Qualifying Exam Syllabus

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Major Topic: Functional Analysis (Modern Analysis)

- Banach Spaces: Baire Category Theorem, Open Mapping Theorem, Closed Graph Theorem, Uniform Boundedness Principle, Hahn-Banach Theorem, Banach-Alaoglu Theorem, weak and weak-* topologies
- Topological Vector Spaces: seminorms, locally convex spaces, Krein-Milman Theorem
- Hilbert Spaces: sesquilinear forms, inner products, Cauchy-Schwartz inequality, Riesz Representation Theorem, orthonormal bases
- Operators on Hilbert Spaces: topologies on $B(H)$, polar decomposition, compact operators, trace-class operators, Hilbert-Schmidt operators, normal operators
- Examples: $C_?(X)$, $B_?(H)$, $L^p(X,\mu)$

References: Pederson, *Analysis Now*, Chapters 2-3

Major Topic: Operator Algebras (Modern Analysis)

- Banach Algebras: spectrum, spectral radius formula, characters, maximal ideals, Gelfand transform
- C^*-Algebras: adjoining an identity, spectral permanence, commutative C^*-algebras, continuous functional calculus, approximate identities, ideals and quotients, states, GNS construction, Gelfand-Naimark Theorem
- Von Neumann Algebras: spectral theorem for normal operators; Bicommutant Theorem; Kaplansky Density Theorem; comparison of projections in factors; definitions and examples of factors of types I, II, and III; the coupling constant; index of subfactors
- Examples: $C_?(X)$, $B_?(H)$, $L^\infty(X,\mu) \rtimes_\alpha G$ (G countable and discrete)

References: Pederson, *Analysis Now*, Chapter 4; Davidson, *C^*-Algebras by Example*, Chapters 1-2.2; Jones' notes on von Neumann algebras, Chapters 2-11

Minor Topic: Algebraic Topology (Geometry and Topology)

- homotopy, fundamental group, Van Kampen's Theorem
- fiber bundles, principal G-bundles, covering spaces, classification theorem for covering spaces
- CW-complexes, Δ-complexes
- singular, simplicial, and cellular homology

References: Hatcher, *Algebraic Topology*, Chapters 0-2; Bredon, *Topology and Geometry*, Sections 2.13, 3.1-9

Here are the questions (that I can remember) from my Qualifying Exam:

Major Topic: Operator Algebras

(1) (Rieffel) What is the GNS-construction?
Here I assumed A was a C^*-algebra, which prompted:

- (Rieffel) What if A is just a normed $*$-algebra?
- (Rieffel) What does it mean for a linear functional to be positive on a normed $*$-algebra?
- (Rieffel) Where would you use the fact that A is a C^*-algebra during the constuction?

(2) (Rieffel) Compute some examples of the GNS-construction.

- (Rieffel) What about the non-commutative case? Try $M_n(\mathbb{C})$.
  I picked the trace, but Jones asked me to pick another state, so I picked a vector state.
- (Rieffel) What else can you say about this state?
- (Rieffel) What is a density matrix?
- (Rieffel) Now what about computing the GNS-construction for an infinite dimensional example.
  I picked $\mathrm{vN}(S_\infty)$.
- (Rieffel) What do you know about $\mathrm{vN}(S_\infty)$?
- (Jones) What is a factor? How do you know $\mathrm{vN}(S_\infty)$ is a factor?
- (Jones) What do you get from doing the GNS-construction in this case?

Major Topic: Functional Analysis

(1) (Teichner) Are the L^p spaces locally convex?

(2) (Teichner) Why do we care about local convexity?

(3) (Teichner) Is L^p (I interpreted as $L^p[0,1]$) isomorphic to l^p?

Minor Topic: Algebraic Topology

(1) (Teichner) Can you compute as many homotopy groups as possible of $\mathbb{CP}^n$ as possible?
I got a lot of help in this question, and it went pretty well. This question was really cool as it connected ideas in previous subjects to alegraic topology. For example, I talked about the set of rank one projections in $\mathbb{C}^n$ in connection to $\mathbb{CP}^n$, and these projections give density matrices which correspond to pure states: If $P = |x\rangle\langle x|$, then $T \mapsto \mathrm{tr}(PT) = \langle Tx, x\rangle$ is a pure state (and also a vector state).

(2) (Ganor) Can you compute the homology of the Klein Bottle?