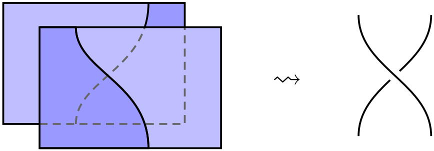
Unit object. A monoidal 2-category has a *unit object* I, represented by a ‘blank’ region.

II.5 The periodic table

Something interesting happens when we consider interchangers in the context of the unit object.

Consider the interchanger diagram, but with all 4 planar regions labelled by the unit object:



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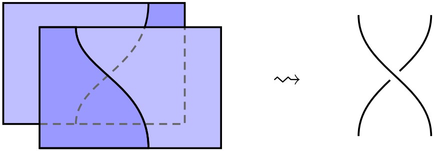

We obtain the graphical representation of a *braiding*.

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The emerging pattern here is called the *periodic table*, and was predicted by Baez and Dolan in 1995.

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0	Set					
1						
2						

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2						

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		k					
		0	1	2	3	4	5
n	0	Set	Monoid	Commutative monoid	. . .		
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	2						

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n \ k	0	1	2	3	4	5
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2						

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n \ k	0	1	2	3	4	5
-2	Point	...				
-1	Truth value	Point	...			
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Conjecture (String Diagram Hypothesis.) String diagrams of n-dimensional structures in $(n+k)$-dimensional space give a sound and complete calculus for k-tuply monoidal n-categories.

Practical — *http://homotopy.io*

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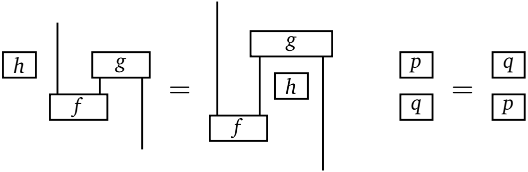
Monoidal



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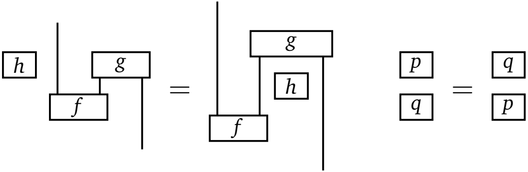

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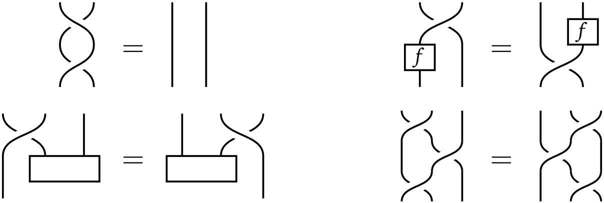

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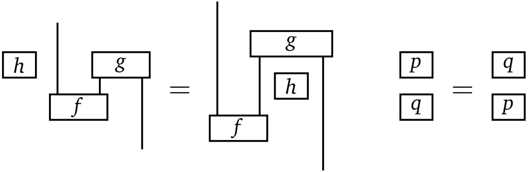

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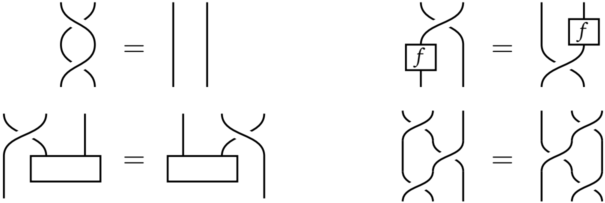
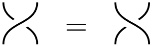


Symmetric

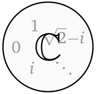


Part III

Higher vector spaces

III.1. 2–vector spaces

$$\begin{matrix} & 1 & \sqrt{2}-i \\ 0 & & \\ & i & \ddots \end{matrix}$$


1
0
ℂ
√2−i
i

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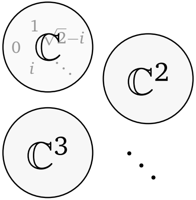

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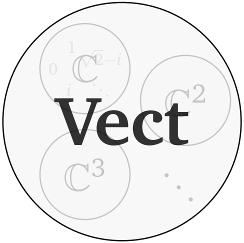

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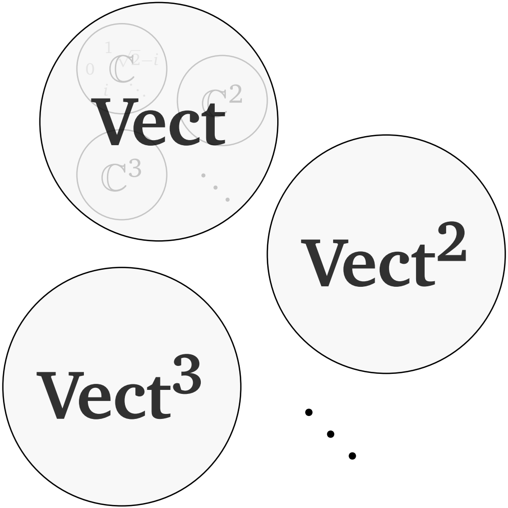

III.1. 2–vector spaces


2Vect
Vect
ℂ
ℂ2
ℂ3
Vect2
Vect3

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Several more 'coordinate-independent' definitions of semisimplicity: For example, a k-linear category is finite semisimple if it is abelian, every object is a finite direct sum of simple objects and there are only finitely many isomorphism classes of simple objects.

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- vector spaces have sums of elements $v + w$, while 2–vector spaces have biproducts $A \oplus B$;
- in a vector space we can multiply a vector by any element of the field k, while in a 2–vector space we can multiply an object by any vector space.

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There is a matrix calculus, just as for ordinary vector spaces.

Definition 32. The symmetric monoidal 2-category $\mathbf{Mat}(\mathbf{Vect}_k)$ is built from the following structures:

- 0-cells are natural numbers;
- 1-cells are matrices of finite-dimensional vector spaces;
- 2-cells are matrices of linear maps.

The canonical 2-functor $\mathbf{Mat}(\mathbf{Vect}_k) \to \mathbf{2Vect}$ is an equivalence.

III.1. 2–vector spaces

$$\begin{pmatrix} V_{11} & \cdots & V_{1n} \\ \vdots & \ddots & \vdots \\ V_{m1} & \cdots & V_{mn} \end{pmatrix}$$

(a) A 1-cell $V : n \to m$

$$\begin{pmatrix} V_{11} \xrightarrow{\phi_{11}} V'_{11} & \cdots & V_{1n} \xrightarrow{\phi_{1n}} V'_{1n} \\ \vdots & \ddots & \vdots \\ V_{m1} \xrightarrow{\phi_{m1}} V'_{m1} & \cdots & V_{mn} \xrightarrow{\phi_{mn}} V'_{mn} \end{pmatrix}$$

(b) A 2-cell $\phi : V \Rightarrow V'$

III.1. 2–vector spaces

$$\begin{pmatrix} V_{11} & \cdots & V_{1n} \\ \vdots & \ddots & \vdots \\ V_{m1} & \cdots & V_{mn} \end{pmatrix} \qquad \begin{pmatrix} V_{11} \xrightarrow{\phi_{11}} V'_{11} & \cdots & V_{1n} \xrightarrow{\phi_{1n}} V'_{1n} \\ \vdots & \ddots & \vdots \\ V_{m1} \xrightarrow{\phi_{m1}} V'_{m1} & \cdots & V_{mn} \xrightarrow{\phi_{mn}} V'_{mn} \end{pmatrix}$$

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Vertical composition of 2-cells:
entry-wise composition of linear maps

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The dual of a 1-cell F:
'conjugate transpose' with conjugate $\overline{(-)}$ replaced by dual $(-)^*$

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Composition of 1*-cells:*
matrix product with $(+, \cdot)$ replaced by $(\oplus, \otimes)$

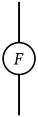
The dual of a 1*-cell F:*
'conjugate transpose' with conjugate $\overline{(-)}$ replaced by dual $(-)^*$

Monoidal product of 1*-cells:*
Kronecker product with $\cdot$ replaced by $\otimes$

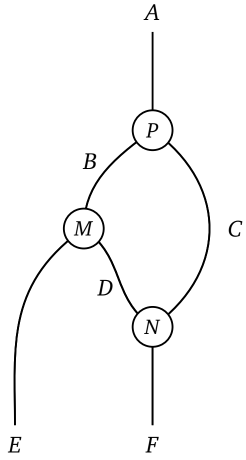
III.1. 2–vector spaces

A direct perspective on $\mathbf{2Vect}_k$.

vector space
V



linear map
$F : V \to W$



A (composed) linear map
$L : E \otimes F \to A$

III.1. 2–vector spaces

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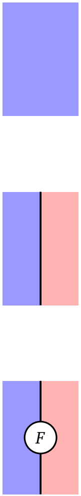
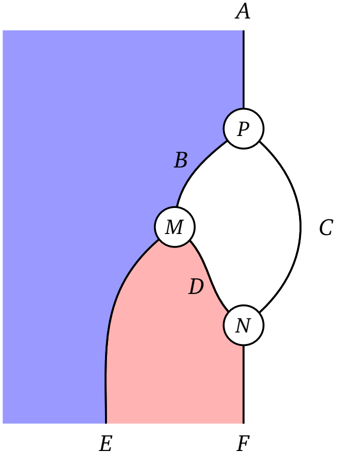
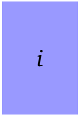


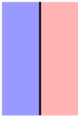


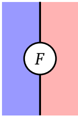
III.1. 2–vector spaces

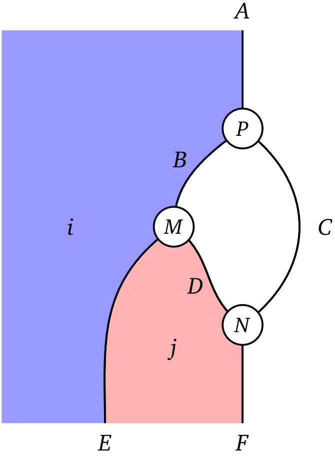
A direct perspective on **2Vect**$_k$.



indexing set
$i \in S$

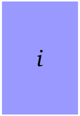






III.1. 2–vector spaces

A direct perspective on $\mathbf{2Vect}_k$.



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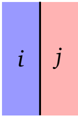
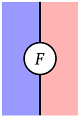
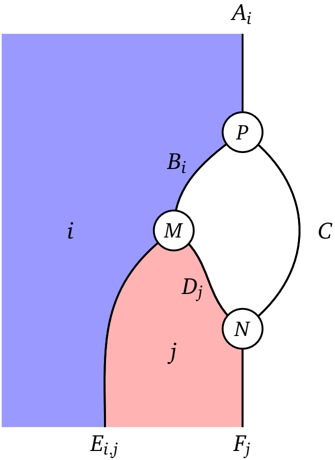


family of vector
spaces $V_{i,j}$





III.1. 2–vector spaces

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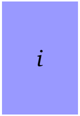
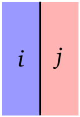
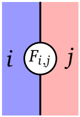


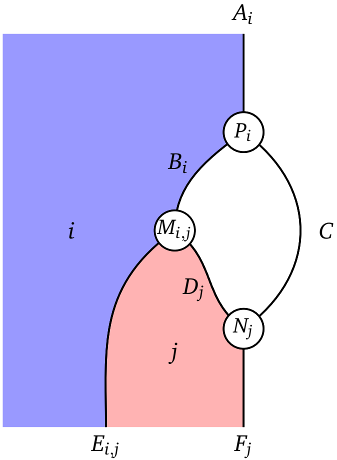
indexing set
$i \in S$



family of vector spaces $V_{i,j}$

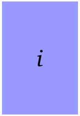


family of linear maps
$F_{i,j} : V_{i,j} \to W_{i,j}$



III.1. 2–vector spaces

A direct perspective on $\mathbf{2Vect}_k$.



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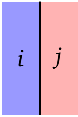

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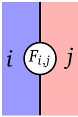

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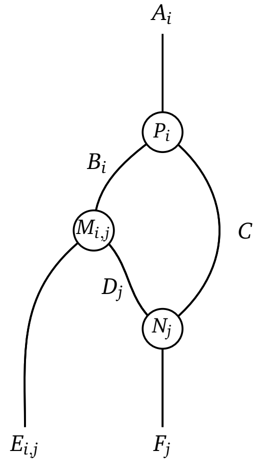
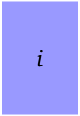


A family of linear maps, indexed by i and j

III.1. 2–vector spaces

A direct perspective on $\mathbf{2Vect}_k$.



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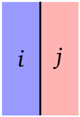

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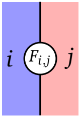

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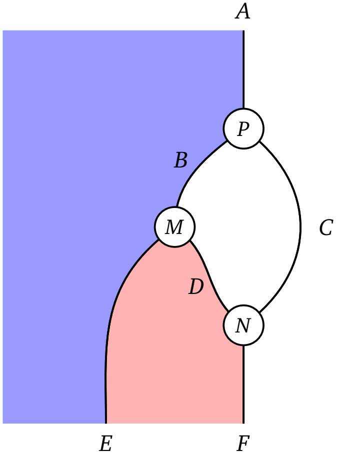

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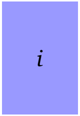

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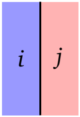

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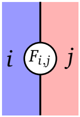

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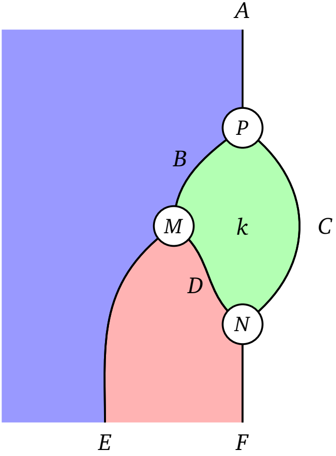

A family of linear maps, indexed by i and j
$\sum_k L^k_{i,j} : E_{i,j} \otimes F_j \to A_i$

III.2. 2–Hilbert spaces

2-*Hilbert spaces* are the ‘dagger’ versions of 2-vector spaces.

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Definition 33. A finite-dimensional 2-Hilbert space is a $\mathbb{C}$-linear dagger category equivalent to $\mathbf{Hilb}^n$ for some $n \in \mathbb{N}$.

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There is an equivalent matrix calculus $\mathbf{Mat}(\mathbf{Hilb})$ with matrices of finite-dimensional Hilbert spaces.

III.3. 2Vect & graph planar algebras

The following three structures are equivalent:

- isomorphism classes of 1-cells in **2Vect**;
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Let Γ be a bipartite graph with corresponding 1-cell $[\Gamma] : n \to m$.

$$[\Gamma]_{a,b} \cong k\langle \text{edges between } a \text{ and } b \rangle \cong k\langle \text{paths } a \rightsquigarrow b \text{ of length one} \rangle$$

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More generally,

$$([\Gamma]^* \circ [\Gamma] \circ \cdots \circ [\Gamma]^* \circ [\Gamma])_{a,b} \cong \bigoplus_{x_1,\ldots,x_{2n}} [\Gamma]^*_{x_1,a} \otimes [\Gamma]_{x_1,x_2} \otimes \cdots \otimes [\Gamma]^*_{x_{2n},x_{2n-1}} \otimes [\Gamma]_{x_{2n},b}$$

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$\Rightarrow$ The full sub-2-category T_Γ of **2Vect** generated from $[\Gamma]$ and $[\Gamma]^*$ is the *graph planar algebra* associated to Γ (after a choice of a pivotal structure on T_Γ).

III.4. . . . towards higher vector spaces

There is yet another perspective on 2-vector spaces:

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For 2-Hilbert spaces, we replace f.d. semisimple algebras by finite-dimensional C^*-algebras and bimodules by Hilbert bimodules.

III.4. . . . towards higher vector spaces

					objects
k					elements of k

III.4. . . . towards higher vector spaces

				objects	morphisms
k					elements of k
Vect				f.d. vector spaces	linear maps

III.4. ...towards higher vector spaces

			objects	1-morphisms	2-morphisms
k					elements of k
Vect				f.d. vector spaces	linear maps
2Vect		Mod(−)	f.d. semisimple algebras	f.d. bimodules	intertwiners
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III.4. . . . towards higher vector spaces

		objects	1-morphisms	2-morphisms	3-morphisms
k					elements of k
Vect				f.d. vector spaces	linear maps
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		$\cong$	finite semisimple categories	linear functors	natural transformations
3Vect		multifusion categories	finite semisimple bimodule categories	intertwining functors	natural transformations

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For $n = 1, 2, 3$, **nVect** is a symmetric monoidal n-category with duals.

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$$\mathbf{3Vect}(\mathrm{I},\mathrm{I}) \cong \mathbf{2Vect} \qquad \mathbf{2Vect}(\mathrm{I},\mathrm{I}) \cong \mathbf{Vect} \qquad \mathbf{Vect}(\mathrm{I},\mathrm{I}) = k$$

III.4. …towards higher vector spaces

		objects	1-morphisms	2-morphisms	3-morphisms
k					elements of k
Vect				f.d. vector spaces	linear maps
2Vect		Mod(−) $\cong$	f.d. semisimple algebras	f.d. bimodules	intertwiners
			finite semisimple categories	linear functors	natural transformations
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Conjecture: 4Vect is a symmetric monoidal 4-category with duals.

III.4. . . . towards higher vector spaces

An emerging big picture on **nVect**:

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Definition 36. The *idempotent completion* $\widehat{\mathcal{C}}$ of a category $\mathcal{C}$ has

- objects: idempotents $p : A \to A$ in $\mathcal{C}$;
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Exm: Algebra $A \rightsquigarrow$ category $\mathrm{B}A$ with one object $*$ and $\mathrm{End}(*) = A$.
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Conclusion:
Studying **nVect** is about studying 'higher idempotents' in **Vect**.

Part IV

Dualizability and topological quantum field theory

IV.1 Outline and motivation

In physics, a *quantum field theory* in n-dimensions allows us to evaluate a "partition function" Z on a closed n-manifold M, to obtain a complex number $Z(M)$.

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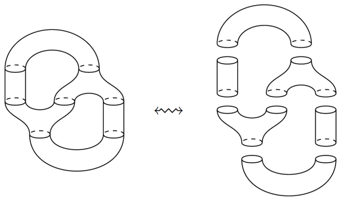
André Henrique's course next week will look at *conformal* field theories, which adds further geometrical structure.

IV.1 Outline and motivation

To enforce some degree of locality, we can imagine *cutting* n-manifolds along codimension-1 boundaries, illustrated here for $n = 2$:



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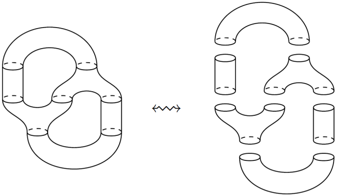

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Definition 37. The symmetric monoidal category $\mathbf{Bord}^{\mathrm{or}}_{1,2}$ of *once-extended 2-dimensional oriented bordisms* is defined as follows:

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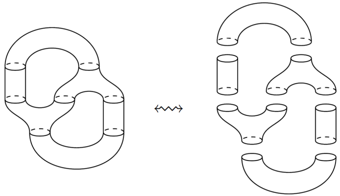

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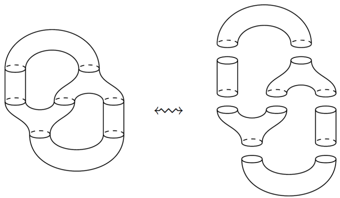

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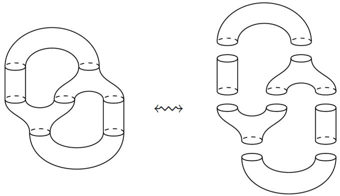

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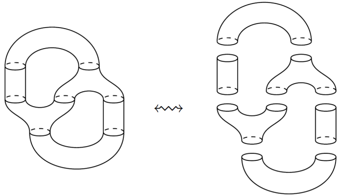

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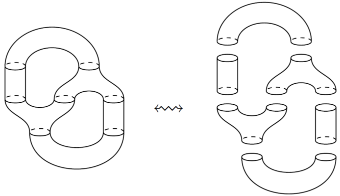

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This sends the circle to a Hilbert space of *boundary conditions*.

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Plan. Show $\mathbf{Bord}^{\mathcal{S}}_{n-k,\ldots,n}$ is sometimes *free* on some structure. TQFTs are then just *instances* of this structure in $n\mathbf{Vect}_k$.

IV.2 Duals in a monoidal category

Definition 41. An object L is *left dual* to an object R, and R is *right dual* to L, written $L \dashv R$, when there is a unit morphism $I \xrightarrow{\eta} R \otimes L$ and a counit morphism $L \otimes R \xrightarrow{\varepsilon} I$ such that:

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$$\begin{array}{ccccc}
L & \xrightarrow{\rho_L^{-1}} & L \otimes I & \xrightarrow{\mathrm{id}_L \otimes \eta} & L \otimes (R \otimes L) \\
{\scriptstyle \mathrm{id}_L}\downarrow & & & & \downarrow{\scriptstyle \alpha^{-1}_{L,R,L}} \\
L & \xleftarrow[\lambda_L]{} & I \otimes L & \xleftarrow[\varepsilon \otimes \mathrm{id}_L]{} & (L \otimes R) \otimes L
\end{array}$$

$$\begin{array}{ccccc}
R & \xrightarrow{\lambda_R^{-1}} & I \otimes R & \xrightarrow{\eta \otimes \mathrm{id}_R} & (R \otimes L) \otimes R \\
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This notation is chosen because of the attractive form it gives to the duality equations:

They are also called the *snake equations*.

IV.2 Duals in a monoidal category

The monoidal category **FHilb** has all duals. Every finite-dimensional Hilbert space H is both right dual and left dual to its dual Hilbert space H^*, in a canonical way.

Of course, this is the origin of the terminology.

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This is an *entangled state* of $H^* \otimes H$.

So category theory can express important *logical* properties of linear algebra, which we can use to study quantum information.

IV.2 Duals in a monoidal category

Duality is a *property-like* structure, in the following sense.

Lemma 42. *In a monoidal category with $L \dashv R$, then $L \dashv R'$ if and only if $R \simeq R'$. Similarly, if $L \dashv R$, then $L' \dashv R$ if and only if $L \simeq L'$.*

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Proof. If $L \dashv R$ and $L \dashv R'$, define maps $R \to R'$ and $R' \to R$ as follows:

The snake equations imply that these are inverse. Conversely, if $L \dashv R$ and $R \xrightarrow{f} R'$ is invertible, we can construct a duality $L \dashv R'$:

□

IV.2 Duals in a monoidal category

If the monoidal category has a braiding then a duality $L \dashv R$ gives rise to a duality $R \dashv L$, as the next lemma investigates.

Lemma 43. *In a braided monoidal category,* $L \dashv R \;\Rightarrow\; R \dashv L$.

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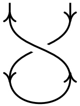
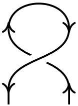
Lemma 43. *In a braided monoidal category,* $L \dashv R \;\Rightarrow\; R \dashv L$.

Proof. Construct a new duality as follows:



$I \xrightarrow{\eta'} L \otimes R$



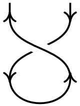
$R \otimes L \xrightarrow{\varepsilon'} I$

IV.2 Duals in a monoidal category

If the monoidal category has a braiding then a duality $L \dashv R$ gives rise to a duality $R \dashv L$, as the next lemma investigates.

Lemma 43. *In a braided monoidal category, $L \dashv R \;\Rightarrow\; R \dashv L$.*

Proof. Construct a new duality as follows:



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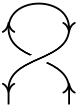
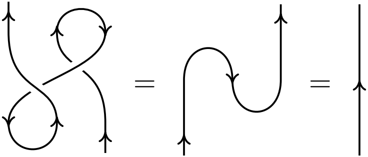


$R \otimes L \xrightarrow{\varepsilon'} I$

We can then test the snake equations:



The other snake equation can be proved in a similar way. □

IV.2 Duals in a monoidal category

We can use dual objects to characterize the oriented bordism categories in dimension 1.

Theorem 44. *The symmetric monoidal category* $\mathbf{Bord}^{\mathrm{or}}_{0,1}$ *is equivalent to the free symmetric monoidal category on an object with a right dual.*

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As a consequence, 1-dimensional oriented TQFTs

$$Z : \mathbf{Bord}^{\mathrm{or}}_{0,1} \to \mathbf{Hilb}$$

are given up to isomorphism by Hilbert spaces that have duals.

These are exactly the finite-dimensional Hilbert spaces.

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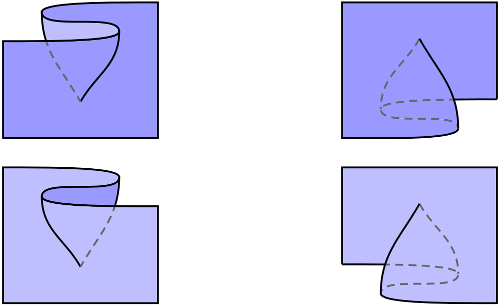
These are exactly the finite-dimensional Hilbert spaces.

If we choose a Hilbert space H and a compact oriented 1-manifold M, then we obtain $Z(M) = \dim(H)^{\text{components}(M)}$, which is diffeomorphism-invariant as required.

IV.3 Duals in a monoidal 2-category

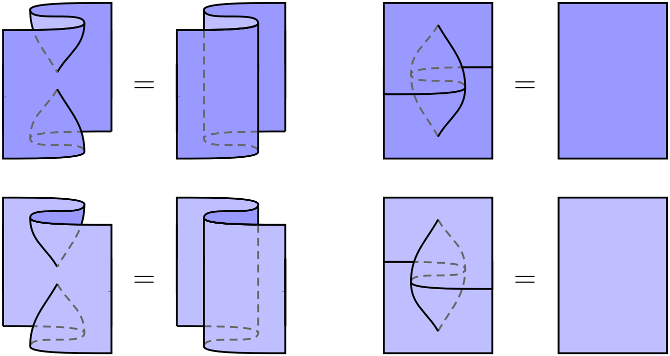
Definition 45. In a monoidal 2-category, an object L has a *right dual* R when it can be equipped with 1-morphisms called *folds*

and invertible 2-morphisms called *cusps*:



IV.3 Duals in a monoidal 2-category

The invertibility equations look like this:

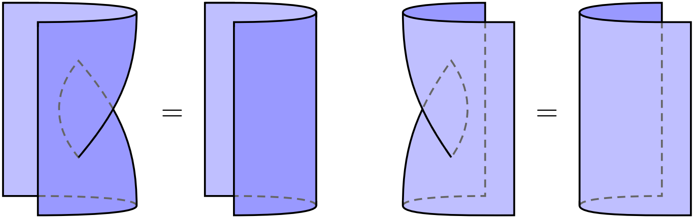


These are equations we would expect to be satisfied by surfaces embedded in $\mathbb{R}^3$!

IV.3 Duals in a monoidal 2-category

Definition 46. In a monoidal 2-category, a duality of objects $L \dashv R$ is *coherent* when it satisfies the four *swallowtail equations*.

Here are two of them:



There are 2 more which are similar.

IV.3 Duals in a monoidal 2-category

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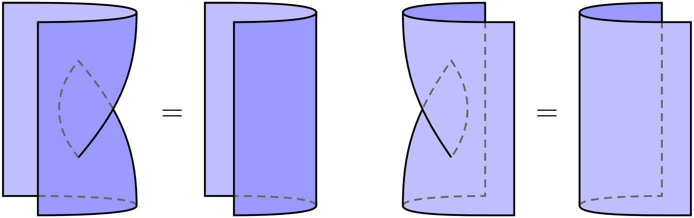

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Note the interchangers playing a key role in these equations.

IV.3 Duals in a monoidal 2-category

Theorem 47 (Pstragowski). *In a monoidal 2-category, every dual pair of objects gives rise to a coherent dual pair.*

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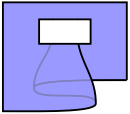
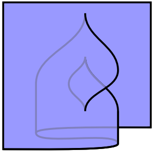


$=$



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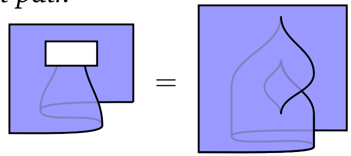

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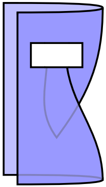

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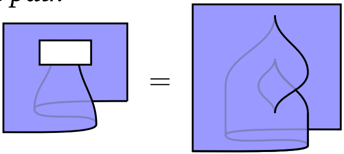

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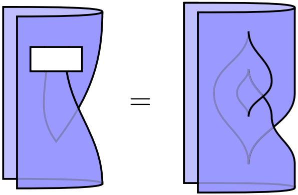

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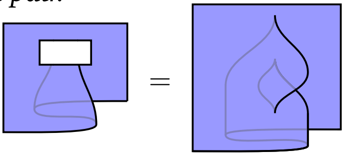

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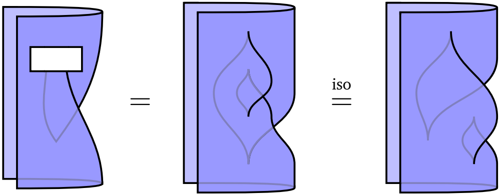

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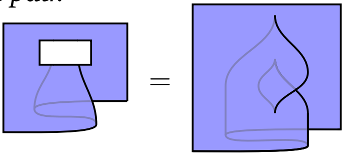

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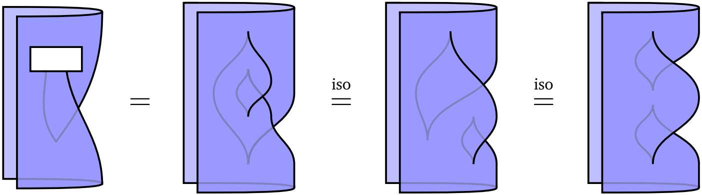

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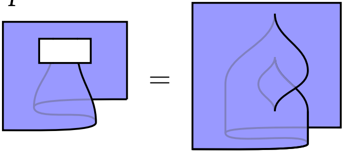

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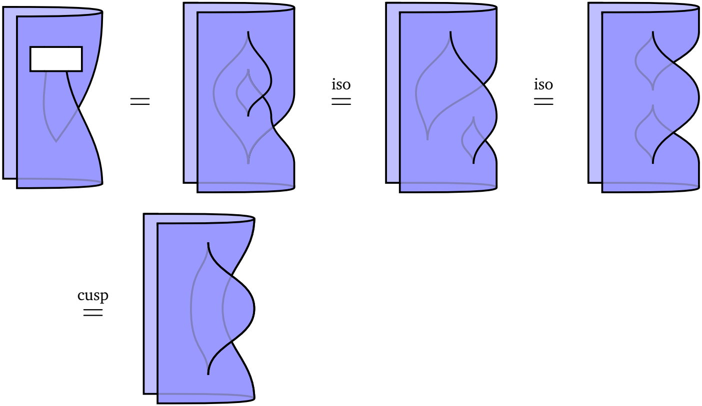

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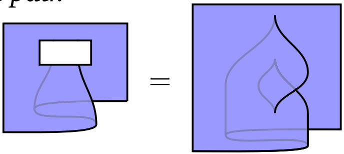

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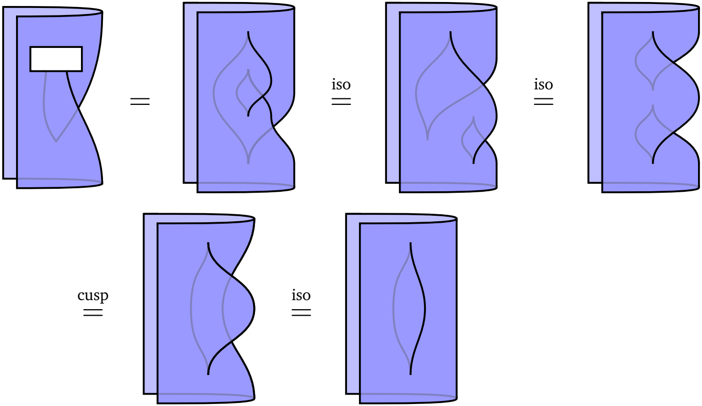

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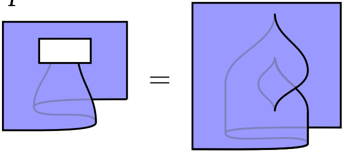

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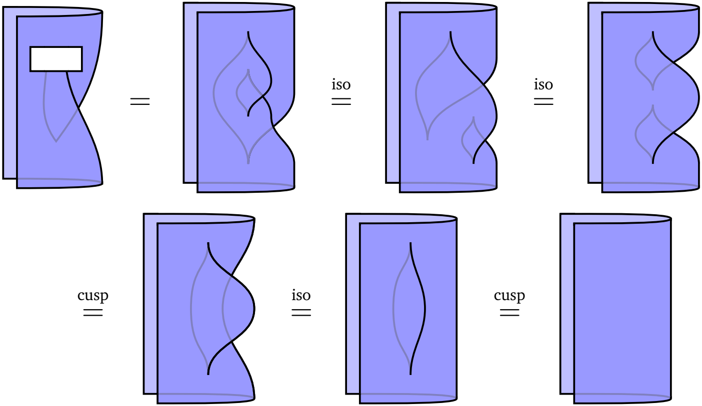

□

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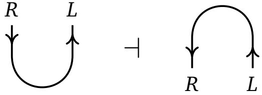
Let’s go further and imagine a duality of 1-morphisms like this:



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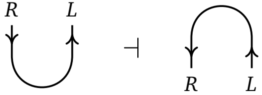

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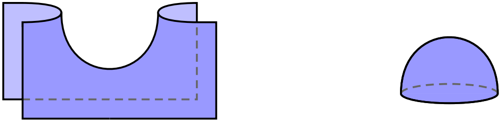

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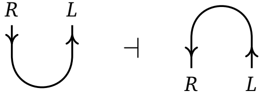

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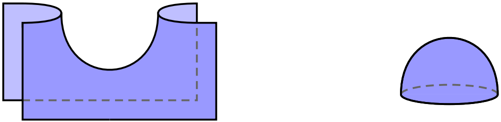
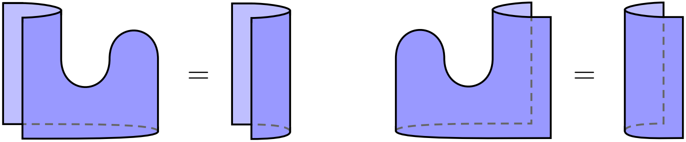


The snake equations for the duality would then look like this:



This gives all of the structure of *framed* 2-manifolds.

IV.3 Duals in a monoidal 2-category

We have motivated the following result.

Theorem 48 (Schommer-Pries). *The symmetric monoidal 2-category* $\mathbf{Bord}^{\mathrm{fr}}_{0,1,2}$ *is equivalent to the symmetric monoidal 2-category freely generated by one object with all coherent left and right duals, such that the cups and caps also have left and right duals.*

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In principle, this makes such TQFTs much easier to find.

However, beware the following:

- We still need "coherent" in the statement of Theorem 48.
- Computing an actual topological invariant still requires a coherent duality structure on our chosen object in **C**.

IV.4 The cobordism hypothesis

We now generalize this idea to arbitrary dimension, and give a statement of the *cobordism hypothesis*, one of the most exciting ideas in higher category theory.

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We begin with a *coinductive* definition of equivalence in n-categories.

Definition 49. In an n-category, k-morphisms $F, G : A \to B$ are *equivalent*, written $F \simeq G$, just when:

- if $k = n$, then $F = G$;
- if $k < n$, then there are $(k+1)$-morphisms $P : F \to G$ and $Q : G \to F$ with $Q \circ P \simeq \mathrm{id}_G$ and $P \circ Q \simeq \mathrm{id}_F$.

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The equivalences form the *core* of the n-category.

Definition 50. Given an n-category $\mathbf{C}$, its *core*, written $\mathrm{Core}(\mathbf{C})$, is the sub-$n$-groupoid containing all the objects and all the equivalences.

IV.4 The cobordism hypothesis

We can also give a general coinductive definition of duality.

Definition 51. In an n-category, given k-morphisms $F : A \to B$ and $G : B \to A$ with $k < n$, a *duality* $F \dashv G$ comprises $(k+1)$-morphisms called the *unit* $\eta : \mathrm{id}_A \to G \circ F$ and *counit* $\varepsilon : F \circ G \to \mathrm{id}_B$, satisfying the snake equations up to equivalence:

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This agrees with the definitions we have already seen for dual objects in monoidal 1- and 2-categories.

IV.4 The cobordism hypothesis

We now strengthen the idea of duality as follows.

Definition 52. In an n-category, a k-morphism $F : A \to B$ is *fully dualizable* when $k = n$, or there is an infinite chain of k-morphisms

$$\cdots \dashv {}^{**}F \dashv {}^{*}F \dashv F \dashv F^{*} \dashv F^{**} \dashv \cdots$$

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This makes fully-extended TQFTs relatively easy to *find*.

However, they are still hard to *evaluate*.

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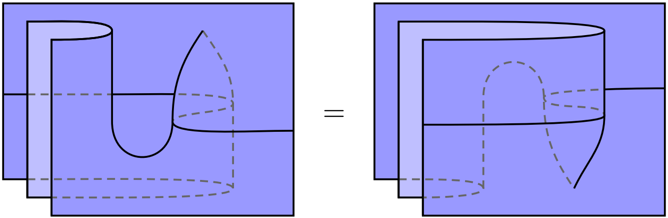
- such that the *cusp flip* holds, along with flipped variants:



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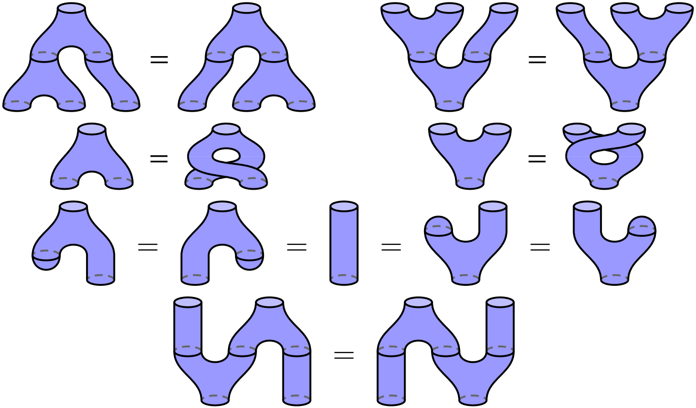

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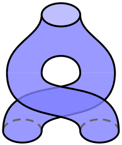

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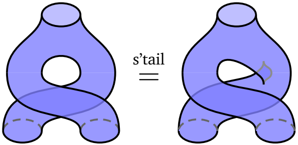

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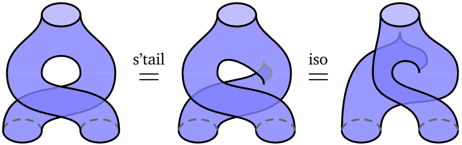

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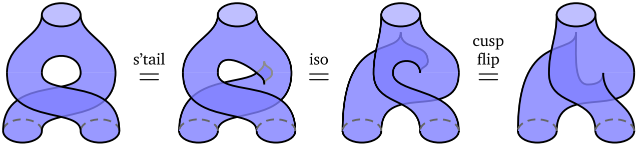

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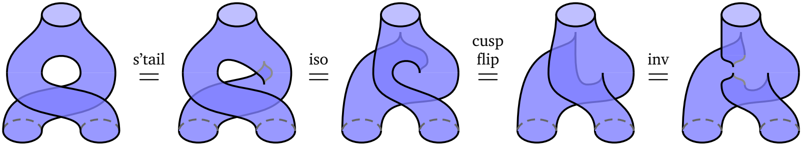

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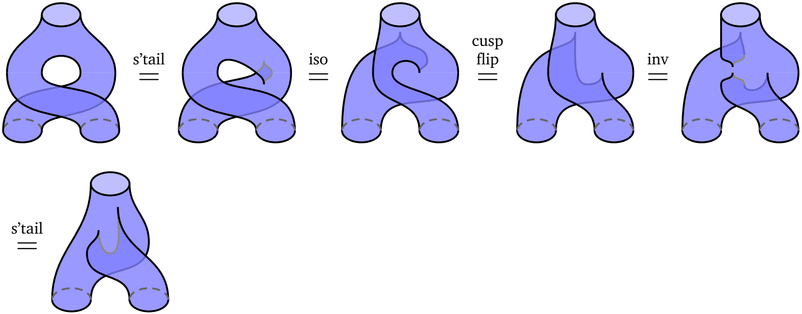

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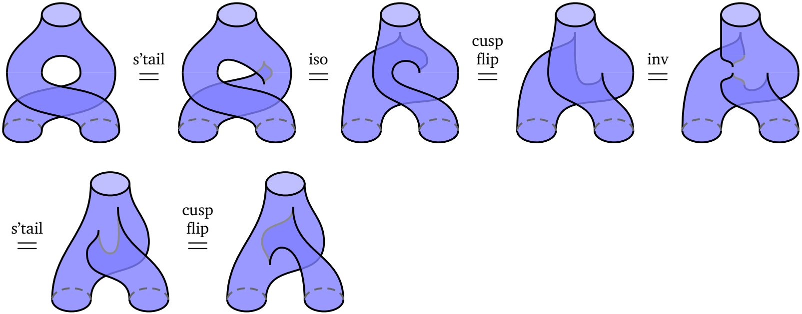

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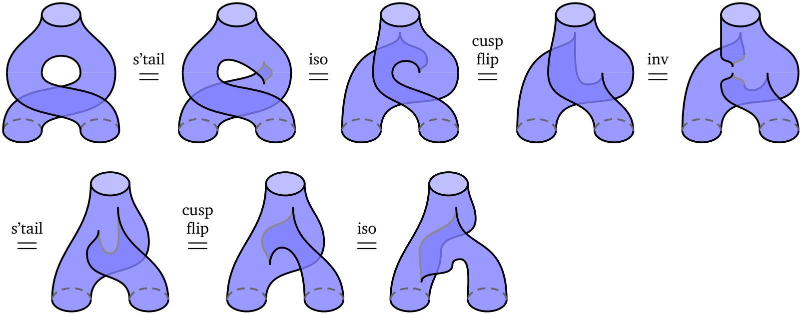

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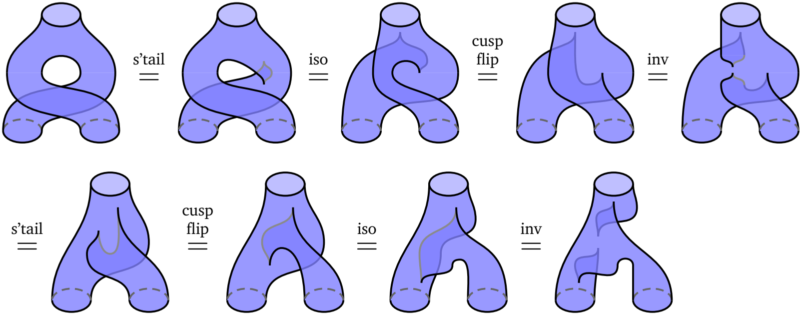

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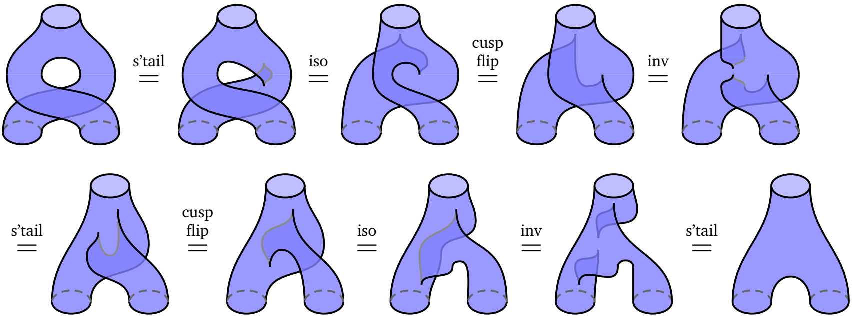

□

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Theorem 59 (Bartlett, Douglas, Schommer-Pries, V, arXiv:1509.06811). *Symmetric monoidal functors* $Z : \mathbf{Bord}^{\mathrm{or}}_{1,2,3} \to \mathbf{2Hilb}$ *correspond up to equivalence to modular multifusion categories equipped with a square root of the global dimension in each factor.*

Part V

Quantum Information

V.1. Quantum teleportation in 2-categories

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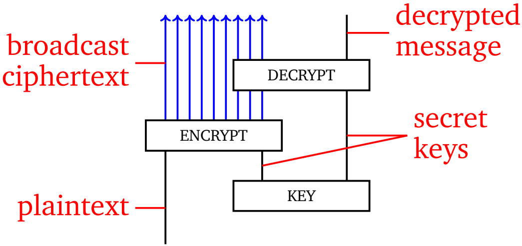

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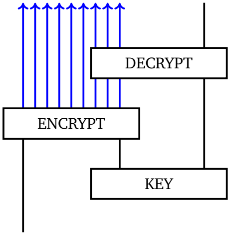

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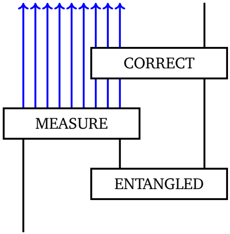

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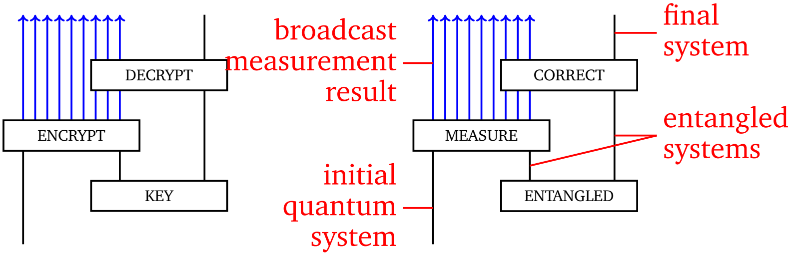

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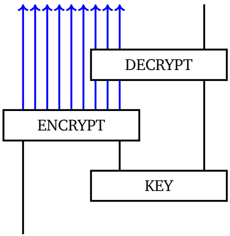

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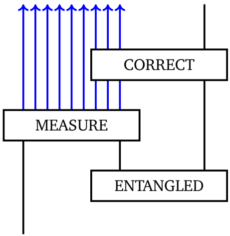

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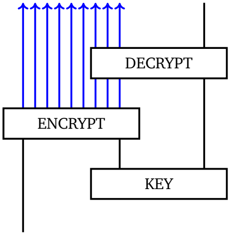

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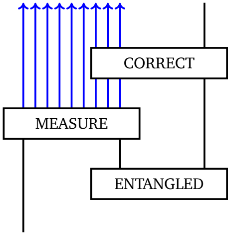

Quantum teleportation

We can make this precise using 2-categories.

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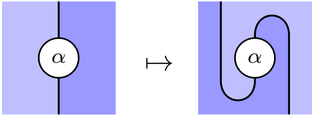

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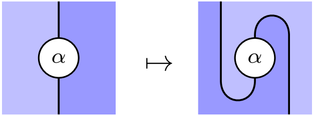

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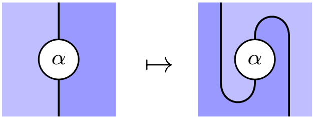

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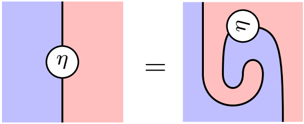
This leads to a very flexible graphical calculus:



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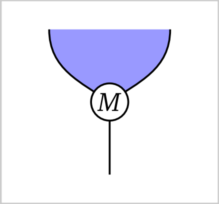

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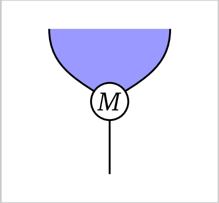

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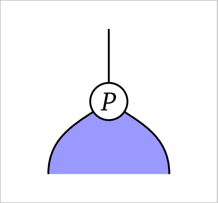

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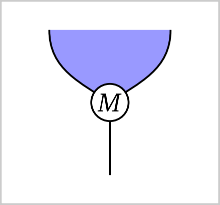

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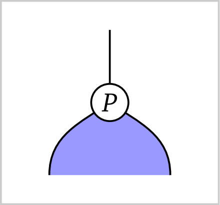

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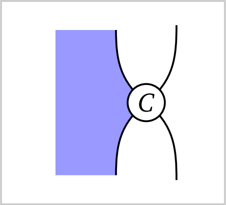

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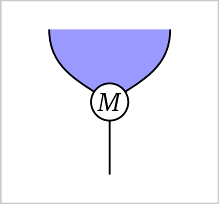

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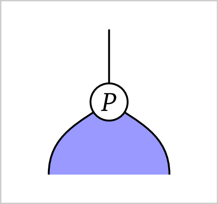

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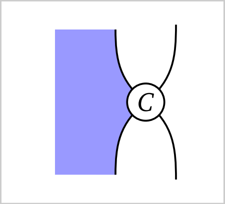

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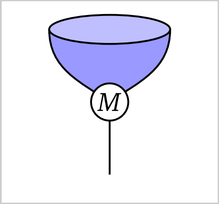

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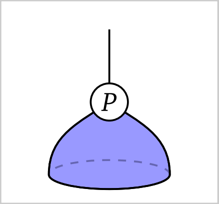

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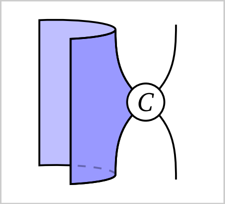

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Since copying classical information is a commutative operation, we may also model this interaction as a 012 TQFT with defects.

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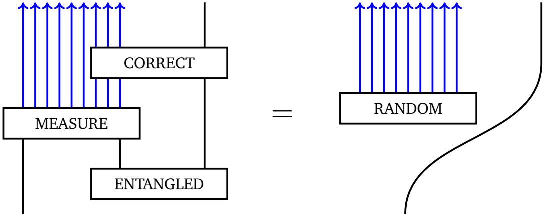
Here is the heuristic quantum teleportation diagram:



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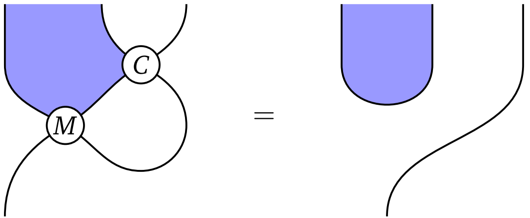

We make it rigorous with this 2-categorical equation.

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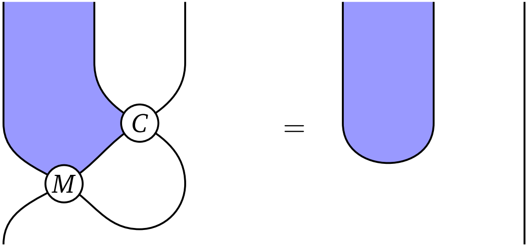
We begin with the definition of quantum teleportation:



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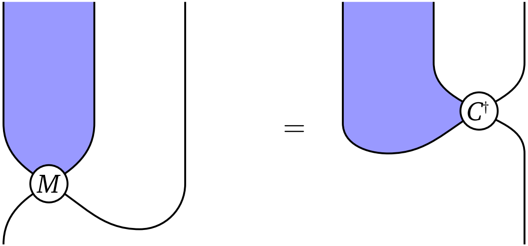
Apply $C^\dagger$:



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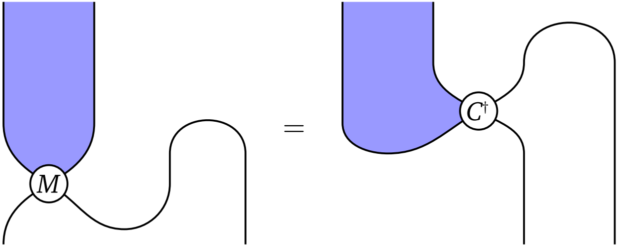
Bend down a wire:



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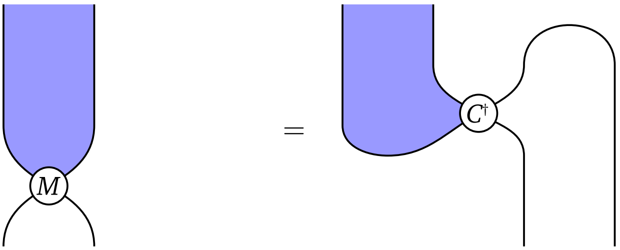

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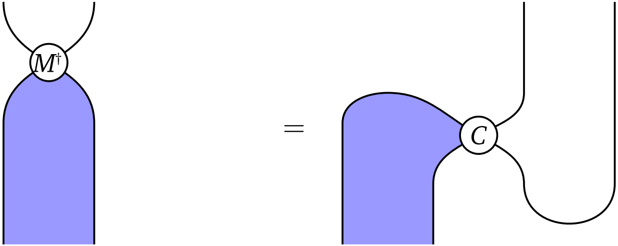
Take adjoints:



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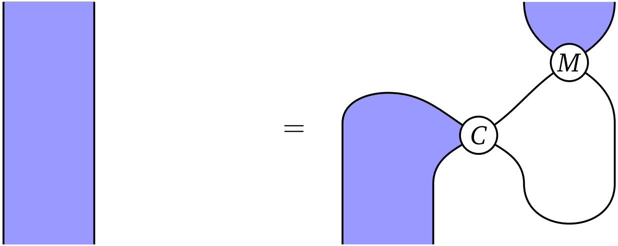

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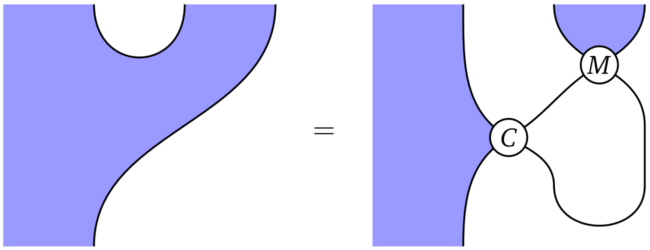
Bend up the surface:



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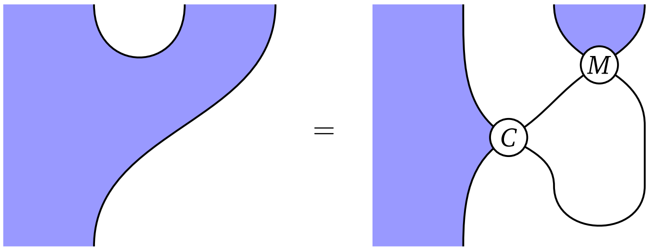

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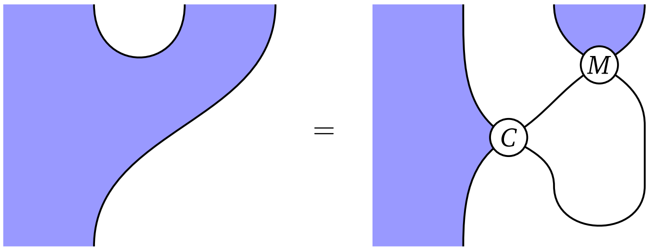

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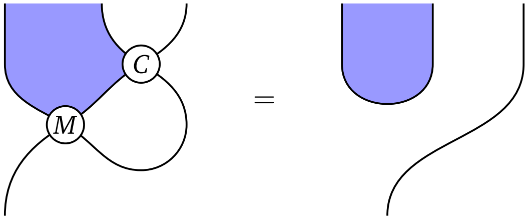
We have just seen a *2-categorical* proof of equivalence with teleportation, independent of the Hilbert space formalism.

V.1. Quantum teleportation in 2-categories

Theorem 61. *Solutions to the teleportation equation in* **2Hilb** *correspond exactly to quantum teleportation schemes.*



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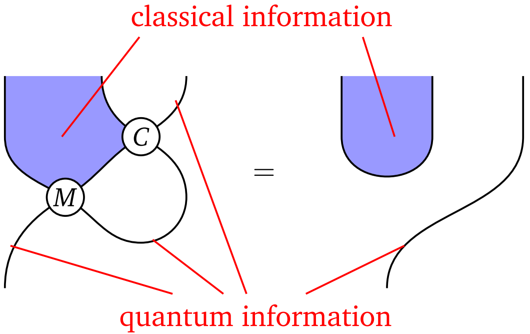

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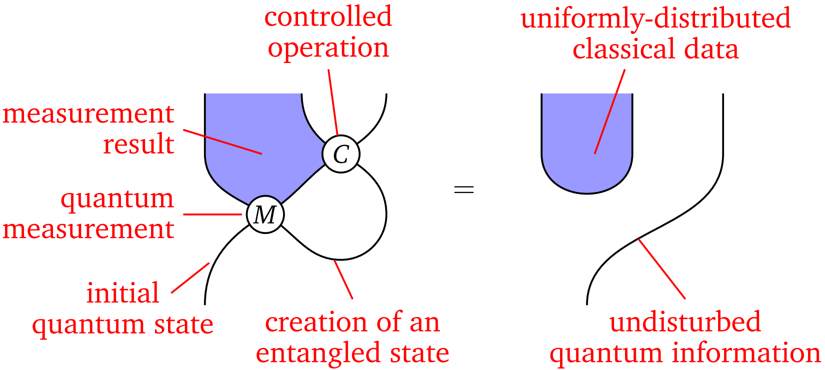

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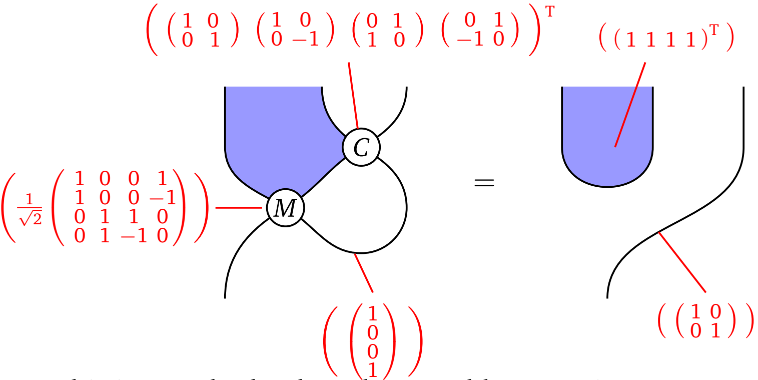

This is exactly the data that would appear in a quantum information textbook.

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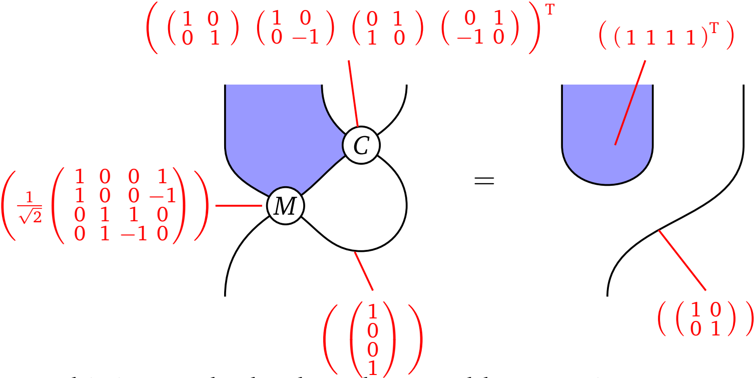

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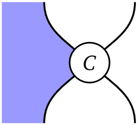
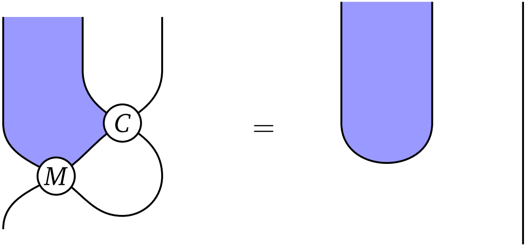
In the rest of this lecture, we investigate these solutions in **2Hilb**.

V.2. Biunitaries as quantum structures

A controlled operation vertex



is part of a quantum teleportation protocol if and only if it is unitary, and there exists a unitary measurement vertex *M* such that



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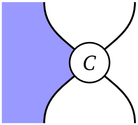

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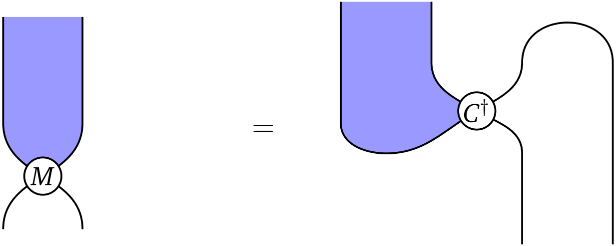
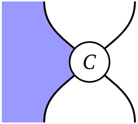
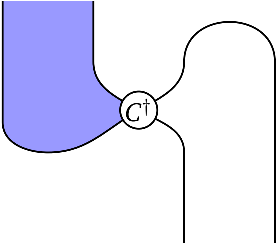


V.2. Biunitaries as quantum structures

A controlled operation 2-morphism

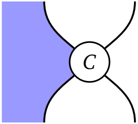


is part of a quantum teleportation protocol if and only if it is unitary and the following 2-morphism is unitary:



V.2. Biunitaries as quantum structures

A controlled operation 2-morphism



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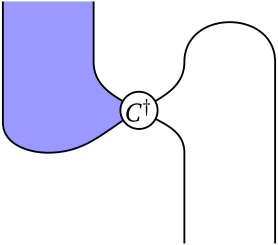

Such 2-morphisms are known as *biunitaries*.

V.2. Biunitaries as quantum structures

Definition 62. In **2Hilb**, a 4-valent vertex is *biunitary* if it is

V.2. Biunitaries as quantum structures

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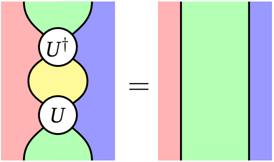

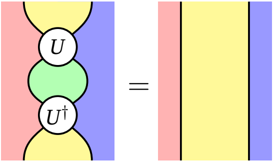

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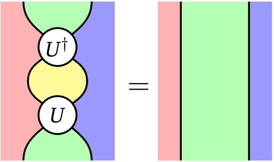
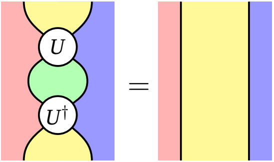
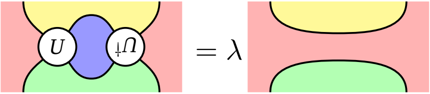
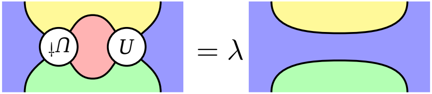




- *horizontally unitary*:





V.2. Biunitaries as quantum structures

Definition 62. In **2Hilb**, a 4-valent vertex is *biunitary* if it is

- *(vertically) unitary*:

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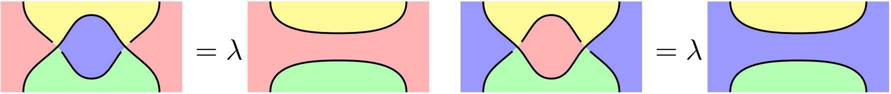
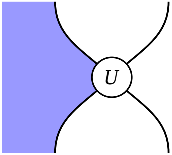


These look just like the *second Reidemeister move*.

V.2. Biunitaries as quantum structures

Corollary 63. *Biunitaries in* **2Hilb** *of type*



(1)

correspond to teleportation protocols.

V.2. Biunitaries as quantum structures

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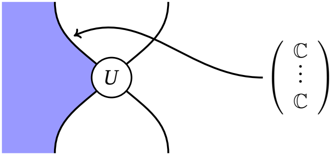

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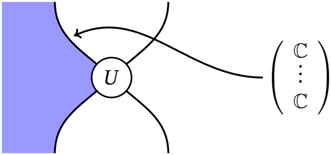

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Definition 64. A *unitary error basis* (UEB) is a family of n^2 unitary $n \times n$-matrices $\{U_i\}_{1 \leq i \leq n^2}$ such that

$$\mathrm{Tr}(U_i^\dagger U_j) = n\delta_{i,j}$$

V.2. Biunitaries as quantum structures

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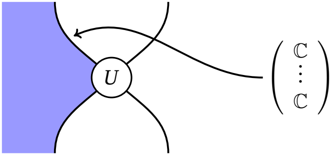

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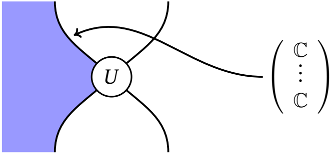
Theorem 65. *Biunitaries of type* (1) *in* **2Hilb** *are UEBs.*
Proof. Next slide.

V.2. Biunitaries as quantum structures

Corollary 63. *Biunitaries in* **2Hilb** *of type*



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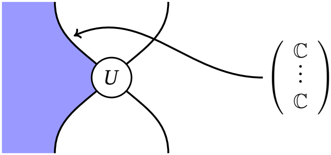
Corollary 66. *Quantum teleportation protocols are classified by UEBs.*

V.2. Biunitaries as quantum structures

Corollary 63. *Biunitaries in* **2Hilb** *of type*



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Proof. Next slide.

Corollary 66. *Quantum teleportation protocols are classified by UEBs.*

In conventional quantum information theory this is originally due to Werner. We have just seen a 2-categorical proof.

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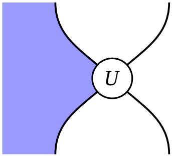
A biunitary of the following type is a unitary error basis:



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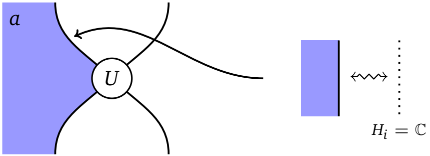

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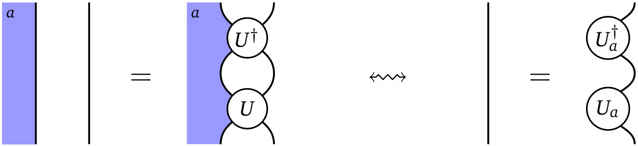

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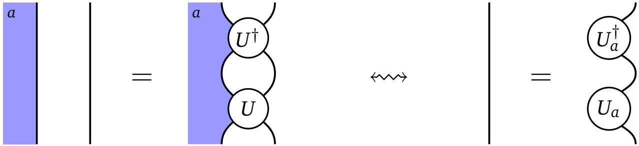

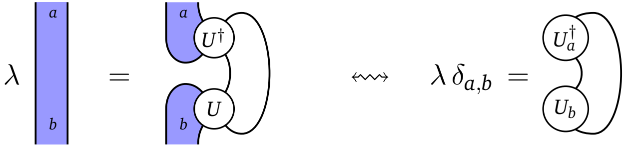

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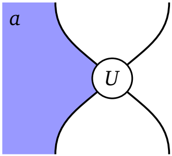
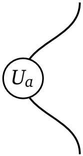


$\leftrightsquigarrow$



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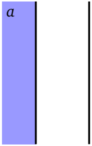
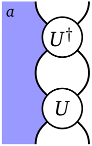
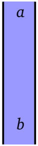


$=$



$\leftrightsquigarrow$ $\mathbb{1} = U_a^\dagger U_a$

λ



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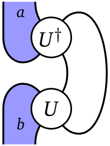

$\leftrightsquigarrow$ $\lambda\,\delta_{a,b} = \mathrm{Tr}(U_a^\dagger U_b)$

V.2. Biunitaries as quantum structures

Other biunitaries also play important roles in quantum information.

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Other biunitaries also play important roles in quantum information.

complex Hadamard matrices

$n \times n$-matrix $\{H_{i,j}\}_{1\leq i,j,\leq n}$

$|H_{i,j}|^2 = 1 \qquad H^\dagger H = n\mathbb{1}$

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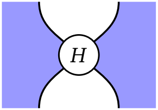

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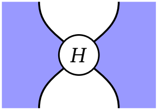

Play key roles in quantum information ...

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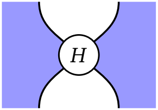

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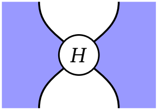

Play key roles in quantum information ... but hard to construct.

Only a handful of known constructions, for example:

Hadamard + Hadamard + Hadamard $\rightsquigarrow$ UEB

$$(U_{ab})_{c,d} = \frac{1}{\sqrt{n}} A_{a,d} B_{b,c} C_{c,d}$$

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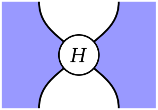

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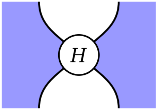

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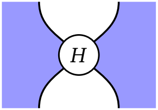

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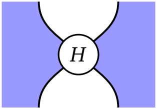
Why do they work? Where do they come from?

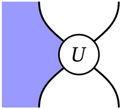
How can we find them?

V.3. Composing quantum structures

Hadamards and UEBs are biunitaries of the following type:



Hadamard



UEB

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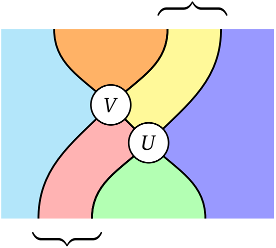
Hadamard   UEB

Theorem 67. *We can compose biunitaries diagonally:*



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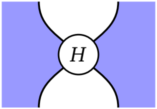

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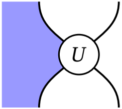
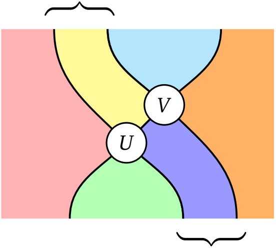
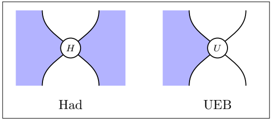


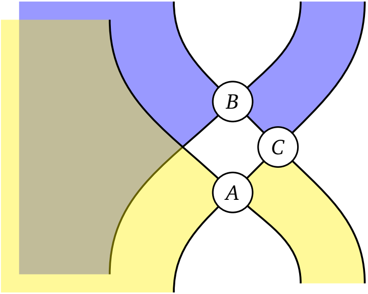
UEB

Theorem 67. *We can compose biunitaries diagonally:*

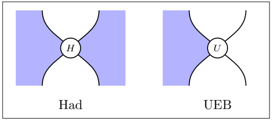


V.3. Composing quantum structures

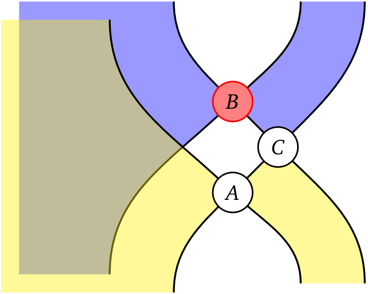




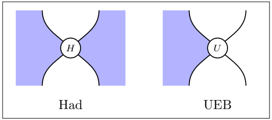
V.3. Composing quantum structures



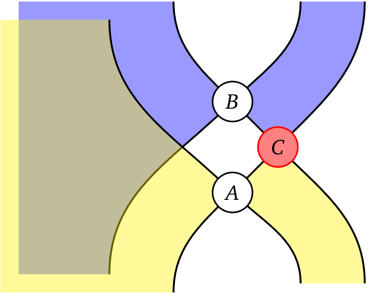
Had



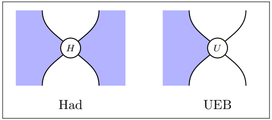
V.3. Composing quantum structures



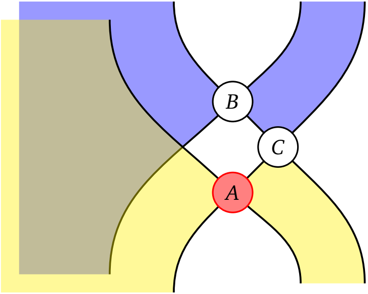
Had + Had



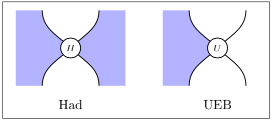
V.3. Composing quantum structures



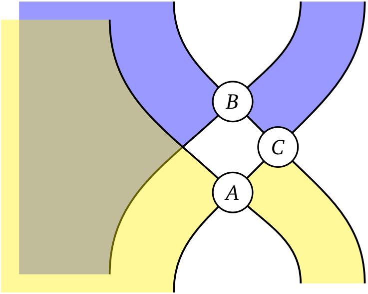
Had + Had + Had



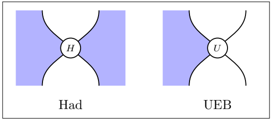
V.3. Composing quantum structures



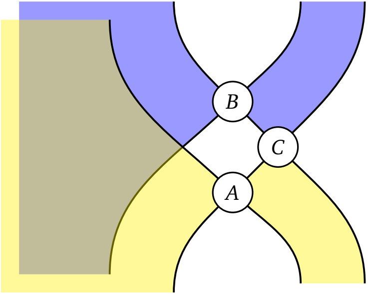
$$\text{Had} + \text{Had} + \text{Had} \rightsquigarrow \text{UEB}$$



V.3. Composing quantum structures

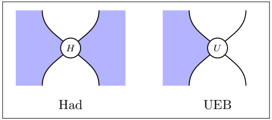


Had + Had + Had ⇝ UEB

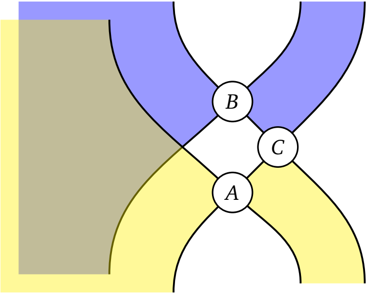


$$(U_{ab})_{c,d} = \frac{1}{\sqrt{n}} A_{a,d} B_{b,c} C_{c,d}$$

V.3. Composing quantum structures

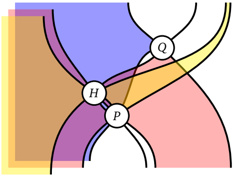


Had + Had + Had ⇝ UEB

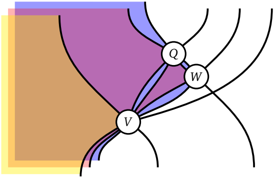


$$(U_{ab})_{c,d} = \frac{1}{\sqrt{n}} A_{a,d} B_{b,c} C_{c,d}$$ ✓

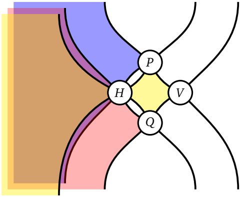
V.3. Composing quantum structures



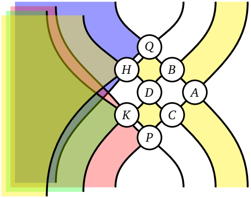
$U_{abc,de,fg}=H^{b,c}_{a,eg}P^{c,g}_{e,b,f}\, Q_{c,g,d}$



$U_{abc,def,gh}:=\sum_r V^{b,c}_{a,rf,g}Q^{c}_{b,r,d}W_{rc,e,h}$

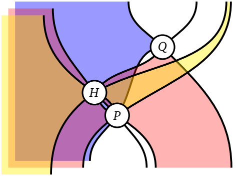


$U_{abc,de,fg}=\sum_r H^{b,c}_{a,r}P_{c,r,d}Q_{r,b,f}V_{r,e,g}$

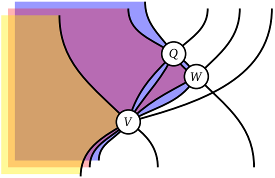


$U_{abcd,ef,gh}=\frac{1}{n}\sum_{r,s} A_{f,h}B_{s,f}C_{r,h}D_{s,r}H^{d}_{a,s}K^{c}_{b,r}Q_{d,s,e}P_{r,c,g}$

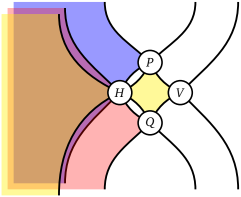
V.3. Composing quantum structures



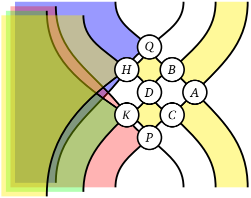
$U_{abc,de,fg}=H^{b,c}_{a,eg}P^{c,g}_{e,b,f}\,Q_{g,d}$



$U_{abc,def,gh}:=\sum_r V^{b,c}_{a,rf,g}Q^{c}_{b,r,d}W_{r,e,h}$

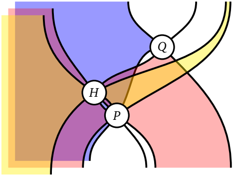


$U_{abc,de,fg}=\sum_r H^{b,c}_{a,r}P_{c,r,d}Q_{r,b,f}V_{r,e,g}$

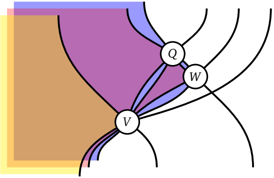


$U_{abcd,ef,gh}=\frac{1}{n}\sum_{r,s}A_{f,h}B_{s,f}C_{r,h}D_{s,r}H^{d}_{a,s}K^{c}_{b,r}Q_{d,s,e}P_{r,e,g}$

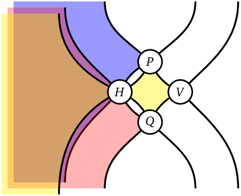
V.3. Composing quantum structures



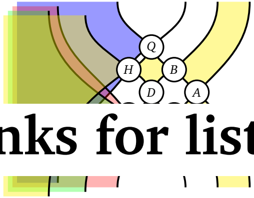
$U_{abc,de,fg} = H^{b,c}_{a,eg} P^{c,g}_{e,b,f}\, Q_{\dots,d}$



$U_{abc,def,gh} := \sum_r V^{b,c}_{a,rf,g} Q^{c}_{b,r,d} W_{r\dots,h}$



$U_{abc,de,fg} = \sum_r H^{b,c}_{a,r} P_{c,r,d} Q_{r,b,f} V_{\dots,g}$



$U_{abcd,ef,gh} = \frac{1}{n} \sum_{r,s} A_{f,h} B_{s,f} C_{r,h} D_{s,r} H^{d}_{a,s} K^{c}_{b,r} Q_{d,s,e} P_{\dots,g}$

Thanks for listening!