A "Subfactor" Planar Algebra Everyone Will Understand

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1 Introduction

Planar algebras were first defined by Jones in [Jon99] to capture the structure of the standard invariant of a finite index, extremal II_1-subfactor. Since their inception, the language of planar algebras has been applied to many branches of mathematics, in particular TQFT's, representation theory, and category theory. Some planar algebras are very easy to understand, such as $TL_\bullet(\mathbb{C}, \delta)$, the Temperley-Lieb planar algebra over $\mathbb{C}$ with modulus δ. Others, such as the standard invariant of a finite index, extremal II_1-subfactor remain more cryptic. We describe a planar algebra obtained from the simplest subfactor: $\mathbb{C}I_2 \subset M_2(\mathbb{C})$. This example is just the planar algbera of the bipartite graph with two vertices connected by two edges first defined by Jones in [Jon00], but we present it here in the language used to describe the standard invariant of a II_1-subfactor.

2 Planar Algebras

2.1 The Planar Operad

Definition 2.1.1. Recall from [Jon99] that a planar k-tangle T consists of the following data:

(1) The skeleton of T, denoted $S(T)$, consisting of:

(a) the closed unit disk D in $\mathbb{C}$, whose boundary is denoted $D_0(T)$,

(b) a finite (possibly empty) set of disjoint subdisks $D_1, \ldots, D_n$ in the interior of D whose boundaries are denoted $D_1(T), \ldots, D_n(T)$,

(c) for $i \geq 0$, an even number $2k_i \geq 0$ of distinct marked points on $D_i(T)$ called the boundary points of $D_i(T)$ with $k = k_0$,

(d) inside D, but outside D_i for $i > 0$, there is a finite set of disjointly smoothly embedded curves called strings which are either closed curves or whose boundaries are marked points of the $D_i(T)$'s and the strings meet each $D_i(T)$ transversally. Each marked point on $D_i(T)$ for $i \geq 1$ meets exactly one string.

(2) The connected components of $D \setminus S(T)$ are called regions and are shaded black and white so that regions whose closures meet have different shadings (often we will call black regions shaded and white regions unshaded),

Definition: A boundary segment of $D_i(T)$, $i = 0, \ldots, n$, is a connected arc on $D_i(T)$ between two boundary points of $D_i(T)$. A simple boundary segment of $D_i(T)$, $i = 0, \ldots, n$, is a boundary segment of $D_i(T)$ in T which touches only two (adjacent) boundary points.

(3) For every $D_i(T)$, $i \geq 0$, there is a distinguished simple boundary segment of $D_i(T)$ whose interior meets an unshaded region.

Remark 2.1.2. The case $k = 0$ is exceptional in that there are two kinds of 0-tangle depending on whether the region meeting $D_0(T)$ is white or black.

Definition 2.1.3. The planar operad $\mathbb{P}$ is the set of all orientation-preserving diffeomorphism classes of planar k-tangles, k being arbitrary, The diffeomorphisms preserve D, but may move the D_i's, $i \geq 1$.

Definition 2.1.4. Given a planar k-tangle T, a k'-tangle S, and a disk D_i of T with $i > 0$ with $k_i = k'$, we define the k-tangle $T \circ_i S$ by isotoping S so that $D_0(S)$, together with the marked points, coincides with that $D_i(T)$, and the chosen simple boundary segments for $D_i(T)$ and $D_0(S)$ share a boundary segment. The strings may then be joined at $D_i(T)$ and smoothed, and $D_i(T)$ is removed to obtain $T \circ_i S$ whose diffeomorphism class only depends on those of T and S.

2.2 Planar Algebras

Definition 2.2.1. A planar algebra is a sequence (V_k) of $\mathbb{C}$-vector spaces for $k \geq 1$ and two $\mathbb{C}$-vector spaces V_0^{white} and V_0^{black} of dimension 1 together with a representation of the planar operad as multiplinear maps among the V_i's, $Z\colon \mathbb{P} \to ML(V_i)$, i.e. to each k-tangle T in $\mathbb{P}$, there is a multilinear map

$$Z(T)\colon \prod_{i=1}^{n} V_{k_i} \longrightarrow V_k$$

such that $Z(T \circ_i S) = Z(T) \circ_i Z(S)$ where $\circ_i$ on the right hand side is composition of multilinear maps in the obvious way.

Remark 2.2.2. There is a unique way to identify each V_0 as a $\mathbb{C}$-algebra, and Z of the empty picture in each case is 1. There are two scalars associated to a general planar algebra, δ_1, δ_2, which correspond to the closed loop parameters.

3 The Simplest "Subfactor" Planar Algebra

3.1 A "Finite Index" Subfactor

The index of a II_1-subfactor was first defined by Jones in [Jon83]. The simplest examples of factors are type I_n-factors for $n \in \mathbb{N}$, which are (non-canonically) isomorphic to $M_n(\mathbb{C})$. Therefore the simplest example of a subfactor is $\mathbb{C}I_2 \subset M_2(\mathbb{C})$ where I_2 is the 2×2 identity matrix. One should think of this subfactor of having "index" four as $\mathbb{C}I_2$ is one dimensional and $M_2(\mathbb{C})$ is four dimensional, so we set $\delta = 2 = \sqrt{4}$. Let $\operatorname{tr}\colon M_2(\mathbb{C}) \to \mathbb{C}$ be the normalized trace. There is a unique linear map $E\colon M_2(\mathbb{C}) \to \mathbb{C}I_2$ fixing $\mathbb{C}I_2$ such that $\operatorname{tr} \circ E = \operatorname{tr}$. It is given by

$$E\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a+d}{2} & 0 \\ 0 & \frac{a+d}{2} \end{pmatrix} = \frac{1}{2}\operatorname{trace}\begin{pmatrix} a & b \\ c & d \end{pmatrix} I_2 = \operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} I_2$$

Note that the map $\langle \cdot, \cdot \rangle\colon M_2(\mathbb{C}) \times M_2(\mathbb{C}) \to \mathbb{C}$ given by

$$\langle x, y \rangle = \frac{1}{\delta}\operatorname{trace}(y^*x) = \operatorname{tr}(y^*x)$$

defines an inner product on $M_2(\mathbb{C})$ (here y^* is the adjoint (conjugate transpose) of y). We fix the folowing orthonormal basis

$$B = \left\{ \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \right\}$$

of $M_2(\mathbb{C})$, but our construction will be independent of B.

We need the following result to define our planar algebra, whose proof is left to the reader:

Lemma 3.1.1 (Key Lemma)**.** *Suppose $x \in M_2(\mathbb{C})$.*

(1) $x = \sum_{b \in B} E(xb)b^* = \sum_{b \in B} bE(xb^*)$.

(2) $4I_2 = \sum_{b \in B} bb^*$.

(3) $\sum_{b \in B} xb \otimes b^* = \sum_{b \in B} b \otimes b^*x$.

3.2 Planar Structure

Definition 3.2.1. Set $V_0^{\text{white}} = \mathbb{C}$ and V_0^{black} equal to $\mathbb{C}$. For $k \geq 1$, set

$$V_k = \bigotimes_{\mathbb{C}}^{k} M_2(\mathbb{C}).$$

Define the maps $d_i, s_i, \partial_i, \sigma_i$ for $i = 1, \dots, k$ and s_0 by the following formulas where $\xi = x_1 \otimes \cdots \otimes x_k \in V_k$ and $\delta = 2$:

$$\begin{aligned}
d_1(\xi) &= x_1 x_2 \otimes x_3 \otimes \cdots \otimes x_k \\
d_i(\xi) &= x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes \cdots \otimes x_k \text{ for } 1 < i < k \\
d_k(\xi) &= x_k x_1 \otimes x_2 \otimes \cdots \otimes x_{k-1} \\
s_0(\xi) &= I_2 \otimes x_1 \otimes \cdots \otimes x_k \\
s_i(\xi) &= x_1 \otimes \cdots \otimes x_i \otimes I_2 \otimes x_{i+1} \otimes \cdots \otimes x_k \text{ for } 0 < i < k \\
s_k(\xi) &= x_1 \otimes \cdots \otimes x_k \otimes I_2 \\
\partial_1(\xi) &= \delta E(x_1) x_2 \otimes x_3 \otimes \cdots \otimes x_k \\
\partial_i(\xi) &= \delta x_1 \otimes \cdots \otimes x_{i-1} \otimes E(x_i) x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_k \text{ for } 0 < i < k \\
\partial_k(\xi) &= \delta x_1 \otimes x_2 \otimes \cdots \otimes x_{k-2} \otimes x_{k-1} E(x_k) = \delta E(x_k) x_1 \otimes x_2 \otimes \cdots \otimes x_{k-1} \\
\sigma_i(\xi) &= \frac{1}{\delta} \sum_{b \in B} x_1 \otimes \cdots \otimes x_i b \otimes b^* \otimes x_{i+1} \otimes \cdots \otimes x_k \\
&= \frac{1}{\delta} \sum_{b \in B} x_1 \otimes \cdots \otimes x_i \otimes b \otimes b^* x_i \otimes x_{i+1} \otimes \cdots \otimes x_k \text{ for } i < k \\
\sigma_k(\xi) &= \frac{1}{\delta} \sum_{b \in B} x_1 \otimes \cdots \otimes x_k b \otimes b^* = \frac{1}{\delta} \sum_{b \in B} x_1 \otimes \cdots \otimes x_{k-1} \otimes b \otimes b^* x_k,
\end{aligned}$$

and for $\eta = y_0 \otimes \cdots y_l \in V_l$, define the map $\mu_{\eta,i}$ for $i = 0, \dots, k$ by

$$\begin{aligned}
\mu_{\eta,0}(\xi) &= y_1 \otimes \cdots \otimes y_l \otimes x_1 \otimes \cdots \otimes x_k \\
\mu_{\eta,i}(\xi) &= x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_l \otimes x_{i+1} \otimes \cdots \otimes x_k \\
\mu_{\eta,k}(\xi) &= x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_l.
\end{aligned}$$

Definition 3.2.2. Given a planar k-tangle $T \in \mathbb{P}$, a standard k-picture of T, denoted $\theta(T)$, is obtained from T as follows: cut $D_i(T)$ in the distinguished simple boundary segment of $D_i(T)$, and apply an orientation-preserving diffeomorphism to $\mathbb{C}$ (which will from now on be identified with $\mathbb{R}^2$) which straightens each $D_i(T)$ into a into a horizontal line, denote $L_i(T)$ so that

(1) the y-coordinates of each $L_i(T)$ and every local maximum and local minimum of the strings are all distinct and form a finite set called the set of critical points of $\theta(T)$,

(2) the y-coordinate of $L_0(T)$ is greater than the y-coordinate of $L_i(T)$, $i \geq 1$, and strings go downward from $L_0(T)$, and

(3) all strings come from the top of the $L_i(T)$, $i \geq 1$.

Denote the union of the $L_i(T)$'s and the strings of T by $S'(T)$. Note that the unbounded component of $\mathbb{C} \setminus S'(T)$ is unshaded, as is the region directly below each $L_i(T)$, $i \geq 1$.

Definition 3.2.3. Given a planar k-tangle $T \in \mathbb{P}$ and a standard k-picture $\theta(T)$, one should think that each shaded region is a matrix and that each unshaded region between two shaded region is a $\otimes$ symbol. We define a map

$$Z_\theta(T)\colon \prod_{i=1}^{n} V_{k_i} \to V_k$$

as follows, where $\eta_i \in V_{k_i}$ for each $i = 1, \ldots, n$:

(1) Beginning below all of the critical points of $\theta(T)$, and start with the value $1_\mathbb{C}$.

(2) Traveling upwards, we perform the following maps to in the event of such a critical point:

(a) If we pass an $L_i(T)$ and there are j shaded regions to the left of $L_i(T)$, apply the map μ_{j,y_i}. This critical point adds k_i more shaded regions and is "labeled" by η_i, so we should insert the k_i matrices ξ_i.

(b) If we pass a local maximum and the region above the maximum is shaded and there are j shaded regions to the left right before the minimum, we apply the map d_j. This critical point joins two shaded regions, so we should multiply two matrices.

(c) If we pass a local minimum and the region above the minimum is shaded and there are j shaded regions to the left of the minimum, we apply the map s_j. This critical point adds another shaded region, so we should add another matrix, namely I_2.

(d) If we pass a local maximum and the region above the maximum is unshaded and there are j shaded regions to the left right before the minimum, we apply the map ∂_j. This critical point takes away one shaded region, so we need to get rid of a matrix by applying E.

(e) If we pass a local minimum and the region above the minimum is unshaded and there are j shaded regions to the left right after the minimum, we apply the map σ_j. This critical point takes one shaded region and divides it into two, so we use the orthonormal basis to "divide up" the matrix.

Theorem 3.2.4 ([Jon99])**.** *The map $Z_\theta(T)$ is independent of the choice of standard k-picture $\theta(T)$ as well as the choice B of orthonormal basis.*

Remark 3.2.5. It is a direct result of the key lemma 3.1.1 that the proof given by Jones in [Jon99] works in this case as well.

3.3 Some Calculations

We should convince the reader at this part that these definitions actually give us a well defined map $Z\colon \mathbb{P} \to ML(V_i)$.

References

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[Jon00] ______, *The planar algebra of a bipartite graph*, Knots in Hellas '98, World Scientific, 2000, pp. 94–117.