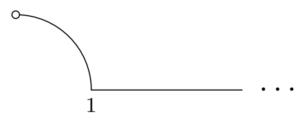
Jones' index rigidity theorem

1 Quantum Integers

Definition 1. Consider $Q = \left\{e^{i\theta} \middle| \theta \in (0, \frac{\pi}{2})\right\} \cup [1, \infty)$



For $q \in Q$, we define quantum n by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Quantum integers satisfy the following formulas:

- $[1] = 1$
- $[2][1] = [2] = q + q^{-1}$
- $[2][n] = [n+1] + [n-1]$.

Lemma 2. *Suppose $q = e^{i\theta}$ for some $\theta \in (0, \frac{\pi}{2})$, where $\theta \neq \frac{2\pi}{2n}$ for some $n \geq 3$, i.e., q is not a primitive $(2n)$-th root of unity for some $n \geq 3$. Let $k \geq 2$ be minimal such that $\theta > \frac{2\pi}{2(k+1)}$. Then $[1], [2] \ldots, [k] > 0$, but $[k+1] < 0$.*

Proof. Note that since $q = e^{i\theta}$,

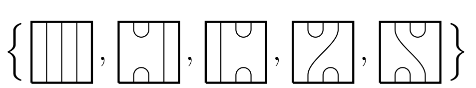
$$[j] = \frac{e^{ij\theta} - e^{-ij\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(j\theta)}{\sin(\theta)}.$$

Since $\sin(\theta) > 0$, we only care about $\sin(j\theta)$. Since $\frac{\pi}{k} > \theta > \frac{\pi}{k+1}$, we have that $\sin(1\theta), \ldots, \sin(k\theta) > 0$, but $\sin((k+1)\theta) < 0$. $\square$

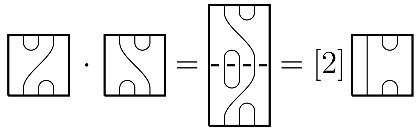
Remark 3. Note that $[n] = 0$ if and only if q is a $(2n)$-th root of unity. If $q = e^{i\theta}$, then $[2] = q + q^{-1} = 2\cos(\theta)$.

2 The Temperley-Lieb algebras

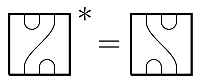
Definition 4. For $n \geq 0$, let $TL_n(q)$ be the complex vector space whose basis is the set of non-intersecting string diagrams (up to isotopy) on a rectangle with n boundary points on the top and bottom. For example, the basis for $TL_3(q)$ is given by



On $TL_n(q)$, we define a multiplication by (the bilinear extension of) stacking boxes, removing the middle line segment, and smoothing the strings, and removing any closed loops and multiplying by a factor of [2], e.g.

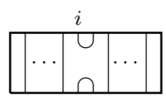


We define an involution by (the anti linear extension of) reflection about a horizontal line, e.g.

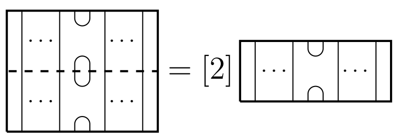


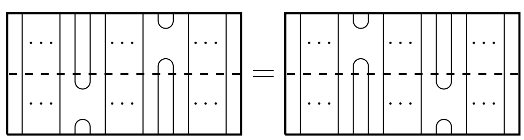
The multiplication and the adjoint make $TL_n(q)$ a complex $*$-algebra.

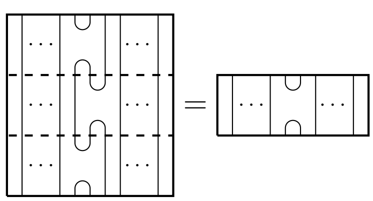
Proposition 5. *For $i = 1, \ldots, n-1$, the elements*

$E_i =$ 

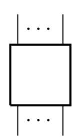
satisfy the following relations:

(1) $E_i^2 =$  $= [2]E_i = [2]E_i^*$,

(2) $E_iE_j =$  $= E_jE_i$ *if* $|i-j| > 1$, *and*

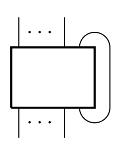
(3) $E_iE_{i\pm1}E_i =$  $= E_i$.

Definition 6. The inclusion tangle

$i_n =$ 

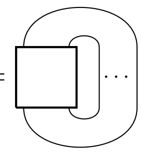
is a unital, injective $*$-algebra homomorphism $TL_n(q) \to TL_{n+1}(q)$.

The conditional expectation tangle

$\mathcal{E}_{n+1} =$ 

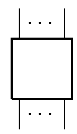
is a surjective map of $\mathbb{C}$-vector spaces $TL_{n+1}(q) \to TL_n(q)$.

The trace tangle

$\mathrm{tr}_n =$ 

defines a map $TL_n(q) \to TL_0(q)$. Note that $TL_0(q) \cong \mathbb{C}$ via the map which sends the empty diagram to 1. Using tr_n, we can define a sesquilinear form on $TL_n(q)$ by $\langle x, y \rangle_n = \mathrm{tr}_n(xy^*)$.

Of course, the identity map id_n is given by the following diagram:



Proposition 7. *The maps $i_n, \mathcal{E}_{n+1}$, tr_n, and id_n satisfy the following relations:*

(1) $\mathcal{E}_{n+1} \circ i_n = [2]\,\mathrm{id}_n$,

(2) $\mathrm{tr}_{n+1} = \mathrm{tr}_n \circ \mathcal{E}_{n+1}$,

(3) $(i_n \circ i_{n-1} \circ \mathcal{E}_n(x))E_n = E_n i_n(x) E_n$ *for all* $x \in TL_n(q)$,

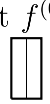
(4) $\mathrm{tr}_n(xy) = \mathrm{tr}_n(yx)$ *for all* $x, y \in TL_n(q)$,

(5) (Markov property) $\mathrm{tr}_{n+1}(i_n(x) \cdot E_n) = \mathrm{tr}_n(x)$ *for all* $x \in TL_n(q)$, *and*

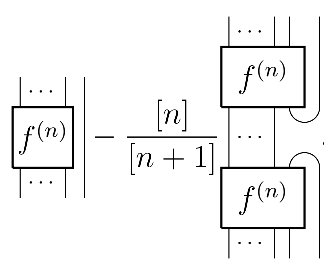
(6) $\mathrm{tr}_n(\mathcal{E}_{n+1}(x) \cdot y) = \mathrm{tr}_{n+1}(x \cdot i_n(y))$ *for all* $x \in TL_{n+1}(q)$ *and* $y \in TL_n(q)$.

Proof. The proof is a good exercise in drawing diagrams and is left to the reader. □

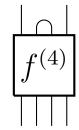
3 The Jones-Wenzl projections

Definition 8. Let $f^{(0)} \in TL_0(q)$ be the empty diagram. Let $f^{(1)} \in TL_1(q)$ be the strand, i.e., $f^{(1)} =$ . If $[n+1] \neq 0$, we inductively define

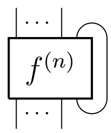
$$f^{(n+1)} = i_n(f^{(n)}) - \frac{[n]}{[n+1]} i_n(f^{(n)}) E_n i_n(f^{(n)}) =$$



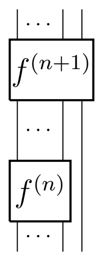
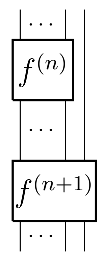
Proposition 9. *Suppose* $[1], \cdots, [n] \neq 0$ *so that* $f^{(0)}, f^{(1)}, \ldots, f^{(n)}$ *are well-defined. Then* $f^{(n)}$ *satisfies the following properties:*

(1) *Capping any two strands on top of* $f^{(n)}$ *gives zero, e.g.,*  $= 0$,

(2) $f^{(n)}$ *is an orthogonal projection, i.e.,* $f^{(n)} = (f^{(n)})^* = (f^{(n)})^2$,

(3) $\mathcal{E}_n(f^{(n)}) =$  $= \frac{[n+1]}{[n]} f^{(n-1)}$,

(4) $\mathrm{tr}_n(f^{(n)}) = [n+1]$, *and*

(5) $(i_n(f^{(n)}))f^{(n+1)} =$  $= f^{(n+1)}(i_n(f^{(n)}) =$  $= f^{(n+1)}$.

Proof. The proof is a simple induction on n. We show property (4) holds for $f^{(n)}$ (assuming all the properties hold for $f^{(n-1)}$), since we will use it below.

$$\begin{aligned}
\mathrm{tr}_n(f^{(n)}) &= \mathrm{tr}_n(i_{n-1}(f^{(n-1)})) - \frac{[n-1]}{[n]}\,\mathrm{tr}_n(i_{n-1}(f^{(n-1)})E_{n-1}i_{n-1}(f^{(n-1)})) \\
&= [2]\,\mathrm{tr}_{n-1}(f^{(n-1)}) - \frac{[n-1]}{[n]}\,\mathrm{tr}_{n-1}(f^{(n-1)}) \\
&= [2][n] - [n-1] \\
&= [n+1].
\end{aligned}$$

□

Theorem 10 (Jones' index rigidity)**.** *Suppose* $\langle \cdot, \cdot \rangle_j$ *is positive semidefinite for all* $j \geq 0$. *Then either* $q \geq 1$, *or* q *is a primitive* $(2n)$*-th root of unity for some* $n \geq 3$. *Hence*

$$[2] = q + q^{-1} \in \left\{ 2\cos\left(\frac{\pi}{n}\right) \middle| n \geq 3 \right\} \cup [2, \infty).$$

Proof. If q is not of this form, then let k be as in Lemma 2. We see that since $[1], [2], \ldots, [k] \neq 0$, $f^{(k)}$ is well-defined. However,

$$\langle f^{(k)}, f^{(k)} \rangle_k = \mathrm{tr}_k(f^{(k)}) = [k+1] < 0,$$

which is a contradiction. □