Subfactors, tensor categores, module categories, and algebra objects in tensor categories

These notes were taken by Dave Penneys at Noah Snyder's talk on 2/20/10 at the Subfactor Tahoe Retreat. The material is mostly taken from:

- Mueger's "From Subfactors to Categories and Topology I" (arXiv:math/0111204)
- Ostrik's "Module categories, weak Hopf algebras and modular invariants" (arXiv:math/0111139)
- Kirillov and Ostrik's "On q-analog of McKay correspondence and ADE classification of $\hat{\mathfrak{sl}}^2$ conformal field theories" (arXiv:math/0101219)

Remark 1. Our tensor categores are assumed to have duals and a trivial object 1.

Module and tensor categories from a subfactor

Let $N \subset M$ be a finite index II_1-subfactor. We get four categories of bimodules: ${}_N\mathsf{Mod}_N, {}_N\mathsf{Mod}_M, {}_M\mathsf{Mod}_N, {}_M\mathsf{Mod}_M$. The objects are the bimodules which occur as submodules of an iterated basic construction of $N \subset M$, and the morphisms are bimodule intertwiners, i.e., bimodule maps.

Fact 2. ${}_N\mathsf{Mod}_N, {}_M\mathsf{Mod}_M$ *are tensor categories. They are fusion categories if* $N \subset M$ *is finite depth.*

Note that there is a functor

$$\otimes_N \colon {}_N\mathsf{Mod}_N \otimes {}_N\mathsf{Mod}_M \longrightarrow {}_N\mathsf{Mod}_M$$

satisfying associativity axioms. Heuristically, one should think of this functor as a categorification of a ring action on a module, e.g., $\lambda\colon A \otimes X \to X$. The associativity of the action means the following diagram commutes:

$$\begin{array}{ccc} A \otimes A \otimes X & \xrightarrow{m \otimes \mathrm{id}_X} & A \otimes X \\ \downarrow{\scriptstyle \mathrm{id}_A \otimes \lambda} & & \downarrow{\scriptstyle \lambda} \\ A \otimes X & \xrightarrow{\lambda} & X \end{array}$$

where $m\colon A \otimes A \to A$ is the multiplication map. This means we have to have some type of associator isomorphisms in the categorified version.

Definition 3. A left module category over a tensor category C is a category M and a functor $\mathsf{C} \otimes \mathsf{M} \to \mathsf{M}$ satisfying some associativity axioms up to an associator.

Fact 4. *A finite index II_1-subfactor gives two tensor categories and two module categories over them:*

$$ {}_N\mathsf{Mod}_N \underset{{}_M\mathsf{Mod}_N}{\overset{{}_N\mathsf{Mod}_M}{\rightleftarrows}} {}_M\mathsf{Mod}_M $$

Remark 5. We can hide the right half of the above diagram by using the notion of the dual. It turns out that if M is a module category over a tensor category C, we can form the dual tensor category $\mathsf{C}^*_{\mathsf{M}}$ of C with respect to M. If $\mathsf{C} = {}_N\mathsf{Mod}_N$ and $\mathsf{M} = {}_N\mathsf{Mod}_M$, then $\mathsf{C}^*_{\mathsf{M}} \cong {}_M\mathsf{Mod}_M$, and the opposite category $\mathsf{M}^{\mathsf{op}} \cong {}_M\mathsf{Mod}_N$ gives the other module category.

Algebra objects and subfactors

A complex algebra is a complex vector space A with a map $m\colon A \otimes A \to A$ such that the following diagram commutes:

$$\begin{CD} A \otimes A \otimes A @>{m \otimes \mathrm{id}_A}>> A \otimes A \\ @V{\mathrm{id}_A \otimes m}VV @VV{m}V \\ A \otimes A @>{m}>> A. \end{CD}$$

Definition 6. An algebra object in a tensor category C is an object $A \in \mathsf{C}$ and a map $m\colon A \otimes A \to A$ satisfying the associativity axiom up to the associator.

An algebra object $A \in \mathsf{C}$ is called a Frobenius algebra object if it comes with a map $\mathrm{tr}\colon A \to 1$ satisfying a certain nondegeneracy axiom (the categorified "bilinear form" $A \otimes A^* \to$ has a biadjoint) where $1 \in \mathsf{C}$ is the trivial object

Examples 7.

(1) Let G be a finite group. Let C be the category of G-graded vector spaces, i.e., vector spaces V which are the direct sum of vector spaces V_g for each $g \in G$:

$$V = \bigoplus_{g \in G} V_g.$$

C is a tensor category where $\otimes$ is given by

$$(V \otimes W)_g = \bigoplus_{hk=g} V_h \otimes_{\mathbb{C}} W_k.$$

The group algebra $\mathbb{C}G$ is an algebra object in this category.

(2) ${}_N M_N \in {}_N\mathsf{Mod}_N$ is a Frobenius algebra object.

Exercise 8. Show that the multiplications induce the algebra object structures in the above examples.

Theorem 9. *If $X \in {}_N\mathsf{Mod}_N$ is a simple Frobenius algebra object, then X comes from a factor P where $N \subset P$. Moreover, any unitary tensor category with simple 1 can be realized as a category of bimodules over a factor N (see Yamagami). This means every algebra object can be realized as a subfactor.*

Remark 10. The index of the subfactor coming from an algebra object in a tensor category is the Frobenius-Perron dimension of the object, not the square of the dimension.

Algebra objects and module categories

Given an algebra object $A \in \mathsf{C}$, we can make a left module category as follows: set M equal to the category of *right* A-module objects, i.e., those objects $X \in \mathsf{C}$ with a map $\rho\colon X \otimes A \to X$ satisfying the associativity axiom up to an associator:

$$\begin{CD} X \otimes A \otimes A @>{\mathrm{id}_X \otimes m}>> X \otimes A \\ @V{\mathrm{id}_A \otimes \rho}VV @VV{\rho}V \\ X \otimes A @>{\rho}>> X \end{CD}$$

Note that if X is a right A-module object and $Y \in \mathsf{C}$, then $Y \otimes X$ is also a right A-module object with the map $\mathrm{id}_Y \otimes \rho$.

Conversely, the internal Hom construction of Ostrik gives algebra objects from a module category. Heuristically, internal Hom is a way of creating objects in a category in a natural way from two given objects. In the category of vector spaces, $\mathrm{Hom}(X, Y)$ is a complex vector space.

Definition 11. Given a module category M, internal Hom is a bifunctor $\underline{\mathrm{Hom}}\colon \mathsf{M} \otimes \mathsf{M} \to \mathsf{C}$ such that for each $X, Y, Z \in \mathsf{M}$, the composition axiom holds up to isomorphism:

$$\underline{\mathrm{Hom}}(X, Y) \otimes \underline{\mathrm{Hom}}(Y, Z) \cong \underline{\mathrm{Hom}}(X, Z)$$

where the isomorphism is natural.

Example 12.

(1) Let G be a finite group. The category $\mathsf{Rep}(G)$ of finite dimensional complex representations of G thought of as $G - \{e\}$-bimodules where $\{e\}$ is the trivial group is a module category over G-graded vector spaces, and $\underline{\mathrm{Hom}}(X, Y) = Y \otimes X^*$.

(2) If $X, Y \in {}_N\mathsf{Mod}_M$, then $\underline{\mathrm{Hom}}(X, Y) = Y \otimes X^*$.

Fact 13. *Given a module category M over C and an object $X \in \mathsf{M}$, $\underline{\mathrm{Hom}}(X, X)$ is an algebra object in C.*

Remarks 14.

(1) In the subfactor setting, we want $X \in \mathsf{M}$ to be a simple object.

(2) Just as ${}_N M_M$ is the preferred object in the module category ${}_N\mathsf{Mod}_M$, if we have an algebra object $A \in \mathsf{C}$, the preferred object in the left module category of right A-module objects is A as a right A module.

Summary so far

The following three things are basically the same (up to unitarity and simplicity assumptions):

1. A subfactor
2. An algebra object in a tensor category
3. A tensor category with a module category over it and a fixed choice of object in the module category

GHJ subfactors

So what if I have a tensor category and a module category over it, but I haven't fixed a choice of object in the module category? Then I have lots of possible choices, each of which will give me a (possibly) different subfactor! In particular, you can perform the following switcheroo: pick a tensor category C and an algebra object $A \in \mathsf{C}$ that yields a module category M where you can then pick whichever simple object X you like and get a new algebra object $\underline{\mathrm{Hom}}(X, X)$. Subfactors constructed in this way are called Goodman-de la Harpe-Jones subfactors, or GHJ subfactors.

If you have a tensor category C and a module category M over it, and you have a favorite object $V \in \mathsf{C}$ (this is different from subfactors which give you a favorite object in M) then you can ask about the fusion graph for tensoring with V in M. For a Temperley-Lieb tensor category your favorite object is the single strand, and module categories over Temperley-Lieb at special values of d (less than 2) exactly correspond to the ADE Dynkin diagrams.

Example 15. E_6 is a module category over A_{11}. Take the middle vertex X in E_6. The algebra object $A = \mathrm{Hom}(X, X)$ gives the GHJ subfactor of index $3 + \sqrt{3}$.

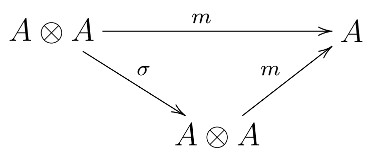
Now we notice something confusing, the GHJ that we constructed corresponding to the module category E_6 has principal graph that isn't E_6! So what on earth is the actual E_6 subfactor? The confusing thing is that the usual E_6 subfactor *is not* part of the whole story I've been telling so far. See the next subsection for how they appear.

Remark 16. The fact that the D_{odd} subfactors don't exist comes from the fact that the algebra object coming from the sum of the first and last vertices of A_{4n-1} is not commutative. The D_{odd}'s do exist as module categories.

Commutative algebra objects

If A is a commutative ring, then we can also make a tensor category out of all A-modules, instead of looking at left, right, and bi- modules.

Definition 17. A commutative algebra object in a *braided* tensor category C is an algebra object where the following diagram commutes:



where σ is the braiding.

Fact 18. *If $A \in \mathsf{C}$ is a commutative algebra object, then the category of A-modules is a tensor category, not just a module category.*

Fact 19. *The ADE subfactors all can be realized as the category of A-modules for A a commutative algebra object in Temperley-Lieb. In fact, all of Ocneanu's "quantum subgroups" arise as the category of A-modules for A a commutative algebra object in*