4/14/08

(1) **Simplicial Resolutions.** Let $F\colon \mathsf{C} \to \mathsf{D}$ be a left adjoint to $U\colon \mathsf{D} \to \mathsf{C}$, and denote by σ (respectively δ) the unit (respectively counit) of the adjunction. So $\sigma(d) \in \mathsf{D}(d, UF(d))$ corresponds to $\mathrm{id}_{F(d)}$ and $\delta(c) \in \mathsf{C}(FU(c), c)$ to $\mathrm{id}_{U(c)}$.

(a) For any object $c \in \mathsf{C}$, show that the following formulas define a simplicial object $\perp_\bullet(c)$ in C: Let $\perp_n(c) = (FU)^n(c)$ and let

$$d_i = (FU)^i\delta((FU)^{n-i}(c))\colon (FU)^{n+1}(c) \longrightarrow (FU)^n(c)$$
$$s_i = (FU)^i F\sigma(U(FU)^{n-i}(c))\colon (FU)^{n+2}(c) \longrightarrow (FU)^{n+1}(c).$$

(b) Show that $\delta(c)$ induces a simplicial map $\epsilon(c)\colon \perp_\bullet(c) \to c_\bullet$, where the right hand side denotes the constant simplicial C-object.

(c) Assume that there is a functor $K\colon \mathsf{D} \to \underline{\mathsf{Set}}$ such that $KU(\perp_\bullet(c))$ is Kan. Show that the augmentation $\epsilon(c)$ gives a weak equivalence $KU(\epsilon(c))$, i.e. it induces an isomorphism on all homotopy groups (that vanish above dimension zero for the constant functor).

(d) Apply this to your favorite pair of adjoint functors and see what you get. For example, you could use $\mathsf{C} = R-\mathsf{Mod}$ or Ring and $\mathsf{D} = \mathsf{Ab}$ or $\underline{\mathsf{Set}}$. If C happens to be an abelian category, one can apply the (normalized) chain complex to $\epsilon(c)$ and get all resolutions we have studied in class so far!

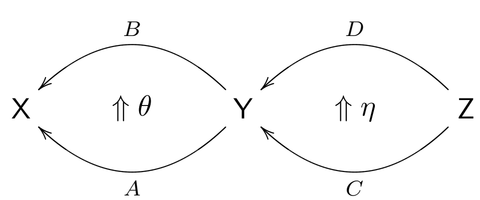
Solution.

(a) We will prove a more general result which will imply the desired result. Our proof will rely on the following facts:

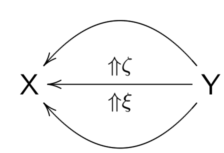
Facts: (i) If X, Y, and Z are categories, we have an equivalence

$$\mathsf{Fun}(\mathsf{X}, \mathsf{Fun}(\mathsf{Y}, \mathsf{Z})) \cong \mathsf{Fun}(\mathsf{X} \times \mathsf{Y}, \mathsf{Z}) \cong \mathsf{Fun}(\mathsf{Y}, \mathsf{Fun}(\mathsf{X}, \mathsf{Z})).$$

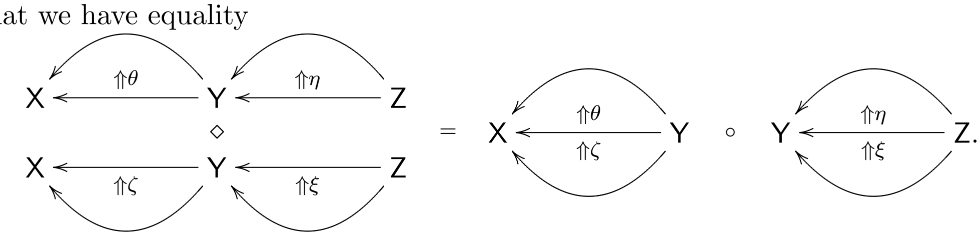
(ii) There is a 2-category Cat whose objects are small categories, 1-morphisms are functors, and 2-morphisms are natural transformations. Recall that there are two compositions $\circ$ and $\diamond$ of 2-morphisms in Cat, given respectively by:



and



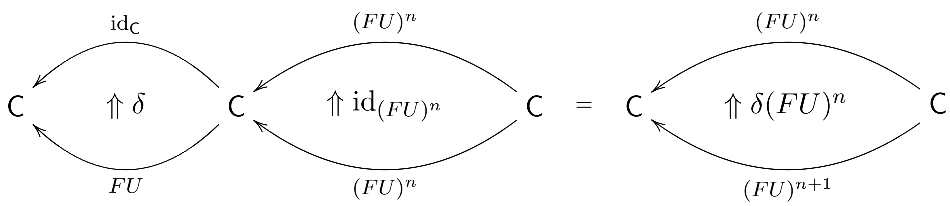
for natural transformations $\theta\colon A \Rightarrow B$, $\eta\colon C \Rightarrow D$, etc. Furthermore, $\underline{\mathfrak{C}}$ is strict in the sense that we have equality



We will construct a simplicial functor $\perp_\bullet \in \mathsf{Fun}(\mathsf{s}\Delta, \mathsf{Fun}(\mathsf{C}, \mathsf{C})) \cong \mathsf{Fun}(\mathsf{C}, \mathsf{Fun}(\mathsf{s}\Delta, \mathsf{C}))$, i.e., $\perp_\bullet(c)$ will be a simplicial object in C. To define this simplicial functor, we will look at a 2-subcategory $\mathfrak{C}$ of

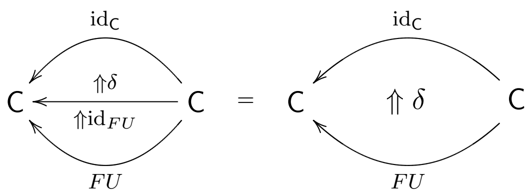
Cat whose objects are the categories C and D; 1-morphisms are composites of F, U, id_{C}, and id_{D}; and 2-morphisms are composites of identity natural transformations, $\sigma\colon \mathrm{id}_{\mathsf{D}} \Rightarrow UF$, and $\delta\colon FU \Rightarrow \mathrm{id}_{\mathsf{C}}$. We will use the following suggestive notation: to say a 1-morphism A is in $\mathfrak{C}$, we will write $A \in \mathfrak{C}$, and to say there is a natural transformation $\eta\colon A \to B$ for $A, B \in \mathfrak{C}$, we will write $\eta \in \mathfrak{C}(A, B)$. Using the definitions of composition of 2-morphisms in $\mathfrak{C}$, one can easily prove the following lemma:

<u>Lemma 1:</u> (i) Consider $(FU)^n \in \mathfrak{C}$. Then



i.e. $\delta \circ \mathrm{id}_{(FU)^n} = \delta(FU)^n$. Similarly, $\mathrm{id}_{(FU)^n} \circ \delta = (FU)^n\delta$, $\sigma \circ \mathrm{id}_{(UF)^n} = \sigma(UF)^n$, and $\mathrm{id}_{(UF)^n} \circ \sigma = (UF)^n\sigma$.

(ii) Consider $FU \in \mathfrak{C}$. Then



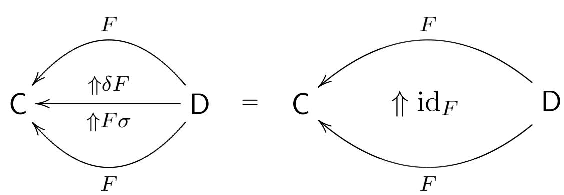
i.e. $\delta \diamond \mathrm{id}_{FU} = \delta$. Similarly, $\sigma \diamond \mathrm{id}_{UF} = \sigma$.

(iii) $\mathrm{id}_A \circ \mathrm{id}_B = \mathrm{id}_{AB}$ and $\mathrm{id}_A \diamond \mathrm{id}_A = \mathrm{id}_A$ for composable $A, B \in \mathfrak{C}$.

Also, since we have an adjunction, we have the following lemma:

<u>Lemma 2:</u> We have the following relations among $F, U, \sigma, \delta, \mathrm{id}_F, \mathrm{id}_U$:

(i) $\delta F \diamond F\sigma = \mathrm{id}_F$:



(ii) $U\delta \diamond \sigma U = \mathrm{id}_U$. The diagram is similar.

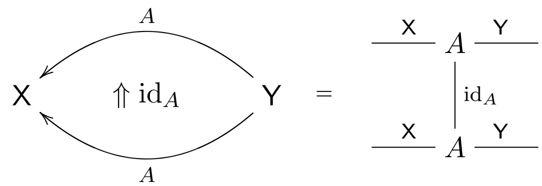
We now introduce a powerful tool: a graphical calculus for working in $\mathfrak{C}$. Usually, $A \in \mathfrak{C}$ is written as an arrow from its source to its target. One could instead write A as a point and its source and target as arrows going in and out of A. This diagram is the dual diagram:

$$\mathsf{X} \xleftarrow{\;A\;} \mathsf{Y} \quad = \quad \xleftarrow{\;\mathsf{X}\;} A \xleftarrow{\;\mathsf{Y}\;} \quad = \quad \overset{\mathsf{X}}{\text{———}}\, A \,\overset{\mathsf{Y}}{\text{———}}$$

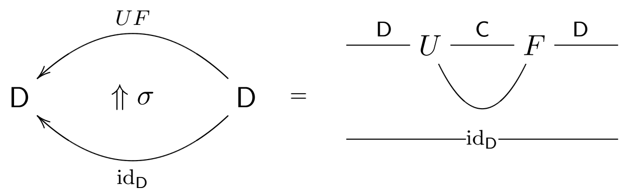
where one often supresses the directions on the arrows when the convention is understood (all arrows will point left in these dual diagrams).

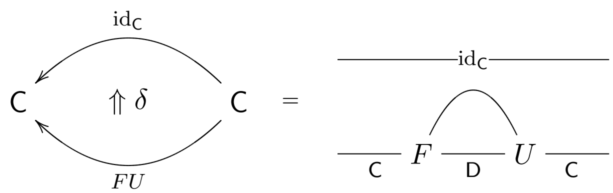
One usually writes natural transformations as 2-cells. Dually, we can write them as pictures from dual diagrams to dual diagrams. This gives the added benefit that we can denote an identity

2-morphisms as a “string” going from the bottom to the top:

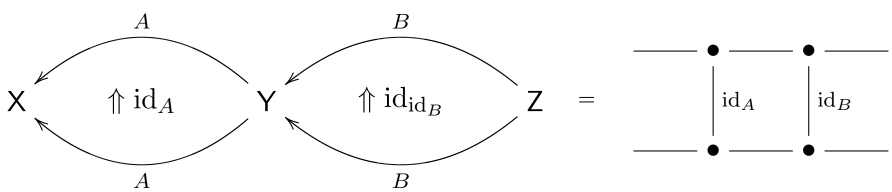


and we can denote the unit or counit by a “cap,” once more reading bottom to top:

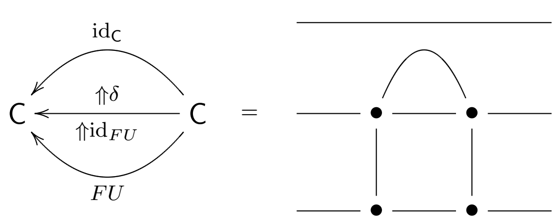




Now ∘-composition of 2-morphisms corresonds in the dual diagram language to splicing pictures together sideways, and ⋄-composition corresponds to stacking. Often we do not label the categories and 1-morphisms as the 2-morphisms encode this data.



When a cap appears, we sometimes do not label the 2-morphisms as they are completely determined. Furthermore, we omit the bullet representing the identity functor:



Using the convention of shading the regions of the diagram which have D along the outer boundary, we get string diagrams in which the lemmas above correspond exactly to isotopy of the strings. In the following diagrams, shaded regions have a $*$ in them:

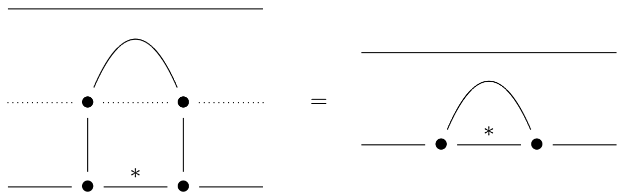


Figure 1: Lemma 1.*ii*, $\delta \diamond \mathrm{id}_{FU} = \delta$.

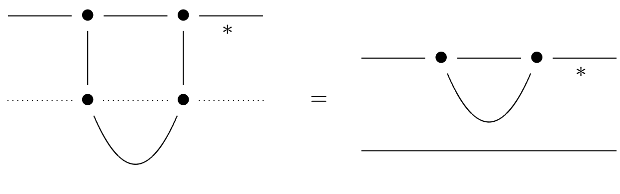


Figure 2: Lemma 1.*ii*, $\sigma \diamond \mathrm{id}_{UF} = \sigma$.

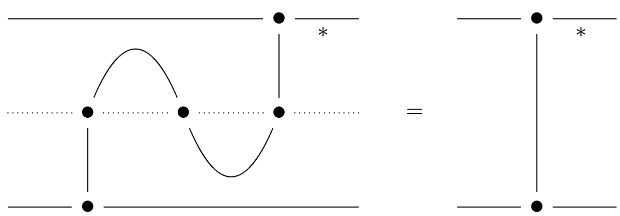


Figure 3: Lemma 2.*i*, $\delta F \diamond F\sigma = \mathrm{id}_F$.

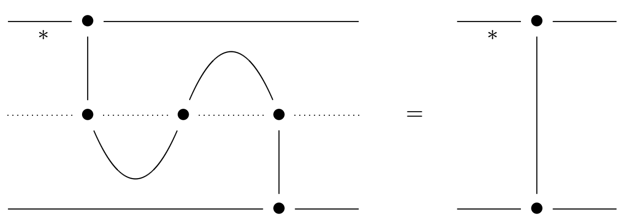


Figure 4: Lemma 2.*ii*, $U\delta \diamond \sigma U = \mathrm{id}_U$.

Now that we have this graphical calculus at our disposal, we can build our simplicial functor $\perp_\bullet$. Set $\perp_n = (FU)^{n+1} \in \mathfrak{C}$ and

$$d_i = (FU)^i \delta (FU)^{n-i} \colon (FU)^{n+1} \longrightarrow (FU)^n$$
$$s_i = (FU)^i F\sigma U (FU)^{n-i} \colon (FU)^{n+2} \longrightarrow (FU)^{n+1}.$$

We can represent these natural transformations diagramatically:

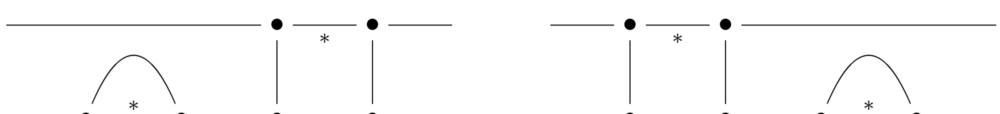


Figure 5: $d_0, d_1 \in \mathfrak{C}((FU)^2, FU)$.

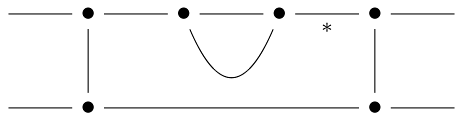
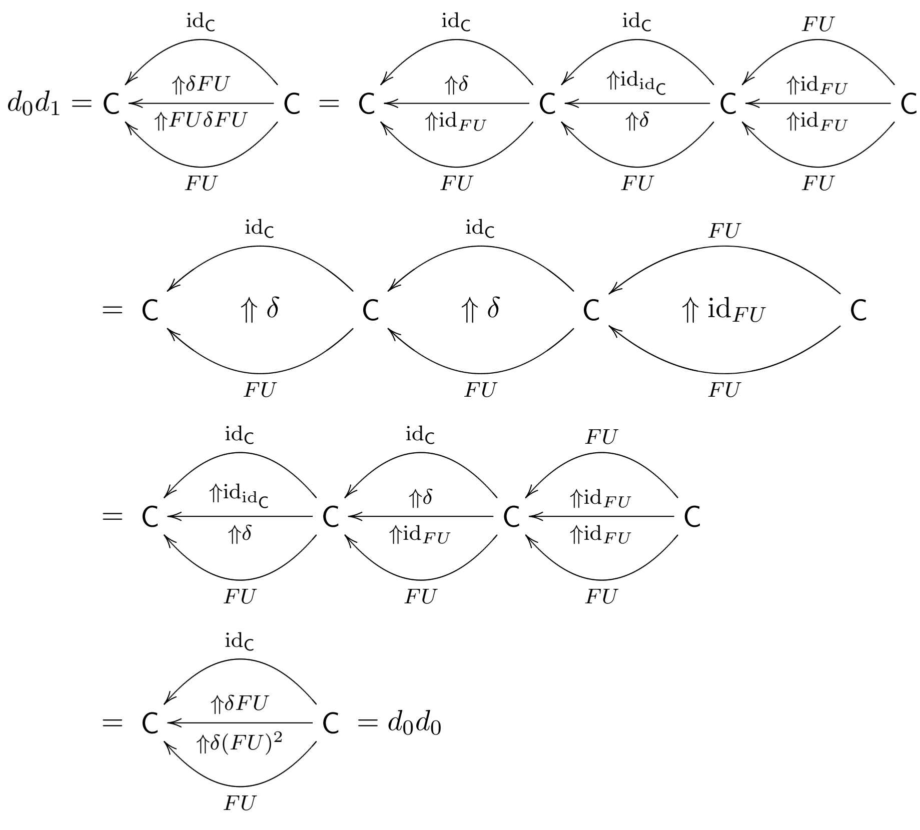


Figure 6: $s_0 \in \mathfrak{C}(FU,(FU)^2)$.

To prove that this defines a simplicial functor, we must show the following relations:

$$d_i d_j = d_{j-1} d_i \ \text{ for } \ i < j$$
$$s_i s_j = s_{j+1} s_i \ \text{ for } \ i \leq j$$
$$d_i s_j = \begin{cases} s_j d_{i-1} & \text{if } \ i < j \\ \text{id} & \text{if } \ i = j, j+1 \\ s_{j-1} d_i & \text{if } \ i > j+1. \end{cases}$$

However, these are straightforward from the earlier lemmas by drawing the appropriate diagrams. We will prove $d_0 d_1 = d_0 d_0 \colon (FU)^3 \to (FU)^1$ first using the usual diagrams in $\mathfrak{C}$, and secondly using the graphical calculus, i.e. the dual diagrams.



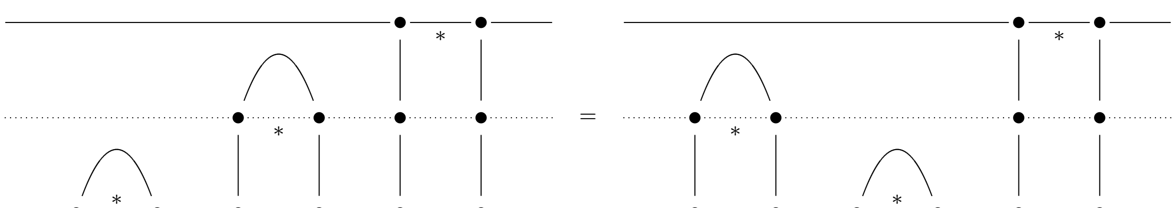


Figure 7: $d_0d_1 = d_0d_0 \in \mathfrak{C}((FU)^3, FU)$.

The only non-obvious relation is $d_is_i = id = d_{i+1}s_i$, but these follow directly from Lemma 2. We will prove $d_1s_1 = \mathrm{id}_{FUFU} = d_2s_1 \in \mathfrak{C}((FU)^2, (FU)^2)$ using the dual diagrams.

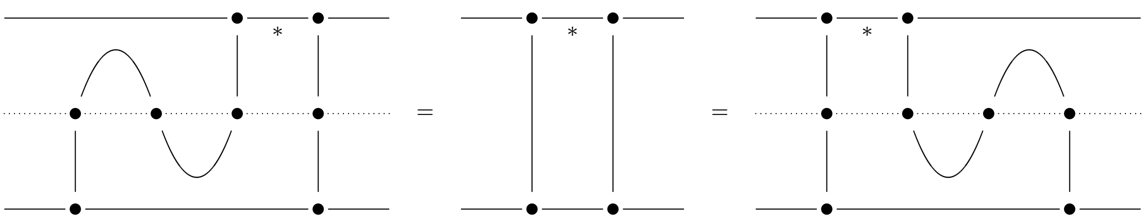


Figure 8: $d_0s_0 = \mathrm{id}_{FU} = d_1s_0 \in \mathfrak{C}(FU, FU)$.

Thus, $\perp_\bullet$ is a simplicial functor in $\mathsf{Fun}(\mathsf{C},\mathsf{C})$, and evaluation at $c \in \mathsf{C}$ gives a simplicial object $\perp_\bullet(c)$ in C.

(b) Define $\perp_{-1} = \mathrm{id}_\mathsf{C}$ and $\epsilon\colon (FU)^n \to \mathrm{id}_\mathsf{C}$ by "capping everything off," i.e. $\delta(d_0)^{n-1}$. Note that this is equivalent to $\delta d_{i_1}\cdots d_{i_{n-1}}$ by the dual diagrams. Next, define $\mathrm{id}_\bullet$ as the constant simplicial functor, i.e. $\mathrm{id}_n = \mathrm{id}_\mathsf{C}$ and d_i, s_j are all $\mathrm{id}_{\mathrm{id}_\mathsf{C}}$. It is obvious that ϵ induces a simplicial map from $\perp_\bullet \to \mathrm{id}_\bullet$, i.e. $d_i\epsilon = \epsilon d_i$, and $s_j\epsilon = \epsilon s_j$, since ϵ is capping everything off. Once more, evaluation at $c \in \mathsf{C}$ gives the simplicial map $\perp_\bullet \to c_\bullet$.

(c) Composition with U gives a simplicial functor $U\perp_\bullet$ in $\mathsf{Fun}(\mathsf{C},\mathsf{D})$. The dual diagrams (for the d_i's and s_j's) are altered by adding one more string to the left. One immediately observes the existence of an extra degeneracy $s_{-1}\colon U(FU)^n \to U(FU)^{n+1}$ by

$$s_{-1} = \sigma U(FU)^{n+1} \in \mathfrak{C}(U(FU)^{n+1}, U(FU)^{n+1}).$$

Dual diagramatically, we have:

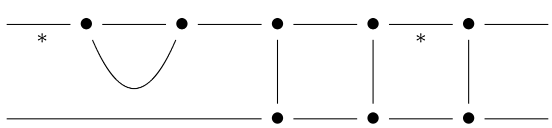


Figure 9: $s_{-1} \in \mathfrak{C}(UFU, U(FU)^2)$.

Now when we apply the functor K, we have a simplicial functor $KU\perp_\bullet$ in $\mathsf{Fun}(\mathsf{C}, \underline{\mathsf{Set}})$ with an extra degeneracy.

At this point, we must evaluate $KU\perp_\bullet$ at an object $c \in \mathsf{C}$ to get a simplicial set for which we can describe the homotopy groups. We show $\epsilon(c)\colon KU\perp_\bullet(c) \to KU\,\mathrm{id}_\bullet(c) = KUc_\bullet$ is a weak equivalence, i.e. all homotopy groups $\pi_n(KU\perp_\bullet(c), *) \cong 0$ for $n > 0$ and $\pi_0(KU\perp_\bullet(c), *) \cong KU(c)$ for a basepoint $* \in KUFU(c)$. Suppose $x, x' \in Z_n(KU\perp_\bullet(c), *)$ for $n > 0$. Recall that a horn h is given by $(n+1)$ n-simplices $y_1, \ldots, y_{i-1}, \widehat{y_i}, y_{i+1}, \ldots, y_{n+2}$ such that $d_iy_j = d_{j-1}y_i$ if $i < j$. Note that we have a horn

$$h = (\widehat{y_0}, y_1 = s_{-1}(x), y_2 = s_{-1}(x'), *, *, \ldots, *) \in KU\perp_{n+1}(c)$$

if $n > 0$ since for all $i > 0$,

$$d_is_{-1}(x) = s_{-1}d_{i-1}(x) = * = s_{-1}d_{i-1}(x') = d_is_{-1}(x').$$

Since $KU \perp_\bullet (c)$ is Kan, we can fill this horn, so there is a $y_0 \in KU \perp_{n+1} (c)$ such that $d_i y_0 = *$ for $i > 1$,

$$d_0 y_0 = d_0 y_1 = d_0 s_{-1}(x) = x \text{ and}$$
$$d_1 y_0 = d_0 y_2 = d_0 s_{-1}(x') = x'.$$

Hence y_0 is a homotopy $x \sim x'$, and $\pi_n(KU \perp_\bullet (c), *) = 0$ for all $n > 0$.

We show $\pi = \pi_0(KU \perp_\bullet (c), *) \cong KU(c)$. We have an extra degeneracy $s_{-1}\colon KU(c) \to KU \perp_0 (c)$ which satisfies $\epsilon s_{-1} = \mathrm{id}_{KU(c)}$:

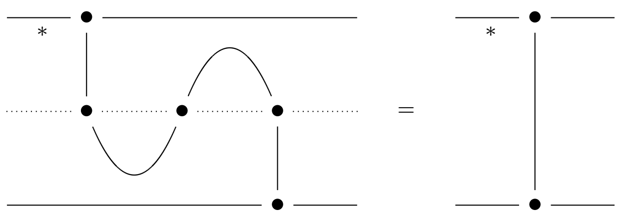
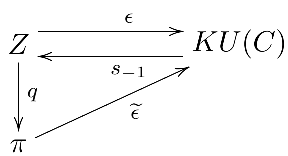


Figure 10: $\epsilon s_{-1} = \mathrm{id}_U \in \mathfrak{C}(U, U)$.

Hence s_{-1} is injective and ϵ is surjective. Now $Z = Z_0(KU \perp_\bullet (c), *) = KU \perp_\bullet (c)$ and $\epsilon(Z) = KU(c)$. Note that if $x \sim x'$ for $x, x' \in Z$, then there is a $y \in KU \perp_1 (c)$ such that $d_0(y) = x$ and $d_1(y) = x'$. Then $\epsilon(x) = \epsilon(x')$ as $\epsilon d_1 = \epsilon d_0$ (this is the "capping off" trick discussed in (b)). Thus, ϵ induces a map $\widetilde{\epsilon}\colon \pi \to KU(c)$ which must be surjective as $\epsilon = \widetilde{\epsilon} q$ where $q\colon Z \to \pi$ is the canonical epimorphism.



Moreover $\widetilde{\epsilon} q s_{-1} = \epsilon s_{-1} = \mathrm{id}$, so $q s_{-1}$ is injective. Let $x \in Z$ and $y = s_{-1}(x)$. Then $d_0(y) = d_0 s_{-1}(x) = x$ and $d_1(y) = d_1 s_{-1}(x) = s_{-1}\epsilon(x)$.

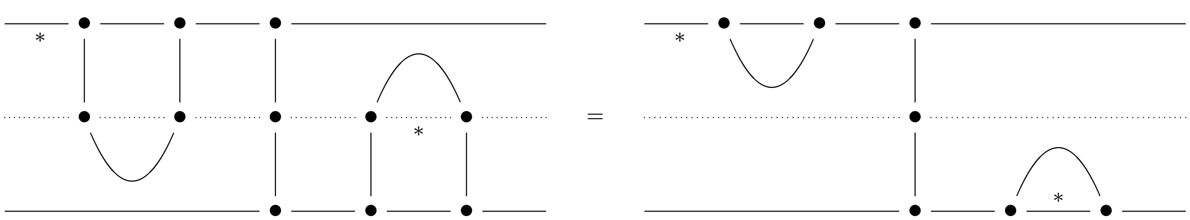


Figure 11: $d_1 s_{-1} = s_{-1}\epsilon \in \mathfrak{C}(UFU, UFU)$.

Hence $x \sim s_{-1}\epsilon(x)$. Thus, $q s_{-1}$ is bijective with inverse $\widetilde{\epsilon}$, so ϵ induces a bijection $\pi \cong KU(c)$, and we are finished.

(d) We will illustrate two examples. First, consider the example $F = \text{Free}\colon \underline{\mathsf{Set}} \to \mathsf{Ab}$ and $U = \text{Forget}\colon \mathsf{Ab} \to \underline{\mathsf{Set}}$. Then $\delta(c)\colon \mathbb{Z}\langle A\rangle \to A$ is evaluation of a formal finite linear combination of elements of $A \in \mathsf{Ab}$. We have a simplicial group

$$\mathbb{Z}\langle A\rangle \underset{d_0, d_1}{\overset{s_0}{\rightleftarrows}} \mathbb{Z}\langle \mathbb{Z}\langle A\rangle\rangle \rightleftarrows \cdots$$

The maps d_i are evaluation of a formal linear combination done at the i^{th} step. For example, if $x \in \mathbb{Z}\langle \mathbb{Z}\langle A\rangle\rangle$, then we have

$$x = \sum_{i=1}^{N_1} n_i \left(\sum_{j=1}^{N_2} m_j a_j \right),$$

where $n_i, m_j \in \mathbb{Z}$ and $a_j \in A$, and we may not distribute the n_i's into the sum over j. Then we have

$$d_0(x) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} n_i m_j a_j \text{ and } d_1(x) = \sum_{i=1}^{N_1} n_i b_i \text{ where } b_j = \sum_{j=1}^{N_2} m_j a_j \in A.$$

Furthermore, $\sigma(S)\colon d \to U(\mathbb{Z}\langle S\rangle)$ is inclusion by $s \mapsto 1 \cdot s$, so if $y \in \mathbb{Z}\langle A\rangle$, then we have

$$y = \sum^{N_3} k_i a_i \text{ and } s_0(y) = \sum^{N_3} k_i (1 \cdot a_i)$$

where $1 \cdot a \in \mathbb{Z}\langle A\rangle$. It is clear that $d_0 s_0 = d_1 s_0 = \text{id}$ in this case.

Consider $F = \mathbb{Z}-\colon \underline{\mathsf{Group}} \to \mathsf{Ring}$, i.e. taking the group ring, and $U = -^{\times}\colon \mathsf{Ring} \to \underline{\mathsf{Group}}$, i.e. taking the group of units. Then $\delta(R)\colon \mathbb{Z}(R^{\times}) \to R$ is once again evaluation and $\sigma(G)\colon G \to (\mathbb{Z}G)^{\times}$ is inclusion. We have a simplicial ring

$$\mathbb{Z}(R^{\times}) \underset{d_0, d_1}{\overset{s_0}{\rightleftarrows}} \mathbb{Z}(\mathbb{Z}(R^{\times}))^{\times} \rightleftarrows \cdots$$

and maps d_i and s_j are defined similarly as before. □