Category theory is an elegant and powerful language for studying various branches of mathematics. Often proofs can be formalized to apply in many different situations, and many definitions apply to many different mathematical structures simultaneously. Deep connections between different areas of mathematics can be seen from the existence of functors (maps) between categories.

2.1. **Categories.**

Definition 2.1.1. A (locally small) *category* has a collection of objects, and between any two objects, a set of morphisms. We write $c \in \mathcal{C}$ to denote an object of a category $\mathcal{C}$, and we write $\mathcal{C}(a \to b)$ or $\operatorname{Hom}(a \to b)$ for the set of morphisms from a to b, and we write $\operatorname{End}(a) := \operatorname{Hom}(a \to a)$. There is a composition law for morphisms: if $f \in \mathcal{C}(a \to b)$ and $g \in \mathcal{C}(b \to c)$, there is a morphism $g \circ f \in \mathcal{C}(a \to c)$. This composition must be associative, and every object $c \in \mathcal{C}$ has an identity morphism $\mathrm{id}_c \in \mathcal{C}(c \to c)$.

An *isomorphism* is an invertible morphism. We say two objects a, b are *isomorphic* if there is an isomorphism $a \to b$. We always assume that $\mathcal{C}$ is *essentially small*, i.e., the isomorphism classes of our category form a set, unless stated otherwise.

Example 2.1.2. Typical examples of categories one works with are:

- Set, the category of sets and functions (which is not essentially small),
- Top, the category of topological spaces and continuous maps (which is not essentially small),
- Vec, the category of finite dimensional complex vector spaces and linear maps,
- Hilb, the category of finite dimensional Hilbert spaces and linear maps,
- $\mathsf{Rep}(G)$, the category of finite dimensional complex representations of a finite group G. In more detail, objects are pairs (V, π) where $V \in \mathsf{Vec}$ and $\pi : G \to \operatorname{End}(V)$ is a homomorphism. The morphisms are G-equivariant maps, i.e., $T : (V, \pi) \to (W, \rho)$ is a linear map $T : V \to W$ such that $T\pi_g = \rho_g T$ for all $g \in G$.
- $\mathsf{Rep}^\dagger(G)$, the category of finite dimensional Hilbert space unitary representations of a finite group G.
- $\mathsf{Rep}(A)$, the category of all finite dimensional complex representations of a finite dimensional unital algebra A.
- $\mathsf{Rep}^\dagger(A)$, the category of all finite dimensional Hilbert space representations of a unitary algebra A.
- $\mathsf{Mod}(A)$, the category of all finite dimensional right A-modules of a finite dimensional unital algebra A. (Observe $\mathsf{Mod}(A) \cong \mathsf{Rep}(A^{\mathrm{op}})$, where A^{op} denotes the opposite algebra with the opposite multiplication $a^{\mathrm{op}} \cdot b^{\mathrm{op}} := (ba)^{\mathrm{op}}$.)
- $\mathsf{Mod}^\dagger(A)$, the category of all unitary finite dimensional right A-modules of a unitary algebra A. (Again, $\mathsf{Mod}^\dagger(A) \cong \mathsf{Rep}^\dagger(A^{\mathrm{op}})$.)

Example 2.1.3. Let M be a monoid. The *delooping* $\mathrm{B}M$ is the category with one object $\star$ and $\operatorname{End}(\star) := M$, where the composition law is multiplication in M and $\mathrm{id}_\star$ is the unit of M. In this sense, one should think of a category as a monoid with more than one object.

Definition 2.1.4. A map of categories is called a *functor*. A functor $F : \mathcal{A} \to \mathcal{B}$ assigns an object $F(a) \in \mathcal{B}$ to each $a \in \mathcal{A}$ and a morphism $F(f) : F(a) \to F(b)$ to each $f \in \mathcal{A}(a \to b)$. We require that F preserves identities and the composition law.

Example 2.1.5. A common type of functor is a *forgetful functor* which forgets extra structure. For example, there are organic forgetful functors $\mathsf{Top} \to \mathsf{Set}$, $\mathsf{Vec} \to \mathsf{Set}$, and $\mathsf{Hilb} \to \mathsf{Vec}$.

Example 2.1.6. (Small) Categories and functors form a category (which is not locally small) $\mathsf{Cat}_{\tau\leq 1}$. The reason for this subscript will become clear in the next chapter.

We call a category $\mathcal{C}$ *pointed* if it comes equipped with a choice of distinguished object $c \in \mathcal{C}$. A functor between pointed categories $(\mathcal{C}, c) \to (\mathcal{D}, d)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(c) = d$. We denote the category of pointed (small) categories by $\mathsf{Cat}^*_{\tau\leq 1}$.

Example 2.1.7. Consider the category Mon of monoids and monoid homomorphisms. Delooping gives a functor $\mathrm{B} : \mathsf{Mon} \to \mathsf{Cat}^*_{\tau\leq 1}$, where a monoid M is mapped to $(\mathrm{B}M, \star)$.

Now for $(\mathcal{C}, c) \in \mathsf{Cat}^*_{\tau\leq 1}$, we can define the *loops* of $\mathcal{C}$ at c by $\Omega(\mathcal{C}, c) := \mathrm{End}_{\mathcal{C}}(c)$, which gives us a monoid. One can check that $\Omega : \mathsf{Cat}^*_{\tau\leq 1} \to \mathsf{Mon}$ is a functor such that $\Omega \circ \mathrm{B} = \mathrm{id}_{\mathsf{Mon}}$.

Definition 2.1.8. The functors from $\mathcal{A} \to \mathcal{B}$ themselves form another category. Given two functors $F, G : \mathcal{A} \to \mathcal{B}$, a *natural transformation* $\rho : F \Rightarrow G$ is an assignment of a map $\rho_a \in \mathcal{B}(F(a) \to G(a))$ for every $a \in \mathcal{A}$ such that the following diagram commutes for every $f \in \mathcal{A}(a \to b)$.

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(b) \\ \downarrow{\scriptstyle \rho_a} & & \downarrow{\scriptstyle \rho_b} \\ G(a) & \xrightarrow{G(f)} & G(b) \end{array}$$

Given three functors $F, G, H : \mathcal{A} \to \mathcal{B}$, composition of natural transformations $\rho : F \Rightarrow G$ and $\sigma : G \Rightarrow H$ is given by $(\sigma \cdot \rho)_a := \sigma_a \circ_{\mathcal{B}} \rho_a$. It is straightforward to show $\sigma \cdot \rho$ satisfies the above commutative square.

We denote the category of functors $\mathcal{A} \to \mathcal{B}$ and natural transformations by $\mathsf{Fun}(\mathcal{A} \to \mathcal{B})$.

Exercise 2.1.9. Prove that if $F_1, F_2 : \mathcal{A} \to \mathcal{B}$ and $G_1, G_2 : \mathcal{B} \to \mathcal{C}$ are functors and $\rho : F_1 \Rightarrow F_2$ and $\sigma : G_1 \Rightarrow G_2$ are natural transformations, find another composite $\sigma \circ \rho : G_1 \circ F_1 \Rightarrow G_2 \circ F_2$.

Definition 2.1.10. An *equivalence* between categories $\mathcal{C} \to \mathcal{D}$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$, a functor $G : \mathcal{D} \to \mathcal{C}$, and natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$.

Example 2.1.11. The forgetful functor $\mathsf{Hilb} \to \mathsf{Vec}$ is an equivalence.

Exercise 2.1.12. Show that there is an equivalence of categories $\mathsf{Fun}(\mathrm{B}G \to \mathsf{Hilb}) \cong \mathsf{Rep}^\dagger(G)$ where G is a finite group and $\mathrm{B}G$ is its delooping from Example 2.1.3.

Definition 2.1.13. A functor $F : \mathcal{C} \to \mathcal{D}$ is called:

- *faithful* if the function $\mathcal{C}(a \to b) \to \mathcal{D}(F(a) \to F(b))$ is injective,
- *full* if the function $\mathcal{C}(a \to b) \to \mathcal{D}(F(a) \to F(b))$ is surjective,
- *essentially surjective* if every object $d \in \mathcal{D}$ is isomorphic to an object of the form $F(c)$ for some $c \in \mathcal{C}$.

Exercise 2.1.14. Prove that a functor $F : \mathcal{C} \to \mathcal{D}$ can be augmented to an equivalence if and only if F is fully faithful and essentially surjective.

The following exercise is essential to universal properties in the sections below.

2.2. **Linear categories and linear functors.**

Definition 2.2.1. We call a category $\mathcal{C}$ *linear* if it is enriched in finite dimensional complex vector spaces. This means that $\mathcal{C}(a \to b)$ is a finite dimensional complex vector space for each $a, b \in \mathcal{C}$, and pre- and post-composition by any composable morphism in $\mathcal{C}$ is a linear operation. In other words, the composition operation

$$- \circ_{\mathcal{C}} - : \mathcal{C}(b \to c) \otimes \mathcal{C}(a \to b) \longrightarrow \mathcal{C}(a \to c) \qquad \text{given by} \qquad g \otimes f \mapsto g \circ f$$

is a linear map.

A functor $F : \mathcal{C} \to \mathcal{D}$ between linear categories is called *linear* if it is linear on hom spaces, i.e., for all $f, g \in \mathcal{C}(a \to b)$ and $\lambda \in \mathbb{C}$, $F(\lambda f + g) = \lambda F(f) + F(g)$.

Two linear categories $\mathcal{C}, \mathcal{D}$ are *equivalent* if there is an equivalence consisting of linear functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.

Unless stated otherwise, all categories and functors in this section are linear.

Example 2.2.2. Let A be a unital finite dimensional complex algebra. The category $\mathrm{B}A$, called the *delooping* of A, has one object $\star$ and $\operatorname{End}(\star) := A$. In this sense, one should think of a linear category as an algebra with more than one object.

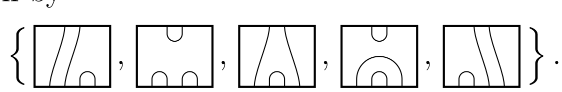
Example 2.2.3. Both Vec and Hilb are linear categories. Moreover, the forgetful functor $\mathsf{Hilb} \to \mathsf{Vec}$ is an equivalence of linear categories.

Example 2.2.4. Let S be a set. The category $\mathsf{Vec}(S)$ has objects finite dimensional S-graded complex vector spaces $V = \bigoplus_{s \in S} V_s$ and grading-preserving linear maps, i.e., if $T : V \to W$, then $T(V_s) \subset W_s$.

Example 2.2.5. Let G be a finite group. Both $\mathsf{Rep}(G)$ and $\mathsf{Rep}^{\dagger}(G)$ are linear categories.

Example 2.2.6. When $\mathcal{C}, \mathcal{D}$ are linear categories, $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ is also a linear category, except the spaces $\operatorname{Hom}(F \Rightarrow G)$ of natural transformations may be infinite dimensional.

Example 2.2.7. Let $d \in \mathbb{C}$. The category $\mathsf{TLJ}(d)$ has objects $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathsf{TLJ}(m \to n)$ consists of complex linear combinations of Kauffman diagrams with m boundary points on the bottom and n boundary points on the top. For example, the basis for $\mathsf{TLJ}(d)(4 \to 2)$ is given by



Composition is given by the usual stacking of diagrams and bubble popping relation. **TODO: more details here!**

Definition 2.2.8. Given a linear category $\mathcal{C}$ and objects $a, b \in \mathcal{C}$, the *linking algebra* $L(a, b)$ is

$$L(a, b) := \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(b \to a) \\ \mathcal{C}(a \to b) & \mathcal{C}(b \to b) \end{pmatrix}$$

whose multiplication is given by matrix multiplication and composition in $\mathcal{C}$. More generally, given $a_1, \dots, a_n \in \mathcal{C}$, we can define the n-fold linking algebra $L(a_1, \dots, a_n)$.

Definition 2.2.9. A linear category is called *pre-semisimple* if for every $a_1, \ldots, a_n \in \mathcal{C}$, the linking algebra $L(a_1, \ldots, a_n)$ is a finite dimensional semisimple complex algebra.

We call a pre-semisimple category *finite* if there is a global bound on the dimensions of the centers of all linking algebras. That is, there is a $K > 0$ such that $\dim(Z(L(a_1, \ldots, a_n))) < K$ for all $a_1, \ldots, a_n \in \mathcal{C}$.

2.3. **Direct sums.** In this section, unless stated otherwise, $\mathcal{C}$ denotes a linear category. Later in the section, $\mathcal{C}$ will be a unitary category.

Definition 2.3.1. Given objects $a, b \in \mathcal{C}$, an object $a \oplus b \in \mathcal{C}$ equipped with morphisms $\iota_a : a \to a \oplus b$, $\iota_b : b \to a \oplus b$, $\pi_a : a \oplus b \to a$, and $\pi_b : a \oplus b \to b$ is called the *direct sum* of a and b if

($\oplus$1) $\pi_a \circ \iota_a = \mathrm{id}_a$ and $\pi_b \circ \iota_b = \mathrm{id}_b$, and
($\oplus$2) $\iota_a \circ \pi_a + \iota_b \circ \pi_b = \mathrm{id}_{a \oplus b}$

The direct sum is canonical in the sense that the space of direct sums is contractible, as was the case for Hilb.

Observe there is a canonical isomorphism of algebras

$$\mathrm{End}_{\mathcal{C}}(a \oplus b) \cong L(a, b) = \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(b \to a) \\ \mathcal{C}(a \to b) & \mathcal{C}(b \to b) \end{pmatrix}. \tag{2.3.2}$$

This map and its inverse are given by

$$f \mapsto \begin{pmatrix} \pi_a \circ f \circ \iota_a & \pi_a \circ f \circ \iota_b \\ \pi_b \circ f \circ \iota_a & \pi_b \circ f \circ \iota_b \end{pmatrix}$$

$$\sum_{i,j \in \{a,b\}} \iota_i \circ g_{ij} \circ \pi_j \leftarrow\!\shortmid \begin{pmatrix} g_{aa} & g_{ab} \\ g_{ba} & g_{bb} \end{pmatrix}$$

We say $\mathcal{C}$ *admits direct sums* if $a \oplus b$ exists for all $a, b \in \mathcal{C}$, and there is a zero object $0 \in \mathcal{C}$ which is simultaneously *initial* and *terminal* in $\mathcal{C}$. This means for every $c \in \mathcal{C}$, there are unique morphisms $0 \to c$ and $c \to 0$.

Exercise 2.3.3. Show that any two initial objects in $\mathcal{C}$ are uniquely isomorphic. Then do the same for two terminal objects.

Exercise 2.3.4. Suppose that $(a \oplus b, \iota_a, \iota_b, \pi_a, \pi_b)$ is the direct sum of a and b. Show that $(a \oplus b, \iota_a, \iota_b)$ is the coproduct of a and b, and $(a \oplus b, \pi_a, \pi_b)$ is the product of a and b.

Exercise 2.3.5. Show that direct sums are preserved by all linear functors between linear categories. That is, if $F : \mathcal{C} \to \mathcal{D}$ is a linear functor and $(a \oplus b, \iota_a, \iota_b, \pi_a, \pi_b)$ witnesses the direct sum of $a, b \in \mathcal{C}$, then $(F(a \oplus b), F(\iota_a), F(\iota_b), F(\pi_a), F(\pi_b))$ witnesses the direct sum of $F(a), F(b) \in \mathcal{D}$, i.e., there is a canonical isomorphism $F(a \oplus b) \cong F(a) \oplus F(b)$.

Exercise 2.3.6. Suppose $\mathcal{C}, \mathcal{D}$ are linear categories such that $\mathcal{D}$ admits direct sums. Show that $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ admits direct sums.

Construction 2.3.7. Given a linear category $\mathcal{C}$, the *additive envelope* of $\mathcal{C}$ is the linear category $\mathsf{Add}(\mathcal{C})$ whose objects are *formal tuples* $(a_i)_{i=1}^n$ for $a_1, \ldots, a_n \in \mathcal{C}$, $n \geq 0$, and whose morphism sets are given by matrices of operators:

$$\mathsf{Add}(\mathcal{C})\left((b_j)_{j=1}^n \to (a_i)_{i=1}^m\right) := \{(x_{ij}) | x_{ij} \in \mathcal{C}(b_j \to a_i)\} \tag{2.3.8}$$

where composition is given by $(x_{ij}) \circ (y_{jk}) := (\sum_j x_{ij} \circ y_{jk})$. By convention, 0 is the only morphism to or from the empty tuple.

Observe that $c \mapsto (c)$ for $c \in \mathcal{C}$ and $x \mapsto (x)$ for $x \in \mathcal{C}(a \to b)$ is a fully faithful linear functor $\mathcal{C} \hookrightarrow \mathsf{Add}(\mathcal{C})$.

Lemma 2.3.9. $\mathsf{Add}(\mathcal{C})$ *admits finite direct sums.*

Proof. Consider objects $c_1 = (a_j)_{j=1}^m$ and $c_2 = (a_j)_{j=m+1}^n$ in $\mathsf{Add}(\mathcal{C})$. We define their direct sum $c_1 \oplus c_2$ as the tuple $(a_j)_{j=1}^n$, and we define $\iota_1 : (a_j)_{j=1}^m \to (a_j)_{j=1}^n$ and $\iota_2 : (a_j)_{j=m+1}^n \to (a_j)_{j=1}^n$ by

$$(\iota_1)_{ij} = \begin{cases} \delta_{i=j}\, \mathrm{id}_{a_i} & \text{if } i, j \le m \\ 0 & i > m \end{cases} \qquad (\iota_2)_{ij} = \begin{cases} \delta_{i=j}\, \mathrm{id}_{a_i} & \text{if } i, j > m \\ 0 & i \le m \end{cases}$$

and $\pi_1 : (a_j)_{j=1}^n \to (a_j)_{j=1}^m$ and $\pi_2 : (a_j)_{j=1}^n \to (a_j)_{j=m+1}^n$ by

$$(\pi_1)_{ij} = \begin{cases} \delta_{i=j}\, \mathrm{id}_{a_i} & \text{if } i, j \le m \\ 0 & j > m \end{cases} \qquad (\pi_2)_{ij} = \begin{cases} \delta_{i=j}\, \mathrm{id}_{a_i} & \text{if } i, j > m \\ 0 & j \le m. \end{cases}$$

The rest of the verification is left to the reader. □

The additive envelope $\mathsf{Add}(\mathcal{C})$ with the canonical inclusion $\mathcal{C} \hookrightarrow \mathsf{Add}(\mathcal{C})$ satisfies the following universal property.

Proposition 2.3.10. *For every linear category $\mathcal{D}$ which admits finite direct sums, precomposition with the canonical inclusion $\mathcal{C} \hookrightarrow \mathsf{Add}(\mathcal{C})$ gives an equivalence*

$$\iota^* : \mathsf{Fun}(\mathsf{Add}(\mathcal{C}) \to \mathcal{D}) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathcal{D}).$$

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a linear functor. We define a linear functor $\mathsf{Add}(F) : \mathsf{Add}(\mathcal{C}) \to \mathcal{D}$ by setting $\mathsf{Add}(F)((a_j)_{j=1}^n) := \bigoplus_{j=1}^n F(a_j)$ and $\mathsf{Add}(F)(x_{ij}) = \sum_{i,j} \iota_i \circ F(x_{ij}) \circ \pi_j$ where $(\iota_j, \pi_j)_{j=1}^n$ witness the direct sum of the $F(a_j)$ in $\mathcal{D}$. Clearly F equals the composite of the canonical inclusion followed by $\mathsf{Add}(F)$. Hence ι^* is essentially surjective.

Suppose now that $\alpha : F \Rightarrow G$ is a natural transformation between functors $\mathcal{C} \to \mathcal{D}$. Since $(a_j)_{j=1}^n \in \mathsf{Add}(\mathcal{C})$ is the direct sum of the $(a_j) \in \mathsf{Add}(\mathcal{C})$, we get a natural transformation $\mathsf{Add}(\alpha) : \mathsf{Add}(F) \Rightarrow \mathsf{Add}(G)$ by defining

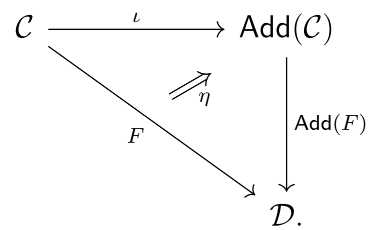
$$\mathsf{Add}(\alpha)_{(a_j)} := \sum_j \iota_j^G \circ \alpha_{a_j} \circ \pi_j^F : \bigoplus_{j=1}^n F(a_j) \to \bigoplus_{j=1}^n G(a_j). \tag{2.3.11}$$

Since, $\mathsf{Add}(\alpha)_{(a)} = \alpha_a : F(a) \to G(a)$ for all $a \in \mathcal{C}$, $\iota^*(\mathsf{Add}(\alpha)) = \alpha$, so ι^* is full.

If $\beta : \mathsf{Add}(F) \Rightarrow \mathsf{Add}(G)$ with $\iota^*(\beta) = 0$, then $\beta_{(a)} : F(a) \to G(a)$ is zero for all $a \in \mathcal{C}$. By naturality, (2.3.11) still holds replacing $\mathsf{Add}(\alpha)$ with β and α_{a_j} with $\beta_{(a_j)}$. Thus $\beta = 0$ and ι^* is faithful. □

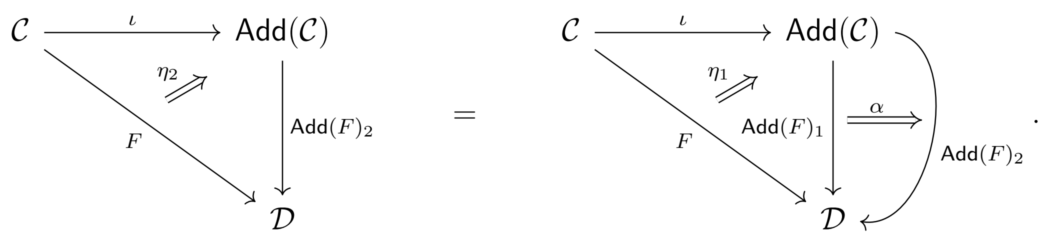
Remark 2.3.12. Unpacking the universal property in Proposition 2.3.10 above, essential surjectivity of ι^* means that for any $F : \mathcal{C} \to \mathcal{D}$, there is a linear functor $\mathsf{Add}(F) : \mathsf{Add}(\mathcal{C}) \to$

$\mathcal{D}$ and a natural isomorphism $\eta : F \Rightarrow \mathsf{Add}(F) \circ \iota$.



(2.3.13)

That ι^* is fully faithful means that whenever $(\mathsf{Add}(F)_i, \eta_i)$ for $i = 1, 2$ both satisfy (2.3.13) above, there is a unique natural transformation $\alpha : \mathsf{Add}(F)_1 \Rightarrow \mathsf{Add}(F)_2$ such that



Corollary 2.3.14. *If $\mathcal{C}$ admits all finite direct sums, then $\mathcal{C}$ is equivalent to $\mathsf{Add}(\mathcal{C})$.*

Proof. This is a formal consequence of the universal property. First, since

$$\iota^* : \mathsf{Fun}(\mathsf{Add}(\mathcal{C}) \to \mathcal{C}) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathcal{C})$$

is an equivalence, there is a functor $\mathsf{Add}(\mathrm{id}_{\mathcal{C}}) : \mathsf{Add}(\mathcal{C}) \to \mathcal{C}$ such that $\mathsf{Add}(\mathrm{id}_{\mathcal{C}}) \circ \iota \cong \mathrm{id}_{\mathcal{C}}$. Next, since

$$\iota^* : \mathsf{Fun}(\mathsf{Add}(\mathcal{C}) \to \mathsf{Add}(\mathcal{C})) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathsf{Add}(\mathcal{C}))$$

is an equivalence and both $\mathrm{id}_{\mathsf{Add}(\mathcal{C})}$ and $\iota \circ \mathsf{Add}(\mathrm{id}_{\mathcal{C}})$ are objects on the left hand side, we see that

$$\iota^*(\iota \circ \mathsf{Add}(\mathrm{id}_{\mathcal{C}})) = \iota \circ \mathsf{Add}(\mathrm{id}_{\mathcal{C}}) \circ \iota \cong \iota \circ \mathrm{id}_{\mathcal{C}} = \iota = \mathrm{id}_{\mathsf{Add}(\mathcal{C})} \circ \iota = \iota^*(\mathrm{id}_{\mathsf{Add}(\mathcal{C})}).$$

Since ι^* is an equivalence, we must have $\mathrm{id}_{\mathsf{Add}(\mathcal{C})} \cong \iota \circ \mathsf{Add}(\mathrm{id}_{\mathcal{C}})$. □

Remark 2.3.15. We will see in the next chapter that using Proposition 2.3.10, we can extend Add to a 2-functor from the 2-category of linear categories to the 2-category of linear categories which admit direct sums. Indeed, given a linear functor $F : \mathcal{C} \to \mathcal{D}$, we can post-compose with the inclusion $\mathcal{D} \hookrightarrow \mathsf{Add}(\mathcal{D})$ and apply Proposition 2.3.10 to obtain a functor $\mathsf{Add}(F) : \mathsf{Add}(\mathcal{C}) \to \mathsf{Add}(\mathcal{D})$.

We leave it to the reader to show that $\mathsf{Add}(\mathrm{id}_{\mathcal{C}}) \cong \mathrm{id}_{\mathsf{Add}(\mathcal{C})}$ and $\mathsf{Add}(F) \circ \mathsf{Add}(G) \cong \mathsf{Add}(F \circ G)$ for composable F, G.

Exercise 2.3.16. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a linear functor and $\mathcal{D}$ admits direct sums. Prove that if F is fully faithful, then so is $\mathsf{Add}(F) : \mathsf{Add}(\mathcal{C}) \to \mathcal{D}$.

2.4. **Multiplicity spaces.** Let $\mathcal{C}$ be a linear category which admits direct sums and let $c \in \mathcal{C}$. We now define a functor $- \otimes c : \mathsf{Vec} \to \mathcal{C}$, which we can think of as a categorification of the scalar product of $\mathbb{C}$ on Vec. Really, the object $V \otimes c$ is a fancy version of the direct sum $\bigoplus_{j=1}^{\dim(V)} c$ which allows us to define a morphism $f \otimes \mathrm{id}_c : V \otimes c \to W \otimes c$ for every linear map $f : V \to W$.

Construction 2.4.1. First, define $\mathbb{C} \otimes c := c$. For a (finite dimensional) vector space $V \in \mathsf{Vec}$, choose an ordered basis $\{v_1, \ldots, v_n\}$, which we may identify as a collection of maps $\mathbb{C} \to V$ given by $1 \mapsto v_i$. Define $V \otimes c := \bigoplus_{j=1}^n c$ with maps ι_j^V, π_j^V for $j = 1, \ldots, n$.

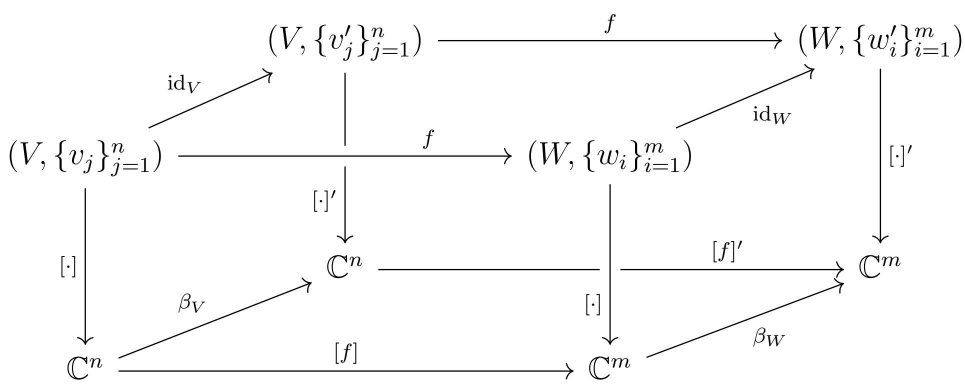
If $W \in \mathsf{Vec}$ with an ordered basis $\{w_1, \ldots, w_m\}$ and $f : V \to W$, recall that $[f] \in M_{m\times n}(\mathbb{C})$ is the matrix such that the following diagram commutes:

$$\begin{CD} (V, \{v_j\}_{j=1}^n) @>{f}>> (W, \{w_i\}_{i=1}^m) \\ @V{[\cdot]}VV @VV{[\cdot]}V \\ \mathbb{C}^n @>{[f]}>> \mathbb{C}^m \end{CD}$$

Letting $W \otimes c = \bigoplus_{i=1}^m c$ with maps ι_i^W, π_i^W for $i = 1, \ldots, m$, we define the map $f \otimes \mathrm{id}_c : V \otimes c \to W \otimes c$ as $\sum_{i=1}^m \sum_{j=1}^n \iota_i^W([f]_{ij} \, \mathrm{id}_{\mathcal{C}})\pi_j^V$.

Exercise 2.4.2. Prove that $(f \otimes \mathrm{id}_c) \circ (g \otimes \mathrm{id}_c) = (f \circ g) \otimes \mathrm{id}_c$ whenever f, g are composable.

Since it is clear that $\mathrm{id}_V \otimes \mathrm{id}_c = \mathrm{id}_{V\otimes c}$ by construction, we can conclude that $-\otimes c : \mathsf{Vec} \to \mathcal{C}$ is a functor. However, we relied on choosing ordered bases for each vector space $V \in \mathsf{Vec}$. If we had chosen different bases, say $\{v_1', \ldots, v_n'\}$ for V and $\{w_1', \ldots, w_m'\}$ for W giving a different coordinate map $[f]' \in M_{m\times n}(\mathbb{C})$, there are unique invertible matrices β_V, β_W making the following diagram commute.



Hence $\beta = \{\beta_V\}_{V\in\mathsf{Vec}}$ gives a canonical natural isomorphism $-\otimes c \Rightarrow -\otimes' c$. Moreover, these canonical natural isomorphisms compose properly, meaning that there is a contractible choice of functor $-\otimes c : \mathsf{Vec} \to \mathcal{C}$.

Definition 2.4.3. Given $V \in \mathsf{Vec}$ and $c \in \mathcal{C}$, we call V the *multiplicity space* for the object $V \otimes c \in \mathcal{C}$.

Given $a, b \in \mathcal{C}$ and $g : a \to b$, we can *amplify* $g : a \to b$ to a map $\bigoplus_{j=1}^n g : \bigoplus_{j=1}^n a \to \bigoplus_{j=1}^n b$ given by $\sum_{j=1}^n \iota_j^b \circ g \circ \pi_j^a$. This means for every vector space V and choice of ordered basis, we get a canonical map $\mathrm{id}_V \otimes g : V \otimes a \to V \otimes b$.

Exercise 2.4.4. Show that $\mathrm{id}_V \otimes g$ is natural in V, i.e., for all linear maps $f : V \to W$, the following diagram commutes.

$$\begin{CD} V \otimes a @>{f\otimes \mathrm{id}_a}>> W \otimes a \\ @V{\mathrm{id}_V \otimes g}VV @VV{\mathrm{id}_W \otimes g}V \\ V \otimes b @>{f\otimes \mathrm{id}_b}>> W \otimes b \end{CD}$$

Deduce that defining $f \otimes g$ by either of the above composites is well-defined, and that the maps $\mathrm{id}_V \otimes g : V \otimes a \to V \otimes b$ compile into a natural transformation $- \otimes g : - \otimes a \Rightarrow - \otimes b$.

Exercise 2.4.5. Show that since $\bigoplus_{j=1}^n h \circ \bigoplus_{j=1}^n g = \bigoplus_{j=1}^n h \circ g$ for composable $g : a \to b$ and $h : b \to c$, the composite

$$- \otimes a \xRightarrow{-\otimes g} - \otimes b \xRightarrow{-\otimes h} - \otimes c$$

is equal to $- \otimes (h \circ g)$.

We summarize the above discussion into the following corollary below, in which $\mathsf{Vec} \times \mathcal{C}$ is the category whose objects are pairs (V, c) with $V \in \mathsf{Vec}$ and $c \in \mathcal{C}$ and whose morphims are pairs of morphisms (f, g) with f in Vec and g in $\mathcal{C}$.

Corollary 2.4.6. *$- \otimes - : \mathsf{Vec} \times \mathcal{C} \to \mathcal{C}$ is a functor.*

Proof. We combine Exercises 2.4.2, 2.4.4, and 2.4.5. First, $\mathrm{id}_V \otimes \mathrm{id}_c = \mathrm{id}_{V \otimes c}$ as $- \otimes c$ is a functor. If $f_1 : U \to V$ and $f_2 : V \to W$, and $g_1 : a \to b$ and $g_2 : b \to c$, then

$$\begin{aligned}
(f_2 \otimes g_2) \circ (f_1 \otimes g_1) &= (f_2 \otimes \mathrm{id}_c) \circ (\mathrm{id}_V \circ g_2) \circ (f_1 \otimes \mathrm{id}_b) \circ (\mathrm{id}_U \otimes g_1) && \text{(Ex. 2.4.4)}\\
&= (f_2 \otimes \mathrm{id}_c) \circ (f_1 \otimes \mathrm{id}_c) \circ (\mathrm{id}_U \circ g_2) \circ (\mathrm{id}_U \otimes g_1) && \text{(Ex. 2.4.4)}\\
&= ((f_2 \circ f_1) \otimes \mathrm{id}_c) \circ (\mathrm{id}_U \otimes (g_2 \circ g_1)) && \text{(Ex. 2.4.2 and 2.4.5)}\\
&= (f_2 \circ f_1) \otimes (g_2 \circ g_1). && \text{(Ex. 2.4.4)}
\end{aligned}$$

□

This corollary is the categorification of a scalar product on a vector space to a Vec-product on linear categories.

Now as linear functors preserve direct sums, each linear functor $F : \mathcal{C} \to \mathcal{D}$ comes equipped with a canonical isomorphism

$$\mu_{V,c} : F(V \otimes c) \longrightarrow V \otimes F(c) \tag{2.4.7}$$

which is natural in both $V \in \mathsf{Vec}$ and $c \in \mathcal{C}$. That is, for every $f : V \to W$ and $g : a \to b$, the following diagrams commute:

$$\begin{array}{ccc}
F(V \otimes c) & \xrightarrow{F(f \otimes \mathrm{id}_c)} & F(W \otimes c) \\
\downarrow{\scriptstyle \mu_{V,c}} & & \downarrow{\scriptstyle \mu_{W,c}} \\
V \otimes F(c) & \xrightarrow{f \otimes \mathrm{id}_{F(c)}} & W \otimes F(c)
\end{array}
\qquad
\begin{array}{ccc}
F(V \otimes a) & \xrightarrow{F(\mathrm{id}_V \otimes f)} & F(V \otimes b) \\
\downarrow{\scriptstyle \mu_{V,a}} & & \downarrow{\scriptstyle \mu_{V,b}} \\
V \otimes F(a) & \xrightarrow{\mathrm{id}_V \otimes F(f)} & V \otimes F(b)
\end{array}$$

2.5. **Idempotent completion.** In this section $\mathcal{C}$ is merely a category. Later in this section, $\mathcal{C}$ will be a unitary category.

Definition 2.5.1. An *idempotent* in $\mathcal{C}$ is a pair (c, e) where $c \in \mathcal{C}$ and $e \in \mathcal{C}(c \to c)$ such that $e \circ e = e$. A *splitting* for an idempotent (c, e) is an triple (a, r, s) where $a \in \mathcal{C}$, $r \in \mathcal{C}(c \to a)$ called a *retract*, and $s \in \mathcal{C}(a \to c)$ such that $s \circ r = e$ and $r \circ s = \mathrm{id}_a$. A linear category $\mathcal{C}$ is called *idempotent complete* if every idempotent admits a splitting.

Exercise 2.5.2. Suppose $(a, r_a, s_a), (b, r_b, s_b)$ are two splittings of (c, e). Show that there is a unique isomorphism $f : a \to b$ which is compatible with (r_a, s_a) and (r_b, s_b).

Exercise 2.5.3. Suppose $\mathcal{C}, \mathcal{D}$ are categories. Show that the property that the idempotent (c, e) admits a splitting is preserved by all functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. That is, if (a, r, s) splits (c, e), prove that $(F(a), F(r), F(s))$ splits $(F(c), F(e))$.

Definition 2.5.4. Suppose $\mathcal{C}, \mathcal{D}$ are categories. A linear functor $F : \mathcal{C} \to \mathcal{D}$ is called *dominant* if for every $d \in \mathcal{D}$, there is a $c \in \mathcal{C}$, a retract $r : F(c) \to d$, and a splitting $s : d \to F(c)$ such that $r \circ s = \mathrm{id}_d$.

Proposition 2.5.5. *Suppose $\mathcal{C}, \mathcal{D}$ are categories with $\mathcal{C}$ idempotent complete. A fully faithful linear functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is dominant.*

Proof. It suffices to show dominant implies essentially surjective. Suppose F is dominant and $d \in \mathcal{D}$. Let $c \in \mathcal{C}$ and $s : d \to F(c)$ and $r : F(c) \to d$ such that $r \circ s = \mathrm{id}_d$. Consider the idempotent $s \circ r \in \mathrm{End}_{\mathcal{D}}(F(c))$. Since F is full, there is an $e \in \mathrm{End}_{\mathcal{C}}(c)$ such that $F(e) = s \circ r$, and since F is faithful, e is also an idempotent. Since $\mathcal{C}$ is idempotent complete, there is a splitting (a, r', s') of (c, e). We claim that $F(a) \cong d$. Indeed, since $s' \circ r' = e$ and $r' \circ s' = \mathrm{id}_a$, we see that the maps

$$F(a) \xrightarrow{F(s')} F(c) \xrightarrow{r} d \qquad \text{and} \qquad d \xrightarrow{s} F(c) \xrightarrow{F(r')} F(a)$$

are mutually inverse. □

Construction 2.5.6. The *idempotent/Karoubi completion* $\mathsf{Idem}(\mathcal{C})$ is the category whose objects are pairs (c, e) where $c \in \mathcal{C}$ and $e \in \mathcal{C}(c \to c)$ is an idempotent. The morphism spaces are given by

$$\mathcal{C}((a, e) \to (b, f)) := \{x \in \mathcal{C}(a \to b) | x = f \circ x \circ e\}.$$

Observe that $\mathcal{C}((a, e) \to (b, f)) \subseteq \mathcal{C}(a \to b)$ is a linear subspace, and if $x \in \mathcal{C}((a, e) \to (b, f))$, then $x = x \circ e = f \circ x$. Composition of morphisms is exactly composition in $\mathcal{C}$, i.e., if $x \in \mathcal{C}((a, e) \to (b, f))$ and $y \in \mathcal{C}((b, f) \to (c, g))$, then $y \circ x \in \mathcal{C}((a, e) \to (c, g))$.

There is a faithful inclusion functor $\mathcal{C} \hookrightarrow \mathsf{Idem}(\mathcal{C})$ given by $c \mapsto (c, \mathrm{id}_c)$.

Proposition 2.5.7. $\mathsf{Idem}(\mathcal{C})$ *is idempotent complete.*

Proof. If $f : (c, e) \to (c, e)$ is an idempotent, then $f : c \to c$ satisfies $fe = ef = f$ and $f^2 = f$. Thus (c, f) is another idempotent. We claim that $r = f : (c, e) \to (c, f)$ and $s = f : (c, f) \to (c, e)$ splits $((c, e), f)$ as an idempotent. Indeed, $rs = f^2 = f = \mathrm{id}_{(c,f)}$ and $sr = f^2 = f \in \mathrm{End}(c, e)$. □

The pair $\mathsf{Idem}(\mathcal{C})$ and the inclusion $\iota : \mathcal{C} \hookrightarrow \mathsf{Idem}(\mathcal{C})$ satisfy the following universal property. We leave it to the reader to unpack this as in Remark 2.3.12.

Proposition 2.5.8. *For every idempotent complete category $\mathcal{D}$, pre-composition with the canonical inclusion $\iota : \mathcal{C} \hookrightarrow \mathsf{Idem}(\mathcal{C})$ gives an equivalence*

$$\iota^* : \mathsf{Fun}(\mathsf{Idem}(\mathcal{C}) \to \mathcal{D}) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathcal{D}).$$

Proof. We define $\mathsf{Idem}(F)(c, e) = d$ where (d, r, s) is any splitting of the idempotent $F(e) : F(c) \to F(c)$, where if $e = \mathrm{id}_c$, we of course pick the trivial splitting $d = F(c), r = s = \mathrm{id}_c$. For $x : (c_1, e_1) \to (c_2, e_2)$ and splittings (d_j, r_j, s_j) of $F(e_j) : F(c_j) \to F(c_j)$ for $j = 1, 2$, $\mathsf{Idem}(x) := r_2 F(x) s_1 : d_1 \to d_2$. One checks that $\mathsf{Idem}(F)$ is a well-defined functor such that $\mathsf{Idem}(F) \circ \iota = F$.

Suppose now $\alpha : F \Rightarrow G$ between functors $\mathcal{C} \to \mathcal{D}$. Suppose (c, e) is an idempotent, and suppose (d_F, r_F, s_F) and (d_G, r_G, s_G) are our chosen splittings of $F(e)$ and $G(e)$ respectively. Since $(c, e) \in \mathsf{Idem}(\mathcal{C})$ splits $e : (c, e) \to (c, e)$ as an idempotent, we get a natural transformation $\mathsf{Idem}(\alpha) : \mathsf{Idem}(F) \Rightarrow \mathsf{Idem}(G)$ by defining

$$\mathsf{Idem}(\alpha)_{(c,e)} := r_G \circ \alpha_c \circ s_F : d_F \to d_G. \tag{2.5.9}$$

Since $\mathsf{Idem}(\alpha)_{(c,\mathrm{id}_c)} = \alpha_c$, we have that $\iota^*(\mathsf{Idem}(\alpha)) = \alpha$, so ι^* is full.

If $\beta^1, \beta^2 : \mathsf{Idem}(F) \Rightarrow \mathsf{Idem}(G)$ with $\iota^*(\beta^1) = \iota^*(\beta^2)$, then $\beta^1_{(c,\mathrm{id}_c)} = \beta^2_{(c,\mathrm{id}_c)} : F(c) \to G(c)$ for all $c \in \mathcal{C}$. By naturality, (2.5.9) still holds replacing $\mathsf{Idem}(\alpha)$ with β^i and α_c with $\beta^i_{(c,\mathrm{id}_c)}$ for $i = 1, 2$. Thus $\beta^1 = \beta^2$ and ι^* is faithful. □

We omit the proof of the next corollary, which is similar to proof of Corollary 2.3.14 via the universal property.

Exercise 2.5.10. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a linear functor and $\mathcal{D}$ is idempotent complete. Prove that if F is fully faithful, then so is $\mathsf{Idem}(F) : \mathsf{Idem}(\mathcal{C}) \to \mathcal{D}$.

Corollary 2.5.11. *If $\mathcal{C}$ is idempotent complete, then $\mathcal{C}$ is equivalent to $\mathsf{Idem}(\mathcal{C})$.*

Similar to Remark 2.3.15, we will see Proposition 2.5.8 above can be used to extend Idem to a 2-functor.

Remark 2.5.12. Suppose $\mathcal{C}$ is a linear category. If (c, e) is an idempotent in $\mathcal{C}$, then $1 - e \in \mathrm{End}(c)$ is also idempotent. If (a, r_a, s_a) is a splitting for (c, e) and (b, r_b, s_b) is a splitting for $(c, 1 - e)$, then we see that c is canonically the direct sum $a \oplus b$ with the maps $\iota_a = s_a$, $\iota_b = s_b$, $\pi_a = r_a$, and $\pi_b = r_b$.

Proposition 2.5.13. *If $\mathcal{C}$ is a linear category that admits direct sums, then $\mathsf{Idem}(\mathcal{C})$ is also.*

Proof. The relation $x = f \circ x \circ e$ is preserved under linear combinations, so $\mathsf{Idem}(\mathcal{C})$ is linear. Suppose $c_1 = (a, e)$ and $c_2 = (b, f)$ are objects in $\mathsf{Idem}(\mathcal{C})$. Using the matrix notation (2.3.2) for morphisms between direct sums, it is easily verified that

$$c_1 \oplus c_2 = \left(a \oplus b, \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right)$$

is an idempotent, and the maps

$$\iota_1 := \begin{pmatrix} e \\ 0 \end{pmatrix} : c_1 \to c_1 \oplus c_2 \qquad \iota_2 := \begin{pmatrix} 0 \\ f \end{pmatrix} : c_2 \oplus c_2 \to c_1$$

$$\pi_1 := \begin{pmatrix} e & 0 \end{pmatrix} : c_1 \oplus c_2 \to c_2 \qquad \pi_2 := \begin{pmatrix} 0 & f \end{pmatrix} : c_1 \to c_1 \oplus c_2$$

witness $c_1 \oplus c_2$ as the direct sum. □

2.6. Cauchy complete linear categories.

Definition 2.6.1. A linear category $\mathcal{C}$ is called *Cauchy complete* if it admits all finite direct sums and it is idempotent complete.

Example 2.6.2. When S is a set, $\mathsf{Vec}(S)$ is Cauchy complete. If $V = \bigoplus_{s \in S} V_s$ and $W = \bigoplus_{s \in S} W_s$, then $V \oplus W = \bigoplus_{s \in S} V_s \oplus W_s$. If $e : V \to V$ is am idempotent, then $e = (e_s)_{s \in S}$ and each $e_s : V_s \to V_s$ is an idempotent. Split each e_s and then take the direct sum to obtain a splitting for e.

Construction 2.6.3. The *Cauchy completion* of a linear category $\mathcal{C}$ is $\cent(\mathcal{C}) := \mathsf{Idem}(\mathsf{Add}(\mathcal{C}))$. Observe that $c \mapsto (c, \mathrm{id}_c)$ gives a faithful linear functor $\mathcal{C} \hookrightarrow \cent(\mathcal{C})$.

Corollary 2.6.4. *$¢(\mathcal{C})$ is Cauchy complete.*

Proof. We know $¢(\mathcal{C})$ is idempotent complete by Proposition 2.5.7. By Proposition 2.5.13, $¢(\mathcal{C})$ also admits finite direct sums. □

Proposition 2.6.5. *For every Cauchy complete linear category $\mathcal{D}$, pre-composition with the canonical inclusion $\mathcal{C} \hookrightarrow ¢(\mathcal{C})$ gives an equivalence*

$$\mathsf{Fun}(¢(\mathcal{C}) \to \mathcal{D}) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathcal{D}).$$

Proof. Define $¢(F) := \mathsf{Idem}(\mathsf{Add}(F))$. We omit the rest of the proof, which is similar to the proofs of Propositions 2.3.10 and 2.5.8. □

Corollary 2.6.6. *If $\mathcal{C}$ is Cauchy complete, then $\mathcal{C}$ is equivalent to $¢(\mathcal{C})$.*

Similar to Remark 2.3.15, We will see Proposition 2.6.5 above can be used to extend $¢$ to a functor.

Exercise 2.6.7. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a linear functor and $\mathcal{D}$ is Cauchy complete. Prove that if F is fully faithful, then so is $¢(F) : ¢(\mathcal{C}) \to \mathcal{D}$.

Theorem 2.6.8. $¢(\mathrm{B}\mathbb{C}) \cong \mathsf{Vec}$.

Proof. We saw that $\mathsf{Add}(\mathrm{B}\mathbb{C})$ has objects finite tuples $(\star, \ldots, \star)$; write $[n]$ for the tuple of length n. Observe that $\mathrm{Hom}([n] \to [m]) = M_{m\times n}(\mathbb{C})$. Thus $\mathsf{Add}(\mathrm{B}\mathbb{C})$ can be identified with the subcategory of Vec whose objects are $\mathbb{C}^n$ for $n \geq 0$, with the convention that $\mathbb{C}^0 = 0$. This subcategory is clearly equivalent to Vec. Since Vec is already idempotent complete, the result follows. □

Exercise 2.6.9. Find an example of a linear category $\mathcal{C}$ such that $\mathsf{Add}(\mathsf{Idem}(\mathcal{C}))$ is not equivalent to $\mathsf{Idem}(\mathsf{Add}(\mathcal{C}))$.
Hint: Try $\mathrm{B}A$ for an algebra A without non-trivial idempotents with projective modules which are not free, e.g., $C(S^2)$.

Proposition 2.6.10. *When $\mathcal{C}$ is linear and $\mathcal{D}$ is Cauchy complete, $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ is Cauchy complete.*

Proof. For linear functors $F, G : \mathcal{C} \to \mathcal{D}$, it is straightforward to verify that setting $(F \oplus G)(c) := F(c) \oplus G(c)$, and similarly for morphisms, gives a direct sum of linear functors.

Suppose now $\pi : F \Rightarrow F$ is an idempotent. Then $\pi_c : F(c) \to F(c)$ is an idempotent in $\mathcal{D}$ for all $c \in \mathcal{C}$. Choose any splitting (a_c, r_c, s_c) for $(F(c), \pi_c)$ in $\mathcal{D}$, and define $A : \mathcal{C} \to \mathcal{D}$ by $A(c) := a_c$ and $A(f : c \to d) := r_d \circ F(f) \circ s_c : a_c \to a_d$. Since $r_c \circ s_c = \mathrm{id}_c$ for all $c \in \mathcal{C}$, A is a functor. Observe that $r = (r_c)_{c\in\mathcal{C}}$ is a natural transformation $F \Rightarrow A$, and similarly and $s = (s_c)_{c\in\mathcal{C}} : A \Rightarrow F$ is a natural transformation such that $r \circ s = \mathrm{id}_A$. Since $s_c \circ r_c = \pi_c$ for all $c \in \mathcal{C}$, we see that $s \circ r = \pi : F \Rightarrow F$ as desired. □

2.7. **Direct sum and Deligne product of linear categories.** We now define the direct sum of two linear categories.

Definition 2.7.1. Given two linear categories $\mathcal{C}, \mathcal{D}$, we define the category $\mathcal{C} \boxplus \mathcal{D}$ whose objects are formal direct sums $c \boxplus d$ with $c \in \mathcal{C}$ and $d \in \mathcal{D}$ and whose morphisms are given by

$$\mathrm{Hom}(c_1 \boxplus d_1 \to c_2 \boxplus d_2) := \mathcal{C}(c_1 \to c_2) \oplus \mathcal{D}(d_1 \to d_2)$$

where the direct sum on the right hand side is the direct sum of vector spaces.

Example 2.7.2. For algebras A, B, every representation of $A \oplus B$ decomposes canonically as a direct sum of representations, giving a canonical equivalence $\mathsf{Rep}(A \oplus B) \cong \mathsf{Rep}(A) \boxplus \mathsf{Rep}(B)$.

Example 2.7.3. $\mathsf{Vec}(S) \cong \mathsf{Vec}^{\boxplus |S|}$.

Exercise 2.7.4. Show that if $\mathcal{C}, \mathcal{D}$ admit all direct sums, are idempotent complete, or are Cauchy complete, then so is $\mathcal{C} \boxplus \mathcal{D}$ respectively.

TODO: universal property, contractible space

Definition 2.7.5. Given two linear categories $\mathcal{C}, \mathcal{D}$, their *Deligne product* is the linear category $\mathcal{C} \boxtimes \mathcal{D}$ whose objects are formal tensors $c \boxtimes d$ with $c \in \mathcal{C}$ and $d \in \mathcal{D}$ and whose morphisms are given by

$$\mathrm{Hom}(c_1 \boxtimes d_1 \to c_2 \boxtimes d_2) := \mathcal{C}(c_1 \to c_2) \otimes \mathcal{D}(d_1 \to d_2)$$

where the tensor product on the right hand side is the tensor product of vector spaces.

Now when $\mathcal{C}, \mathcal{D}$ admit direct sums, are idempotent complete, or are Cauchy complete, it may not be the case that $\mathcal{C} \boxtimes \mathcal{D}$ is as well. We thus *redefine* the Deligne product for these categories as follows: When $\mathcal{C}, \mathcal{D}$:

- admit direct sums, we define $\mathcal{C} \boxtimes_{\mathsf{Add}} \mathcal{D} := \mathsf{Add}(\mathcal{C} \boxtimes \mathcal{D})$.
- are idempotent complete, we define $\mathcal{C} \boxtimes_{\mathsf{Idem}} \mathcal{D} := \mathsf{Idem}(\mathcal{C} \boxtimes \mathcal{D})$.
- are Cauchy complete, we define $\mathcal{C} \boxtimes_{\cent} \mathcal{D} := \cent(\mathcal{C} \boxtimes \mathcal{D})$.

We will usually omit the decoration from the notation on the left hand side and simply use $\boxtimes$ for this redefined Deligne product when working with such linear categories.

TODO: universal property, contractible space

2.8. Right modules over semisimple algebras.

For this section, A is a finite dimensional complex semisimple algebra. Recall that by the Artin-Wederburn Theorem, A is a direct sum of matrix algebras

$$A = \bigoplus_{j=1}^{n} M_{a_j}(\mathbb{C}).$$

Lemma 2.8.1. *Every corner of a semisimple algebra (eAe where $e \in A$ is an idempotent) is semisimple.*

Proof. By taking direct sums, it suffices to consider the case of an idempotent $e \in M_n(\mathbb{C})$. Since $eM_n(\mathbb{C})e = \mathrm{End}(e\mathbb{C}^n) \cong M_{\mathrm{rank}(e)}(\mathbb{C})$, the result follows. □

Suppose $M_n(\mathbb{C})$ acts on a finite dimensional vector space V. Then $V \cong e_{11}V \otimes \mathbb{C}^n$ as A-representations where $M_n(\mathbb{C})$ acts by left multiplication on $\mathbb{C}^n$, e_{11} is the matrix with a one in the $(1,1)$ entry and zeroes elsewhere, and $e_{11}V$ is a *multiplicity space* which carries the trivial $M_n(\mathbb{C})$-action. Indeed, if $\{\xi_i\}$ is a basis for $e_{11}V$ and $\{\epsilon_j\}$ is the standard basis for $\mathbb{C}^n$, then the map $f : \xi_i \otimes \epsilon_j \mapsto e_{ji}\xi_i$ is a linear isomorphism $e_{11}V \otimes \mathbb{C}^n \to V$ which intertwines the $M_n(\mathbb{C})$-actions:

$$e_{k\ell} \cdot f(\xi_i \otimes \epsilon_j) = e_{k\ell}e_{ji}\xi_i = \delta_{\ell=j}e_{ki}\xi_i = \delta_{\ell=j}f(\xi_i \otimes \epsilon_k) = f(\xi_i \otimes e_{k\ell}\epsilon_j) = f(e_{k\ell}(\xi_i \otimes e_{k\ell}\epsilon_j)).$$

We record the above result as the following proposition.

Proposition 2.8.2. $\mathsf{Rep}(M_n(\mathbb{C})) \cong \mathsf{Vec}$.

Corollary 2.8.3. *Suppose A is a finite dimensional complex semisimple algebra with n simple summands. The category $\mathsf{Rep}(A)$ of finite dimensional complex A-representations is equivalent to $\mathsf{Vec}^{\boxplus n}$.*

Proof. Induct on n using Example 2.7.2 and Proposition 2.8.2. $\square$

Similar to $\mathsf{Rep}(A)$, which can also be called the category of *left A-modules*, we can define the category of right A-modules.

Definition 2.8.4. A *right A-module* is a vector space $M \in \mathsf{Vec}$ with a right action of A, i.e., a unital algebra homomorphism $\rho : A^{\mathrm{op}} \to \mathrm{End}(M)$. Here, A^{op} is the *opposite algebra* with multiplication $a \cdot b := ba$. Usually we drop the symbol ρ and write $ma = \rho(a)m$.

We denote the category of right A-modules by $\mathsf{Mod}(A)$.

Since every finite dimensional complex semisimple algebra is isomorphic to its opposite algebra (transpose is such an algebra isomorphism), there is an equivalence of categories $\mathsf{Rep}(A) \cong \mathsf{Mod}(A)$.

Proposition 2.8.5. *Suppose A is a finite dimensional complex semisimple algebra. There is a canonical equivalence $\text{¢}(\mathrm{B}A) \cong \mathsf{Mod}(A)$.*

Proof. This follows immediately from the fact that $\mathsf{Mod}(A) \cong \mathsf{Vec}^{\boxplus n}$ is Cauchy complete. Indeed, we see that $A_A \in \mathsf{Mod}(A)$ has a non-zero component in each of the n components of $\mathsf{Mod}(A) \cong \mathsf{Vec}^{\boxplus n}$. Since any non-zero vector space V satisfies $\text{¢}\{V\} \cong \mathsf{Vec}$, we see that for the full subcategory $\{A_A\} \subset \mathsf{Mod}(A)$, $\text{¢}(\mathrm{B}A) \cong \text{¢}\{A_A\} \cong \mathsf{Mod}(A)$. $\square$

Corollary 2.8.6. *Every finite dimensional right A-module is* projective. *That is, given a right A-module M, there is an $n \in \mathbb{N}$ and an idempotent $e \in M_k(A)$ such that $M \cong eA^k$ as right A-modules.*

Proof. Unpacking Proposition 2.8.5, every right A-module is isomorphic to the image of a projection e in some endomorphism algebra of A^k for some k, i.e., $M \cong eA^k$ as right A-modules. $\square$

2.9. Representable functors and the Yoneda embedding.

For this section, $\mathcal{C}$ is a linear category. Every $c \in \mathcal{C}$ gives two *representable* functors:

$$\mathcal{C}(c \to -) : \mathcal{C} \to \mathsf{Vec} \qquad \text{and} \qquad \mathcal{C}(- \to c) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}.$$

We will focus on functors of the second type in this section, and the functors of the first type can be treated similarly.

Lemma 2.9.1 (Yoneda)**.** *Let $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}$ be a linear functor. For each $c \in \mathcal{C}$, the map*

$$\begin{aligned} \text{よ}(a, F) : \mathrm{Hom}(\mathcal{C}(- \to a) \Rightarrow F) &\longrightarrow F(a) \\ \rho &\longmapsto \rho_a(\mathrm{id}_a) \end{aligned}$$

is an isomorphism which is natural in both c and F when both are considered as bilinear functors $\mathcal{C}^{\mathrm{op}} \times \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}) \to \mathsf{Vec}$.

Proof. Observe that $\rho_a : \mathcal{C}(a \to a) \to F(a)$ is a linear map which can be evaluated at id_a, so the above map is well-defined. Since ρ is natural, we see that for every other $b \in \mathcal{C}$ and every $f \in \mathcal{C}(b \to a)$, the following square commutes.

$$\begin{array}{ccc} \mathcal{C}(a \to a) & \xrightarrow{-\circ_{\mathcal{C}} f} & \mathcal{C}(b \to a) \\ \downarrow{\scriptstyle \rho_a} & & \downarrow{\scriptstyle \rho_b} \\ F(a) & \xrightarrow{F(f)} & F(b) \end{array}$$

This means that $F(f) \circ \rho_a = \rho_b \circ (- \circ f)$. Evaluating at id_a, we see $\rho_b(f) = F(f)(\rho_a(\mathrm{id}_a))$, so ρ_b is completely determined by $\rho_a(\mathrm{id}_a)$. Conversely, given an element $x \in F(a)$, we can define a natural transformation by $\rho_b(f) \coloneqq F(f)(x)$, establishing the isomorphism.

To see naturality in a, for all $f \in \mathcal{C}(b \to a)$, we must show that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{C}(- \to a) \Rightarrow F) & \xrightarrow{-\circ_{\mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})}(f \circ_{\mathcal{C}} -)} & \mathrm{Hom}(\mathcal{C}(- \to b) \Rightarrow F) \\ \downarrow{\scriptstyle \text{よ}(a,F)} & & \downarrow{\scriptstyle \text{よ}(b,F)} \\ F(a) & \xrightarrow{\qquad F(f) \qquad} & F(b) \end{array}$$

Given $\rho : \mathcal{C}(- \to a) \Rightarrow F$, the map right and then down is

$$(\rho \cdot (f \circ_{\mathcal{C}} -))_b(\mathrm{id}_b) = \rho_b((f \circ_{\mathcal{C}} -)_b(\mathrm{id}_b)) = \rho_b(f \circ \mathrm{id}_b) = \rho_b(f),$$

which is exactly the map down and then right by the first part of the argument.

To see naturality in F, we must show the following diagram commutes for all $\sigma : F \Rightarrow G$.

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{C}(- \to a) \Rightarrow F) & \xrightarrow{\sigma \circ_{\mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})} -} & \mathrm{Hom}(\mathcal{C}(- \to a) \Rightarrow G) \\ \downarrow{\scriptstyle \text{よ}(a,F)} & & \downarrow{\scriptstyle \text{よ}(a,G)} \\ F(a) & \xrightarrow{\qquad \sigma_a \qquad} & G(a) \end{array}$$

Given $\rho : \mathcal{C}(- \to a) \Rightarrow F$, the map right and then down is

$$(\sigma \cdot \rho)_a(\mathrm{id}_a) = \sigma_a(\rho_a(\mathrm{id}_a)),$$

which is exactly the map down and then right. ∎

Exercise 2.9.2. Write down the statement and the proof of the Yoneda Lemma 2.9.1 on your own without any references. Repeat this as many times as necessary until you truly understand it.

Corollary 2.9.3. *The* Yoneda embedding *functor*

$$\text{よ} : \mathcal{C} \hookrightarrow \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})$$
$$c \mapsto \mathcal{C}(- \to c)$$

is fully faithful.

Proof. The Yoneda Lemma 2.9.1 states that $\mathrm{Hom}(\mathcal{C}(- \to a) \Rightarrow \mathcal{C}(- \to b))$ is canonically isomorphic to $\mathcal{C}(a \to b)$. ∎

Exercise 2.9.4. Let S be a finite set. Show that the Yoneda embedding $よ : \mathsf{Vec}(S) \hookrightarrow \mathsf{Fun}(\mathsf{Vec}(S)^{\mathrm{op}} \to \mathsf{Vec})$ is an equivalence of linear categories.

Definition 2.9.5. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}$ is called *representable* if there is an $a \in \mathcal{C}$ and a natural isomorphism $\alpha : F \Rightarrow \mathcal{C}(- \to a)$. We call (a, α) a representing pair for F.

Remark 2.9.6. Representing pairs for F form a contractible space (when they exist). Indeed, given (a, α) and (b, β) for F, we get canonical natural isomorphisms

$$\mathcal{C}(- \Rightarrow a) \overset{\alpha^{-1}}{\Longrightarrow} F \overset{\beta}{\Longrightarrow} \mathcal{C}(- \to b) \qquad \text{and} \qquad \mathcal{C}(- \Rightarrow b) \overset{\beta^{-1}}{\Longrightarrow} F \overset{\alpha}{\Longrightarrow} \mathcal{C}(- \to a).$$

By the Yoneda embedding, these natural isomorphisms must come from $\mathcal{C}(a \to b)$, i.e., there is an isomorphism $f : a \to b$ such that $\beta \circ \alpha^{-1}$ is postcomposition with f. Plugging a into the left hand side above, we see that $f \in \mathcal{C}(a \to b)$ is exactly equal to $(\beta_a \circ \alpha_a^{-1})(\mathrm{id}_a)$.

We conclude the object representing a representable functor is uniquely determined (up to a contractible space).

2.10. **Semisimplicity and 2-vector spaces.** In this section, $\mathcal{C}$ is a Cauchy complete linear category.

Definition 2.10.1. Let $\mathcal{C}$ be a Cauchy complete linear category. An object $c \in \mathcal{C}$ is called *simple* if $\mathrm{End}_{\mathcal{C}}(c) = \mathbb{C}\,\mathrm{id}_c$. Two simple objects $a, b \in \mathcal{C}$ are called *distinct* if $\mathcal{C}(a \to b) = (0)$ and $\mathcal{C}(b \to a) = (0)$.

A Cauchy complete linear category $\mathcal{C}$ is called *semisimple* ([BW96, Adapted from Def. 2.10], see also [Müg03]) if there is a set $\mathrm{Irr}(\mathcal{C})$ of pairwise distinct simple objects such that for any $a, b \in \mathcal{C}$, the composition map

$$\bigoplus_{s \in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \to s) \otimes_{\mathbb{C}} \mathcal{C}(s \to b) \longrightarrow \mathcal{C}(a \to b) \tag{2.10.2}$$

is an isomorphism. (The direct sum in (2.10.2) is the direct sum in Vec.)

If $\mathrm{Irr}(\mathcal{C})$ can be chosen to be finite, then $\mathcal{C}$ is called *finite semisimple* or a *2-vector space*.

Example 2.10.3. For a set S, the category $\mathcal{C} = \mathsf{Vec}(S)$ is semisimple with $\mathrm{Irr}(\mathcal{C}) = \{\mathbb{C}_s | s \in S\}$. In particular, $\mathsf{Rep}(A)$ is semisimple when A is a finite dimensional complex semisimple algebra by Corollary 2.8.3.

Remark 2.10.4. Simplicity of objects is defined differently in an abelian category, where kernels and cokernels are already assumed to exist. We include a short section on abelian categories at the end of this chapter.

Lemma 2.10.5 (Schur)**.** *Suppose $\mathcal{C}$ is semisimple. Any two simple objects are either isomorphic or distinct.*

Proof. It suffices to prove that if $a \in \mathcal{C}$ is simple, then there is exactly one $c \in \mathrm{Irr}(\mathcal{C})$ isomorphic to a, and a is distinct from every other $d \in \mathrm{Irr}(\mathcal{C})$. Since composition gives an isomorphism

$$\bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \to c) \otimes_{\mathbb{C}} \mathcal{C}(c \to a) \longrightarrow \mathcal{C}(a \to a) \cong \mathbb{C},$$

we can conclude:

- there is exactly one $c \in \mathrm{Irr}(\mathcal{C})$ such that both $\mathcal{C}(a \to c)$ and $\mathcal{C}(c \to a)$ are nonzero,

- $\mathcal{C}(a \to c) \cong \mathbb{C}$ and $\mathcal{C}(c \to a) \cong \mathbb{C}$, and
- if $f : a \to c$ and $g : c \to a$ are non-zero, then $g \circ f \neq 0$.

Choose $f : a \to c$ and $g : c \to a$ such that $g \circ f = \mathrm{id}_a$. Let $\lambda \in \mathbb{C}$ such that $f \circ g = \lambda \, \mathrm{id}_c$. Precompose both sides with g to obtain $g = \mathrm{id}_a \circ g = g \circ f \circ g = \lambda g$, so $\lambda = 1$ and $a \cong c$.

It remains to prove that for all other $d \in \mathrm{Irr}(\mathcal{C})$ with $d \neq c$, $\mathcal{C}(a \to d) = 0$ and $\mathcal{C}(d \to a) = 0$. For all $h \in \mathcal{C}(a \to d)$, observe that

$$h = h \circ \mathrm{id}_a = h \circ (g \circ f) = \underbrace{(h \circ g)}_{\in \mathcal{C}(c \to d) = 0} \circ f = 0.$$

Similarly, $\mathcal{C}(d \to a) = 0$. □

Facts 2.10.6. We gather a list of elementary properties about a semisimple category $\mathcal{C}$.

(EP1) Let $c \in \mathcal{C}$. By (2.10.2), for each $s \in \mathrm{Irr}(\mathcal{C})$ there are finite sets $\{\lambda_i^s\}_{i=1}^{n_s} \subset \mathcal{C}(s \to c)$ and $\{\rho_i^s\}_{i=1}^{n_s} \subset \mathcal{C}(c \to s)$ such that $\mathrm{id}_c = \sum_{s \in \mathrm{Irr}(\mathcal{C})} \sum_{i=1}^{n_s} \lambda_i^s \circ \rho_i^s$.

(EP2) Since the simples in $\mathrm{Irr}(\mathcal{C})$ are pairwise distinct, the morphisms $e_s := \sum_{i=1}^{n_s} \lambda_i^s \circ \rho_i^s$ for $s \in \mathrm{Irr}(\mathcal{C})$ satisfy $e_s \circ e_t = 0$ when $s \neq t$. Indeed,

$$e_s \circ e_t = \left(\sum_{i=1}^{n_s} \lambda_i^s \circ \rho_i^s \right) \circ \left(\sum_{j=1}^{n_t} \lambda_j^t \circ \rho_j^t \right) = \sum_{i=1}^{n_s} \sum_{j=1}^{n_t} \lambda_i^s \circ \underbrace{\rho_i^s \circ \lambda_j^t}_{\in \mathcal{C}(t \to s) = 0} \circ \rho_j^t = 0.$$

(EP3) Since the left hand side of (2.10.2) is a direct sum and

$$\sum_s e_s = \mathrm{id}_c = \mathrm{id}_c \circ \mathrm{id}_c = \left(\sum_s e_s \right) \circ \left(\sum_t e_t \right) = \sum_{s,t} e_s \circ e_t = \sum_s e_s \circ e_s,$$

each e_s is an idempotent. It is straightforward to show that the idempotents e_s are independent of the choice of $\{\lambda_i^s\}_{i=1}^{n_s} \subset \mathcal{C}(s \to c)$ and $\{\rho_i^s\}_{i=1}^{n_s} \subset \mathcal{C}(c \to s)$. Splitting e_s gives an object in $\mathcal{C}$ called the *s-isotypic component* of c.

(EP4) If $f \in \mathcal{C}(c \to s)$ with $s \in \mathrm{Irr}(\mathcal{C})$, then

$$f = f \circ \mathrm{id}_c = \left(\sum_{t \in \mathrm{Irr}(\mathcal{C})} \sum_{i=1}^{n_t} \lambda_i^t \circ \rho_i^t \right) = \sum_{t \in \mathrm{Irr}(\mathcal{C})} \sum_{i=1}^{n_t} \underbrace{f \circ \lambda_i^t}_{\in \mathcal{C}(t \to s)} \circ \rho_i^t = f \circ e_s,$$

and similarly, if $g \in \mathcal{C}(s \to c)$, then $g = e_s \circ g$.

(EP5) If $f \in \mathcal{C}(c \to c)$, then $e_s \circ f = f \circ e_s$, so each e_s is a central idempotent. Indeed, expanding $f = \sum_{t \in \mathrm{Irr}(\mathcal{C})} \sum_{i=1}^{m_t} g_i^t \circ h_i^t$ with $\{g_i^s\}_{i=1}^{n_t} \subset \mathcal{C}(t \to c)$ and $\{h_i^s\}_{i=1}^{n_t} \subset \mathcal{C}(c \to t)$, we have

$$f \circ e_s = \sum_{t \in \mathrm{Irr}(\mathcal{C})} \sum_{i=1}^{m_t} g_i^t \circ h_i^t \circ e_s = \sum_{t \in \mathrm{Irr}(\mathcal{C})} \sum_{i=1}^{m_t} g_i^t \circ h_i^t \circ e_t \circ e_s = \sum_{i=1}^{m_s} g_i^s \circ h_i^s,$$

and similarly for $e_s \circ f$. We call $e_s \circ f = f \circ e_s$ the *s-isotypic component of f*.

(EP6) Suppose $c, s \in \mathcal{C}$ with $s \in \mathrm{Irr}(\mathcal{C})$ simple. For every non-zero $g \in \mathcal{C}(s \to c)$, there is an $f \in \mathcal{C}(c \to s)$ such that $f \circ g = \mathrm{id}_s$, and for every non-zero $f \in \mathcal{C}(c \to s)$, there is an $g \in \mathcal{C}(s \to c)$ such that $f \circ g = \mathrm{id}_s$, Indeed, since $f \circ e_s = f$ for all $f \in \mathcal{C}(c \to s)$ and $e_s \circ g = g$ for all $g \in \mathcal{C}(s \to c)$, expanding $e_s = \sum_{i=1}^{n_s} \lambda_i^s \circ \rho_i^s$ with $\{\rho_i^s\}_{i=1}^{n_t} \subset \mathcal{C}(c \to s)$ and $\{\lambda_i^s\}_{i=1}^{n_s} \subset \mathcal{C}(s \to c)$ yields this result.

This elementary property has the following two immediate consequences.

(EP7) For any non-zero $f \in \mathcal{C}(s \to b)$ and $g \in \mathcal{C}(a \to s)$, $f \circ g \neq 0$.

(EP8) $\mathcal{C}(c \to s) \neq 0$ if and only if $\mathcal{C}(s \to c) \neq 0$.

(EP9) For each $c \in \mathcal{C}$, (2.10.2) and the last elementary property implies that $\mathcal{C}(s \to c) \neq 0$ for only finitely many $s \in \mathcal{C}$. Hence the object $\bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to c) \otimes s$ is well-defined.

Lemma 2.10.7. *Suppose $\mathcal{C}$ is semisimple and $c \in \mathcal{C}$. There is a canonical isomorphism $\upsilon_c : c \cong \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to c) \otimes s$. Moreover, for all $f \in \mathcal{C}(a \to b)$, the following diagram commutes.*

$$\begin{array}{ccc} a & \xrightarrow{\upsilon_a} & \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to a) \otimes s \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (f\circ -)\otimes \mathrm{id}_s} \\ b & \xrightarrow{\upsilon_b} & \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to b) \otimes s \end{array} \tag{2.10.8}$$

Proof. Since the composition map is natural (2.10.2) is natural in a, we get the following isomorphism between representable functors:

$$\begin{aligned} \mathcal{C}\left(a \to \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to c) \otimes s\right) &\cong \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}\left(a \to \mathcal{C}(s \to c) \otimes s\right) && \text{(Ex. 2.3.5)} \\ &\cong \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \to s) \otimes \mathcal{C}(s \to c) && \text{(2.4.7)} \\ &\cong \mathcal{C}(a \to c) && \text{(2.10.2)}. \end{aligned}$$

Now the Yoneda Lemma 2.9.1 and Remark 2.9.6 gives the desired isomorphism υ_c. The final claim follows either from naturality of the Yoneda isomorphism or from naturality of the composition map (2.10.2) in b. □

Corollary 2.10.9. *A semisimple category $\mathcal{C}$ with distinguished set of simples $S = \mathrm{Irr}(\mathcal{C})$ is canonically equivalent to $\mathsf{Vec}(S)$ via the functor $\mathsf{Vec}(S) \to \mathcal{C}$ given by $\bigoplus_{s\in S} V_s \mapsto \bigoplus_{s\in S} V_s \otimes s$.*

Proof. Essential surjectivity follows the canonical isomorphism $\upsilon_c : c \cong \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to c) \otimes s$. Fully faithful follows from (2.10.8). □

When $\mathcal{C}$ is finite semisimple, the above corollary says that every 2-vector space has a basis, in that every object can be written as a linear Vec-combination of simples, and morphisms just move around the multiplicity spaces. The next proposition shows that functors out of 2-vector spaces are completely determined by where they send a basis.

Proposition 2.10.10. *If $\mathcal{C}$ is semisimple and $\mathcal{D}$ is linear, then every linear functor $F : \mathcal{C} \to \mathcal{D}$ is completely determined up to unique natural isomorphism by where it sends simple objects via the commutative diagram*

$$\begin{array}{ccc} F(a) & \xrightarrow{F(\upsilon_a)} & \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to a) \otimes F(s) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle (f\circ -)\otimes \mathrm{id}_{F(s)}} \\ F(b) & \xrightarrow{F(\upsilon_b)} & \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to b) \otimes F(s) \end{array} \qquad \forall f : a \to b. \tag{2.10.11}$$

Moreover, every natural transformation $\rho : F \Rightarrow G$ is completely determined by $\{\rho_s\}_{s\in\mathrm{Irr}(\mathcal{C})}$. In other words, we have an equivalence of categories

$$\mathsf{Fun}(\mathcal{C} \to \mathcal{D}) \cong \mathsf{Fun}(\mathcal{C}_0 \to \mathcal{D})$$

where $\mathcal{C}_0$ is the full subcategory of $\mathcal{C}$ whose objects are $\mathrm{Irr}(\mathcal{C})$.

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ is linear. Then for each $c \in \mathcal{C}$, combining that linear functors preserve direct sums, the canonical isomorphism and Lemma 2.10.7, we have a canonical isomorphism

$$\begin{aligned} F(c) &\xrightarrow{F(\upsilon_c)} F\left(\bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s\to c)\otimes s\right) && \text{(Lem. 2.10.7)}\\ &\cong \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} F(\mathcal{C}(s\to c)\otimes s) && \text{(Ex. 2.3.5)}\\ &\xrightarrow{\bigoplus_s \mu_{\mathcal{C}(s\to c),s}} \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s\to c)\otimes F(s) && (2.4.7). \end{aligned}$$

By Lemma 2.10.7, for every morphism $f \in \mathcal{C}(a \to b)$, (2.10.11) commutes. We conclude that F is completely determined by where it sends simples.

Now suppose $\rho : F \Rightarrow G$. Using the canonical isomorphism $\upsilon_c : c \to \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s\to c)\otimes s$ and natruality of ρ, the following diagram commutes.

$$\begin{array}{ccccc} F(c) & \xrightarrow{F(\upsilon_c)} & F\left(\bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s\to c)\otimes s\right) & \xrightarrow{\cong} & \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s\to c)\otimes F(s) \\ \downarrow{\scriptstyle \rho_c} & & \downarrow{\scriptstyle \rho_{\bigoplus_{s\in\mathrm{Irr}(\mathcal{C})}\mathcal{C}(s\to c)\otimes s}} & & \downarrow{\scriptstyle \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathrm{id}_{\mathcal{C}(s\to c)}\otimes\rho_s} \\ G(c) & \xrightarrow{G(\upsilon_c)} & G\left(\bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s\to c)\otimes s\right) & \xrightarrow{\cong} & \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s\to c)\otimes G(s) \end{array}$$

Thus ρ_c is completely determined by the ρ_s. □

Remark 2.10.12. By Corollary 2.10.9, when $\mathcal{C}$ is semisimple, $\mathcal{C} = ¢(\mathcal{C}_0)$ where $\mathcal{C}_0$ is the full subcategory of $\mathcal{C}$ whose objects are $\mathrm{Irr}(\mathcal{C})$. When $\mathcal{D}$ admits direct sums, we get a quicker proof of the above corollary by the universal property from Proposition 2.3.10 applied to the functor $F_0 = F|_{\mathcal{C}_0} : \mathcal{C}_0 \to \mathcal{D}$, as $F \cong \mathsf{Add}(F_0)$.

Corollary 2.10.13. *When $\mathcal{C}$ is finite semisimple, the Yoneda embedding よ $: \mathcal{C} \hookrightarrow \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})$ is an equivalence.*

Proof. We already know よ is fully faithful. Suppose $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}$ is a linear functor. By Proposition 2.10.10,

$$F \cong \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(-\to s)\otimes F(s) \cong \mathcal{C}(-\to \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} F(s)\otimes s),$$

which is clearly a representable functor. Thus よ is essentially surjective. □

We summarize the results above in the following theorem.

Theorem 2.10.14 (Fundamental Theorem of semisimple categories)**.** *Suppose $\mathcal{C}$ is a Cauchy complete linear category. The following conditions are equivalent.*

(SS1) $\mathcal{C}$ *is semisimple.*
(SS2) $\mathcal{C} \cong \mathsf{Vec}(S)$ *for some set* S *(cf. Example 2.2.4).*
(SS3) *Every* $c \in \mathcal{C}$ *is isomorphic to a direct sum of simples* $c \cong \bigoplus_{i=1}^{n} s_i^{\oplus m_i}$, *where the* $s_i \in \mathcal{C}$ *are mutually distinct simples.*
(SS4) *For every object* $c \in \mathcal{C}$, *the endomorphism algebra* $\mathcal{C}(c \to c)$ *is a finite dimensional complex semisimple algebra, i.e., a multimatrix algebra.*

Proof.
(SS1)⇒(SS2): This is exactly Corollary 2.10.9.
(SS2)⇒(SS4): Every $V = \bigoplus_{s\in S} V_s \in \mathsf{Vec}(S)$ is isomorphic to $\bigoplus_{s\in S} \mathbb{C}_s^{\oplus \dim(V_s)}$.
(SS3)⇒(SS4): Observe that

$$\mathrm{End}_{\mathcal{C}}\left(\bigoplus_{i=1}^{n} s_i^{\oplus m_i}\right) \cong \bigoplus_{i=1}^{n} \mathrm{End}(s_i^{\oplus m_i}) \cong \bigoplus_{i=1}^{n} M_{m_i}(\mathbb{C})$$

which is semisimple.
(SS4)⇒(SS1): We claim that if every endomorphism algebra is semisimple, then the result of Schur's Lemma 2.10.5 holds. That is, simples are either pairwise isomorphic or distinct. Indeed, if s, t are simples, then semisimplicity of

$$\mathrm{End}_{\mathcal{C}}(s \oplus t) \cong \begin{pmatrix} \mathcal{C}(s \to t) & \mathcal{C}(t \to s) \\ \mathcal{C}(s \to t) & \mathcal{C}(t \to t) \end{pmatrix} \cong \begin{pmatrix} \mathbb{C} & \mathcal{C}(t \to s) \\ \mathcal{C}(s \to t) & \mathbb{C} \end{pmatrix}$$

implies $\mathrm{End}_{\mathcal{C}}(s \oplus t)$ is either $M_2(\mathbb{C})$ or $\mathbb{C} \oplus \mathbb{C}$.

Now let $\mathrm{Irr}(\mathcal{C})$ be a set of representatives for the simple objects in $\mathcal{C}$ under the equivalence relation of isomorphism. Observe that the elements of $\mathrm{Irr}(\mathcal{C})$ are pairwise distinct. We first show we can split $\mathcal{C}(c \to c)$ over simples. For $c \in \mathcal{C}$, we have $\mathrm{End}_{\mathcal{C}}(c) = \bigoplus_{k=1}^{n} M_{m_k}(\mathbb{C})$. For each $k = 1, \dots, n$, let $(e_{ij}^k)_{i,j=1}^{m_j}$ be a system of matrix units. For each k, let (a_k, r_k, s_k) be a splitting of (c, e_{11}^k). Since $\mathrm{End}((c, e_{11}^k)) = e_{11}^k M_{m_k}(\mathbb{C}) e_{11}^k = \mathbb{C}e_{11}^k$, we see that a_k is simple, so without loss of generality, we may assume $a_k \in \mathrm{Irr}(\mathcal{C})$.

We claim that the composition map

$$\bigoplus_{k=1}^{n} \mathcal{C}(c \to a_k) \otimes \mathcal{C}(a_k \to c) \longrightarrow \mathcal{C}(c \to c)$$

is an isomorphism. Surjectivity follows by observing that

$$e_{ij}^k = e_{i1}^k e_{1j}^k = e_{i1}^k e_{1j}^k = e_{i1}^k s_k r_k e_{1j}^k,$$

and injectivity follows from observing that $\{r_k e_{1j}^k\}_{j=1}^{m_k} \subset \mathcal{C}(c \to a_k)$ and $\{e_{i1}^k s_k\}_{i=1}^{m_k} \subset \mathcal{C}(a_k \to c)$ are bases. We prove the first set is a basis and the second is similar. If $\sum_{j=1}^{m_k} \lambda_j r_k e_{1j}^k = 0$, then applying e_{ii}^k on the right shows each $\lambda_i = 0$. If $f : c \to a_k$ is an arbitrary morphism, then since $\mathrm{id}_{a_k} = r_k s_k = r_k e_{11}^k s_k$ and $\mathrm{id}_c = \sum_{\ell=1}^{n} \sum_{j=1}^{m_\ell} e_{jj}^\ell$, we have

$$f = \mathrm{id}_{a_k} \circ f \circ \mathrm{id}_c = r_k e_{11}^k s_k f \sum_{\ell=1}^{n} \sum_{j=1}^{m_\ell} e_{jj}^\ell = \sum_{j=1}^{m_k} r_k e_{11}^k s_k f e_{jj}^k \in \mathrm{span}\{r_k e_{1j}^k\}_{j=1}^{m_k}. \qquad \square$$

Corollary 2.10.15. *The following are equivalent for a Cauchy complete linear category.*
(fSS1) $\mathcal{C}$ *is finite semisimple.*

(fSS2) $\mathcal{C} \cong \mathsf{Vec}^{\oplus n}$ *for some* $n \geq 1$ *(cf. Example 2.7.3).*
(fSS3) $\mathcal{C} \cong \mathsf{Rep}(A)$ *for some finite dimensional complex semisimple algebra* A*.*

Proof.
(fSS2)⇔(fSS1): Clearly when S is finite, $\mathsf{Vec}(S) \cong \mathsf{Vec}^{|S|}$.
(fSS2)⇒(fSS3): If $\mathcal{C} \cong \mathsf{Vec}^{\oplus n}$, set $A = \mathbb{C}^n$ and note $\mathsf{Rep}(A) \cong \mathsf{Vec}^{\oplus n}$.
(fSS3)⇒(fSS2): This is exactly Corollary 2.8.3. □

Remark 2.10.16. Suppose $\mathcal{C}$ is a (finite) semisimple category and $X \in \mathcal{C}$ is any object such that $\mathcal{C}_{\mathcal{C}}(s \to X) \neq 0$ for all $s \in \operatorname{Irr}(\mathcal{C})$. Since $X \cong \bigoplus_{s \in \operatorname{Irr}(\mathcal{C})} \mathcal{C}(s \to X) \otimes s$, we see that $\operatorname{End}_{\mathcal{C}}(X) \cong \bigoplus_{s \in \operatorname{Irr}(\mathcal{C})} \operatorname{End}(\mathcal{C}(s \to X))$, which is a semisimple algebra whose simple summands correspond to simples $s \in \operatorname{Irr}(\mathcal{C})$. Thus $\mathsf{Mod}(\operatorname{End}_{\mathcal{C}}(X)) \cong \mathsf{Vec}(S) \cong \mathcal{C}$.

Corollary 2.10.17. *Suppose* $\mathcal{C}, \mathcal{D}$ *are semisimple and* $F : \mathcal{C} \to \mathcal{D}$ *is linear.*

1. *F is faithful if and only if no simple in $\mathcal{C}$ is sent to $0_{\mathcal{D}}$.*
2. *F is fully faithful if and only if it sends distinct simples to distinct simples.*
3. *F is essentially surjective if and only if each simple in $\mathcal{D}$ is isomorphic to the image of a simple object in $\mathcal{C}$.*

Proof. (1) is immediate from (2.10.11). Next, for simples $a, b \in \operatorname{Irr}(\mathcal{C})$, we have

$$\mathcal{C}(a \to b) \cong \mathcal{D}(F(a) \to F(b))$$

if and only if F maps distinct simples to distinct simples. Since F is completely determined by where it sends simples by (2.10.11), (2) follows. For the forward direction of (3), it suffices to prove that if $c \in \mathcal{C}$ such that $F(c)$ is simple, then c is simple, which is immediate from (2.10.11). The reverse direction follows from the fact that linear functors preserve direct sums together with (SS3). □

Corollary 2.10.18 (Rank-Nullity)**.** *Suppose* $\mathcal{C}$ *is finite semisimple and* $F : \mathcal{C} \to \mathcal{C}$ *is linear. The following are equivalent:*

(RN1) *F is fully faithful.*
(RN2) *F is essentially surjective.*
(RN3) *F is an equivalence.*

Proof. It suffices to prove that (RN1) is equivalent to (RN2).
(RN1)⇒(RN2): If F is fully faithful, then by Corollary 2.10.17, F maps distinct simples to distinct simples. By a counting argument, every simple is in the essential image of F, so F is essentially surjective by Corollary 2.10.17.
(RN2)⇒(RN1): If F is essentially surjective, then every simple is in the essential image of F. Suppose $c \in \mathcal{C}$ such that $F(c)$ is simple. Writing $c \cong \bigoplus_{s \in \operatorname{Irr}(\mathcal{C})} s^{\oplus m_s}$, since linear functors preserve direct sums, $F(c) \cong \bigoplus_{s \in \operatorname{Irr}(\mathcal{C})} F(s)^{\oplus m_s}$. But since c is simple, it must be the case that

TODO: □

Theorem 2.10.19 (Fundamental Theorem of pre-semisimple categories)**.** *Suppose* $\mathcal{C}$ *is a pre-semisimple category. The following conditions are equivalent.*

(pSS1) *$\mathcal{C}$ is semisimple.*
(pSS2) $\mathcal{C} \cong \mathsf{Vec}(S)$ *for some set* S*.*

(pSS3) $\mathcal{C}$ *is Cauchy complete*

If moreover $\mathcal{C}$ *is finite, then the above conditions are equivalent to:*

(pSS4) $\mathcal{C} \cong \mathsf{Vec}^{\boxplus n}$ *for some* $n \geq 1$.
(pSS5) $\mathcal{C} \cong \mathsf{Mod}(A)$ *for some semisimple finite dimensional complex algebra* A.

Proof.
(pSS1)⇔(pSS2): Since semisimple categories are Cauchy complete by Example 2.6.2 and Corollary 2.10.9, the result is immediate from (SS1)⇔(SS2) from Theorem 2.10.14.
(pSS1)⇔(pSS3): Immediate from the equivalence (SS1)⇔(SS4) from Theorem 2.10.14.
Now assume that $\mathcal{C}$ is finite.
(pSS2)⇔(pSS4): Obvious
(pSS4)⇔(pSS5): Since $\mathsf{Mod}(A)$ is Cauchy complete by Proposition 2.8.5, the result follows by (fSS2)⇔(fSS3) from Corollary 2.10.15. □

Corollary 2.10.20. *Suppose* $\mathcal{C}$ *is a finite pre-semisimple category. Then* $\cent(\mathcal{C})$ *is finite semisimple. In particular,*

- *the Yoneda embedding* $よ : \mathcal{C} \hookrightarrow \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})$ *extends to an equivalence* $\cent(よ) : \cent(\mathcal{C}) \to \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})$, *and*
- *there are* $a_1, \ldots, a_n \in \mathcal{C}$ *such that* $\cent(\mathcal{C}) \cong \mathsf{Mod}(L(a_1, \ldots, a_n))$.

Proof. Observe that every endomorphism algebra of $\cent(\mathcal{C})$ is a finite dimensional complex semisimple algebra by Lemma 2.8.1, so $\cent(\mathcal{C})$ is semisimple. Enumerate $\mathrm{Irr}(\cent(\mathcal{C})) = \{s_1, s_2, \ldots, \}$, and pick for each $s_i \in \mathrm{Irr}(\cent(\mathcal{C}))$ an $a_i \in \mathcal{C}$ such that $\cent(\mathcal{C})(s_i \to a_i) \neq 0$. (Why does such an $a_i \in \mathcal{C}$ exist?) We know that there is a global bound

$$\dim(Z(L(a_1, \ldots, a_k))) < K \qquad\qquad \forall\, 1 \leq k \leq |\,\mathrm{Irr}(\cent(\mathcal{C}))|.$$

Claim. $|\,\mathrm{Irr}(\cent(\mathcal{C}))| \leq K$.

Proof of Claim. Setting $X_k := \bigoplus_{i=1}^{k} a_i \in \cent(\mathcal{C})$, we have

$$L(a_1, \ldots, a_k) \cong \mathrm{End}_{\cent(\mathcal{C})}(X_k).$$

Since each s_i admits a non-zero map to X_k for $i = 1, \ldots, k$, decomposing X_k into simples in $\cent(\mathcal{C})$, we have $k \leq \dim Z(\mathrm{End}_{\cent(\mathcal{C})}(X_k)) < K$, and the claim follows. □

The first bullet point now follows from the universal property Proposition 2.6.5 of Cauchy completion together with Corollary 2.10.13. The second bullet point follows from Remark 2.10.16. In more detail, X_k for $k = |\,\mathrm{Irr}(\cent(\mathcal{C}))|$ satisfies the conditions in Remark 2.10.16 so that

$$\cent(\mathcal{C}) \cong \mathsf{Mod}(\mathrm{End}_{\cent(\mathcal{C})}(X_k)) \cong \mathsf{Mod}(L(a_1, \ldots, a_k)).$$ □

3. Unitary categories

TODO: lead in

3.1. **Unitary categories and linking algebras.** For this section, $\mathcal{C}$ is a linear category. A dagger category is basically a complex $*$-algebra with more than one object.

Definition 3.1.1. A *dagger structure* on a linear category $\mathcal{C}$ is a collection of anti-linear maps $\dagger : \mathcal{C}(a \to b) \to \mathcal{C}(b \to a)$ for all $a, b \in \mathcal{C}$ such that:

- For all $f \in \mathcal{C}(a \to b)$ and $g \in \mathcal{C}(b \to c)$, $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$, and
- For all $f \in \mathcal{C}(a \to b)$, $f^{\dagger\dagger} = f$.

Observe these conditions implies $\mathrm{id}_a^\dagger = \mathrm{id}_a$ for all $a \in \mathcal{C}$.

A *dagger category* is a linear category $\mathcal{C}$ equipped with a dagger structure. We write $\mathcal{C}^\natural$ be the underlying linear category of $\mathcal{C}$ where we have forgotten the dagger structure.

Definition 3.1.2. A morphism $u : a \to b$ in a dagger category is called:

- an *isometry* if $u^\dagger u = \mathrm{id}_a$,
- a *coisometry* if $uu^\dagger = \mathrm{id}_b$,
- a *unitary* if u is both an isometry and a cosiometry.

Definition 3.1.3. A $\dagger$-*functor* $\mathcal{C} \to \mathcal{D}$ is a $\dagger$-preserving linear functor, i.e., for $f \in \mathcal{C}(a \to b)$, $\mathcal{F}(f^\dagger) = \mathcal{F}(f)^\dagger$.

Two $\dagger$-categories $\mathcal{C}, \mathcal{D}$ are *equivalent* if there is an equivalence consisting of $\dagger$-functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and *unitary* natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$.

Exercise 3.1.4. Prove that a $\dagger$-functor $F : \mathcal{C} \to \mathcal{D}$ can be augmented to an equivalence if and only if F is fully faithful and *unitarily* essentially surjective, i.e., for every $d \in \mathcal{D}$, there is a $c \in \mathcal{C}$ and a unitary isomorphism $u : F(c) \to d$.

Definition 3.1.5. Given a category $\mathcal{C}$ and objects $a, b \in \mathcal{C}$, the linking algebra

$$L(a,b) = \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(b \to a) \\ \mathcal{C}(a \to b) & \mathcal{C}(b \to b) \end{pmatrix}$$

is a unital $*$-algebra where the $*$ is given by the dagger transpose operation. Similarly, the n-fold linking algebras $L(a_1, \ldots, a_n)$ are also unital $*$-algebras.

We now generalize the definition of unitary algebra (finite dimensional C*-algebra) to a category.

Definition 3.1.6. A *unitary category* is a dagger category $\mathcal{C}$ such that every linking algebra $L(a_1, \ldots, a_n)$ is a unitary algebra.

A unitary category is called *finite* if there is a global bound on the dimensions of the centers of all linking algebras. That is, there is a $K > 0$ such that $\dim(Z(L(a_1, \ldots, a_n))) < K$ for all $a_1, \ldots, a_n \in \mathcal{C}$.

Example 3.1.7. If A is a unitary algebra, its delooping $\mathrm{B}A$ is a finite unitary category.

Example 3.1.8. Let S be a set. The category $\mathsf{Hilb}(S)$ of finite dimensional S-graded Hilbert spaces and grading-preserving maps is a unitary category.

Example 3.1.9. For G a finite group, $\mathsf{Rep}^\dagger(G)$ is a unitary category.

Exercise 3.1.10. Show that if $\mathcal{C}, \mathcal{D}$ are dagger or unitary categories, then so is $\mathcal{C} \oplus \mathcal{D}$ respectively.

Remark 3.1.11. We will see later on that when $\mathcal{C}, \mathcal{D}$ are unitary categories with $\mathcal{C}$ finite, the category $\mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ of $\dagger$-functors and natural transformations is a unitary category where the dagger of $\eta : F \Rightarrow G$ is given by $\eta_c^\dagger := (\eta_c)^\dagger$ for all $c \in \mathcal{C}$. If $\mathcal{C}$ is not finite, we must restrict to *uniformly bounded* natural transformations to obtain a dagger category again, but the hom spaces $\mathrm{Hom}(F \Rightarrow G)$ will no longer be finite dimensional.

Definition 3.1.12 (Polar decomposition)**.** Suppose $\mathcal{C}$ is a unitary category and $f \in \mathcal{C}(a \to b)$. Then considering f as an off-diagonal morphism in the linking algebra $L(a, b)$, we may decompose $f = u|f|$ where $|f| \in \mathcal{C}(a \to a)$ is the unique positive square root of $f^\dagger f$ and $u \in \mathcal{C}(a \to b)$ is the unique partial isometry such that $u^\dagger u = \mathrm{supp}(|f|)$.

Lemma 3.1.13. *Suppose $\mathcal{C}$ is a unitary category. For $f \in \mathcal{C}(a \to b)$, the following are equivalent.*

(1) f has a left inverse.
(2) $f^\dagger f$ is invertible.
(3) In the polar decomposition $f = u|f|$, u is an isometry.

Dually, f has a right inverse if and only if $ff^\dagger$ is invertible if and only if u is a coisometry.

Proof. Consider the unitary linking algebra $L(a, b)$, which comes equipped with a canonical projection

$$p = \begin{pmatrix} \mathrm{id}_a & 0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(b \to a) \\ \mathcal{C}(a \to b) & \mathcal{C}(b \to b) \end{pmatrix} = L(a, b).$$

We may identify the morphisms $f, u, |f|$ in $\mathcal{C}$ respectively with the operators

$$\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} |f| & 0 \\ 0 & 0 \end{pmatrix}$$

in $L(a, b)$. We can reduce to the case $L(a, b) = M_j(\mathbb{C})$, as the case for a general unitary algebra follows immediately by considering tuples in $\bigoplus_{i=1}^k M_{j_i}(\mathbb{C})$.

Now write $j = n + m$ where $n = \dim(p\mathbb{C}^j)$ and $m = \dim((1 - p)\mathbb{C}^j)$. We identify f, u as operators $\mathbb{C}^n \cong p\mathbb{C}^j \to (1-p)\mathbb{C}^j \cong \mathbb{C}^m$, which are $m \times n$ matrices, and similarly $|f| \in M_n(\mathbb{C})$. We can thus think of $f = u|f|$ as the polar decomposition of an operator in $M_{m\times n}(\mathbb{C})$. The result is now a corollary from the last chapter. □

Corollary 3.1.14. *Suppose $\mathcal{C}$ is a unitary category. Objects $a, b \in \mathcal{C}$ are isomorphic if and only if they are unitarily isomorphic.*

Proof. Suppose $f : a \to b$ is an isomorphism. Let $f = u|f|$ be the polar decomposition. Since f is left invertible, $u^\dagger u = \mathrm{id}_a$ by Lemma 3.1.13. But since f is also right invertible, $uu^\dagger = \mathrm{id}_b$ by Lemma 3.1.13. Hence $u : a \to b$ is the desired unitary isomorphism. □

Using Exercise 3.1.4, we get the following immediate corollary.

Corollary 3.1.15. *Suppose $\mathcal{C}, \mathcal{D}$ are unitary categories. A $\dagger$-functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only the underlying linear functor of F is an equivalence. That is, essential surjectivity of F implies unitary essential surjectivity.*

3.2. **Orthogonal direct sums and multiplicity Hilbert spaces.** In this section, unless stated otherwise, $\mathcal{C}$ denotes a dagger category or unitary category.

Definition 3.2.1. Suppose $\mathcal{C}$ is a dagger category. Given $a, b \in \mathcal{C}$, an $a \oplus b \in \mathcal{C}$ equipped with isometries $v_a : a \to a \oplus b$ and $v_b : b \to a \oplus b$ is called the *orthogonal direct sum* of a and b if $v_a \circ v_a^\dagger + v_b \circ v_b^\dagger = \mathrm{id}_{a\oplus b}$, i.e., $(a \oplus b, v_a, v_b, v_a^\dagger, v_b^\dagger)$ is the direct sum of a and b.

Observe that the projections $v_a \circ v_a^\dagger, v_b \circ v_b^\dagger$ are mutually orthogonal, and the canonical isomorphism

$$\mathrm{End}_{\mathcal{C}}(a \oplus b) \cong \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(b \to a) \\ \mathcal{C}(a \to b) & \mathcal{C}(b \to b) \end{pmatrix}$$

is a †-isomorphism.

We say $\mathcal{C}$ *admits orthogonal direct sums* if this orthogonal $a \oplus b$ exists for all $a, b \in \mathcal{C}$, as does a zero object.

There is a similar uniqueness statement for orthogonal direct sums.

Exercise 3.2.2. Suppose $\mathcal{C}, \mathcal{D}$ are dagger categories. Show that orthogonal direct sums are preserved by all †-functors.

Exercise 3.2.3. Show that if $\mathcal{C}$ is a dagger category that admits orthogonal direct sums, then $\mathcal{C}$ is unitary if and only if every endomorphism algebra is unitary.

The following operator algebraic proof of the next proposition was worked out with Quan Chen, Brett Hungar, and Sean Sanford.

Proposition 3.2.4. *A unitary category $\mathcal{C}$ admits all direct sums if and only if it admits all orthogonal direct sums.*

Proof. The reverse direction is clear.

Let $(a \oplus b, \iota_a, \iota_b, \pi_a, \pi_b)$ be the data of a direct sum of a and b, i.e., $\pi_a \iota_a = \mathrm{id}_a$, $\pi_b \iota_b = \mathrm{id}_b$, and $\iota_a \pi_a + \iota_b \pi_b = \mathrm{id}_{a\oplus b}$. (Note this implies $\pi_a \iota_b = 0$ and $\pi_b \iota_a = 0$). By polar decomposition, we can write $\iota_a = v_a |\iota_a|$, where $|\iota_a| = \sqrt{\iota_a^\dagger \iota_a} : a \to a$ is a positive invertible operator and $v_a = \iota_a |\iota_a|^{-1} : a \to a \oplus b$ is an isometry by Lemma 3.1.13. Similarly, we can write $\pi_b^\dagger = v_b |\pi_b^\dagger|$, where $v_b = \pi_b^\dagger |\pi_b^\dagger|^{-1} : b \to a \oplus b$ is an isometry. Observe that $v_b^\dagger v_a = |\pi_b^\dagger|^{-1} \pi_b \iota_a |\iota_a|^{-1} = 0$, so $q := v_a v_a^\dagger + v_b v_b^\dagger$ is an orthogonal projection.

It remains to prove that $q = \mathrm{id}_{a\oplus b}$. Noting that $\pi_b \iota_a = 0$, we can write

$$\begin{aligned} q = v_a v_a^\dagger + v_b v_b^\dagger &= \mathrm{id}_{a\oplus b}(v_a v_a^\dagger + v_b v_b^\dagger)\,\mathrm{id}_{a\oplus b} \\ &= (\iota_a \pi_a + \iota_b \pi_b)(\iota_a (\iota_a^\dagger \iota_a)^{-1} \iota_a^\dagger + \pi_b^\dagger (\pi_b \pi_b^\dagger)^{-1} \pi_b)(\iota_a \pi_a + \iota_b \pi_b) \\ &= \underbrace{\iota_a \pi_a + \iota_b \pi_b}_{\mathrm{id}_{a\oplus b}} + \underbrace{\iota_a (\iota_a^\dagger \iota_a)^{-1} \iota_a^\dagger \iota_b \pi_b + \iota_a \pi_a \pi_b^\dagger (\pi_b \pi_b^\dagger)^{-1} \pi_b}_{=:n}. \end{aligned}$$

Since $n := q - \mathrm{id}_{a\oplus b}$ is self-adjoint and $n^2 = 0$, $n = 0$. □

Construction 3.2.5. When $\mathcal{C}$ is a dagger category, we define $\mathsf{Add}^\dagger(\mathcal{C})$ as the orthogonal direct sum completion, which is the category $\mathsf{Add}(\mathcal{C})$ with the additional dagger structure $(x_{ij})^\dagger := (x_{ji}^\dagger)$. The reader may verify that $\mathsf{Add}^\dagger(\mathcal{C})$ is again a dagger category which admits finite orthogonal direct sums and satisfies a universal property for †-functors similar to Proposition 2.3.10 into dagger categories which admit orthogonal direct sums.

Exercise 3.2.6. Prove that when $\mathcal{C}$ is unitary, so is $\mathsf{Add}^\dagger(\mathcal{C})$.

Remark 3.2.7. By construction, $\mathsf{Add}^\dagger(\mathcal{C})^\natural = \mathsf{Add}(\mathcal{C}^\natural)$.

Remark 3.2.8. Here is a second completely formal categorical proof of Proposition 3.2.4 noticed by Giovanni Ferrer and David Green. If $\mathcal{C}$ admits direct sums, then the inclusion $\mathcal{C} \hookrightarrow \mathsf{Add}(\mathcal{C})$ is an equivalence, so it is fully faithful and essentially surjective. Thus when $\mathcal{C}$ is unitary, the canonical inclusion $\mathcal{C} \hookrightarrow \mathsf{Add}^\dagger(\mathcal{C})$ is a fully faithful dagger functor whose underlying linear functor is essentially surjective. Thus $\mathcal{C}$ is equivalent to $\mathsf{Add}^\dagger(\mathcal{C})$ by Corollary 3.1.15 and thus admits orthogonal direct sums by Exercise 3.2.2.

Let $\mathcal{C}$ be a dagger category which admits orthogonal direct sums. Every $c \in \mathcal{C}$ gives a dagger functor $- \otimes c : \mathsf{Hilb} \to \mathcal{C}$ by choosing for each $H \in \mathsf{Hilb}$ an ordered orthonormal basis and arguing as above. Analogs of the above exercises show that $- \otimes - : \mathsf{Hilb} \times \mathcal{C} \to \mathcal{C}$ is a dagger functor, where the product category $\mathsf{Hilb} \times \mathcal{C}$ has the dagger $(f, g)^\dagger := (f^\dagger, g^\dagger)$. Finally, as †-functors preserve orthogonal direct sums, each †-functor $F : \mathcal{C} \to \mathcal{D}$ comes equipped with a canonical unitary isomorphism

$$\mu_{H,c} : F(H \otimes c) \longrightarrow H \otimes F(c). \tag{3.2.9}$$

3.3. **Projection completion and unitary Cauchy completion.** In this section $\mathcal{C}$ is a dagger category or a unitary category.

Definition 3.3.1. Suppose $\mathcal{C}$ is a dagger category. A(n) *(orthogonal) projection* in $\mathcal{C}$ is a pair (c, p) where $c \in \mathcal{C}$ and $p \in \mathcal{C}(c \to c)$ such that $p \circ p = p = p^\dagger$. An *orthogonal splitting* for a projection (c, p) is a pair (a, v) where $a \in \mathcal{C}$, $v \in \mathcal{C}(a \to c)$ is an isometry such that $v^\dagger \circ v = \mathrm{id}_a$ and $v \circ v^\dagger = p$. A dagger category $\mathcal{C}$ is called *projection complete* if every projection admits an orthogonal splitting.

Exercise 3.3.2. Suppose $(a, v_a), (b, v_b)$ are two orthogonal splittings of (c, p). Show that there is a unique unitary isomorphism $u : a \to b$ which is compatible with v_a, v_b.

Exercise 3.3.3. Suppose $\mathcal{C}$ is a unitary category and $p \in \mathcal{C}(c \to c)$ is an orthogonal projection. Suppose (a, r, s) is a splitting of p as an idempotent, and let $s = v|s|$ be the polar decomposition of s. Prove that (a, v) is an orthogonal splitting of p.

The proof of the following proposition was worked out with David Reutter and Jan Steinebrunner.

Proposition 3.3.4. *A unitary category $\mathcal{C}$ is idempotent complete if and only if it is projection complete.*

Proof. First, suppose $\mathcal{C}$ is idempotent complete and suppose $p \in \mathcal{C}(c \to c)$ is a projection. Suppose (a, r, s) is a splitting of p as an idempotent, i.e., $r \circ s = \mathrm{id}_a$ and $s \circ r = p$. Observe that $p = r^\dagger \circ s^\dagger$ and $\mathrm{id}_a = s^\dagger \circ r^\dagger$, so $s^\dagger s$ is invertible with inverse $rr^\dagger$.

Let $s = v|s|$ be the polar decomposition of s. Since the linking algebra $L(a, c)$ is unitary and s has a left inverse, $v = s|s|^{-1}$ is an isometry Lemma 3.1.13. Finally, we check

$$vv^\dagger = s|s|^{-2}s^\dagger = s(s^\dagger s)^{-1}s^\dagger = srr^\dagger s^\dagger = p.$$

Hence p splits as a projection.

Conversely, suppose $\mathcal{C}$ is projection complete and suppose $e \in \mathcal{C}(c \to c)$ is an idempotent. Letting $e = u|e|$ be the unique polar decomposition, since $\ker(u) = \ker(e)$ and $1 - e$ is an idempotent, $e(1-e) = 0 = u(1-e)$, so $u = ue$. Moreover, since $\mathrm{supp}(e) = u^\dagger u$, so $eu^\dagger u = e$.

As $u^\dagger u$ is an orthogonal projection, suppose (a, v) be an orthogonal splitting of $u^\dagger u$, so $v^\dagger v = \mathrm{id}_a$ and $vv^\dagger = u^\dagger u$. We claim that $(a, r := v^\dagger, s := ev)$ is a splitting of e. Indeed,

$$\begin{aligned} sr &= evv^\dagger = eu^\dagger u = e \\ rs &= v^\dagger ev = v^\dagger vv^\dagger ev = v^\dagger u^\dagger uev = v^\dagger u^\dagger uv = v^\dagger vv^\dagger v = \mathrm{id}_a \,. \end{aligned}$$

□

Exercise 3.3.5. Suppose $\mathcal{C}, \mathcal{D}$ are dagger categories and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a †-functor. Show that if the projection (c, p) admits an orthogonal splitting, then so does $(F(c), F(p))$.

Definition 3.3.6. When $\mathcal{C}$ is a dagger category, we define $\mathsf{Proj}(\mathcal{C})$ as the projection completion. The reader may verify that $\mathsf{Proj}(\mathcal{C})$ is again a dagger category which is projection complete and satisfies a universal property for †-functors into dagger categories similar to Proposition 2.5.8 which are projection complete. Moreover, we can extend Proj to a dagger functor.

Exercise 3.3.7. Prove that when $\mathcal{C}$ is unitary, so is $\mathsf{Proj}(\mathcal{C})$.

Corollary 3.3.8. *If $\mathcal{C}$ is a unitary category,* $\mathsf{Proj}(\mathcal{C})^\natural \cong \mathsf{Idem}(\mathcal{C}^\natural)$.

Proof. Since $\mathsf{Proj}(\mathcal{C})$ is projection complete, it is also idempotent complete by Proposition 3.3.4 and thus equivalent to $\mathsf{Idem}(\mathcal{C})$. Note that $\mathsf{Idem}(\mathcal{C})$ has no dagger structure, and is thus equal to $\mathsf{Idem}(\mathcal{C}^\natural)$. □

We omit the proof of the following corollary, which is similar to the proof of Proposition 2.5.13.

Corollary 3.3.9. *If $\mathcal{C}$ is a unitary category that admits orthogonal direct sums, then* $\mathsf{Proj}(\mathcal{C})$ *is also.*

Definition 3.3.10. A dagger category $\mathcal{C}$ is called *dagger Cauchy complete* if it admits orthogonal direct sums and it is projection complete. When $\mathcal{C}$ is unitary, we will say *unitarily Cauchy complete.*

Proposition 3.3.11. *A unitary category is unitarily Cauchy complete if and only if it is Cauchy complete.*

Proof. Immediate from Propositions 3.2.4 and 3.3.4. □

Construction 3.3.12. The *dagger Cauchy completion* of a dagger category $\mathcal{C}$ is $¢^\dagger(\mathcal{C}) := \mathsf{Proj}(\mathsf{Add}^\dagger(\mathcal{C}))$. Observe that $c \mapsto (c, \mathrm{id}_c)$ gives a faithful †-functor $\mathcal{C} \hookrightarrow ¢^\dagger(\mathcal{C})$. There is a similar universal property, and $¢^\dagger$ also extends to a dagger functor.

We omit the proof of the following corollary, which is similar to the proof of Corollary 2.6.4 using Corollary 3.3.9.

Corollary 3.3.13. $¢^\dagger(\mathcal{C})$ *is dagger Cauchy complete.*

Theorem 3.3.14. $¢^\dagger(\mathrm{B}\mathbb{C}) \cong \mathsf{Hilb}$.

Proof. As in the proof of Theorem 2.6.8, $\mathsf{Add}^\dagger(\mathrm{B}\mathbb{C})$ has objects $[n]$ for $n \geq 0$, $\mathrm{Hom}([n] \to [m]) = M_{m\times n}(\mathbb{C})$, and the dagger is conjugate transpose. We can thus identify $\mathsf{Add}^\dagger(\mathrm{B}\mathbb{C})$ with the dagger subcategory of Hilb whose objects are $\mathbb{C}^n$, which is clearly equivalent to Hilb. Since Hilb is already projection complete, we are finished. ☐

Corollary 3.3.15. *If $\mathcal{C}$ is a unitary category, $\cent^\dagger(\mathcal{C})^\natural \cong \cent(\mathcal{C}^\natural)$.*

Proof. Combine Remark 3.2.7 and Corollary 3.3.8 to obtain

$$\cent^\dagger(\mathcal{C})^\natural = \mathsf{Proj}(\mathsf{Add}^\dagger(\mathcal{C}))^\natural \underset{\text{(Cor. 3.3.8)}}{\cong} \mathsf{Idem}(\mathsf{Add}^\dagger(\mathcal{C})^\natural) \underset{\text{(Rem. 3.2.7)}}{\cong} \mathsf{Idem}(\mathsf{Add}^\dagger(\mathcal{C}^\natural)) = \cent(\mathcal{C}^\natural).$$ ☐

We end this section with a discussion of direct sum and Deligne product for dagger/unitary categories.

We remark that when $\mathcal{C}, \mathcal{D}$ are dagger/unitary, so is $\mathcal{C} \boxplus \mathcal{D}$ with dagger given by $(f \boxplus g)^\dagger = f^\dagger \boxplus g^\dagger$, and so is $\mathcal{C} \boxtimes \mathcal{D}$ with $(f \boxtimes g)^\dagger = f^\dagger \boxtimes g^\dagger$.

3.4. Semisimple unitary categories.

Here is the unitary analog of Theorem 2.10.14.

Theorem 3.4.1 (Fundamental Theorem of semisimple unitary categories)**.** *Suppose $\mathcal{C}$ is a unitary category. The following conditions are equivalent.*

(SS†1) *$\mathcal{C}$ is semisimple.*
(SS†2) *$\mathcal{C} \cong \mathsf{Hilb}(S)$ for some set S.*
(SS†3) *$\mathcal{C}$ is Cauchy complete*

If moreover $\mathcal{C}$ is finite, then the above conditions are equivalent to:

(SS†4) *$\mathcal{C} \cong \mathsf{Hilb}^{\oplus n}$ for some $n \geq 1$.*
(SS†5) *$\mathcal{C} \cong \mathsf{Rep}^\dagger(A)$ for some finite dimensional unitary algebra A.*

Proof.
(SS†1)⇒(SS†2): If $\mathcal{C}$ is unitary semisimple, then following square commutes.

$$\begin{array}{ccc} \mathsf{Hilb}(S) & \xrightarrow{\bigoplus H_s \mapsto \bigoplus H_s \otimes s} & \mathcal{C} \\ \big\downarrow{\scriptstyle \text{Forget}} & & \big\downarrow{\scriptstyle \text{Forget}} \\ \mathsf{Vec}(S) & \xrightarrow{\bigoplus V_s \mapsto \bigoplus H_s \otimes s} & \mathcal{C}^\natural \end{array}$$

Hence the dagger functor $\mathsf{Hilb}(S) \to \mathcal{C}$ is an equivalence on underlying linear categories and thus an equivalence by Corollary 3.1.15.
(SS†2)⇒(SS†3): Just observe $\mathsf{Hilb}(S)$ is unitarily Cauchy complete.
(SS†3)⇒(SS†1): Since $\mathcal{C}$ is unitary, every endomorphism algebra is a unitary algebra, which is semisimple, and thus (SS†3) holds.

Now assume that $\mathcal{C}$ is finite unitary.
(SS†4)⇔(SS†2): Clearly when S is finite, $\mathsf{Hilb}(S) \cong \mathsf{Hilb}^{|S|}$.
(SS†4)⇔(SS†5): If $\mathcal{C} \cong \mathsf{Hilb}^{\oplus n}$, set $A = \mathbb{C}^n$ and note $\mathsf{Rep}^\dagger(A) \cong \mathsf{Hilb}^{\oplus n}$.
(SS†5)⇒(SS†4): This is the unitary version of Corollary 2.8.3, which is also true. ☐

We have the following immediate corollary.

Corollary 3.4.2. *Suppose $\mathcal{C}$ is a semisimple unitary category. Every $c \in \mathcal{C}$ is unitarily isomorphic to an orthogonal direct sum of simples $c \cong \bigoplus_{i=1}^n s_i^{\oplus m_i}$, where the $s_i \in \mathcal{C}$ are mutually distinct simples.*

Proof. Note $\mathcal{C} \cong \mathsf{Hilb}(S)$, for which the result holds. □

We remark that even though a unitary semisimple category $\mathcal{C}$ is dagger equivalent to $\mathsf{Hilb}(\mathrm{Irr}(\mathcal{C}))$, we cannot at this time canonically identify each object $c \in \mathcal{C}$ with $\bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to c) \otimes s$ as $\mathcal{C}(s \to c)$ is not a Hilbert space. The missing ingredient is a unitary version of the Yoneda Lemma. The problem here is that $\mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})$ is not a dagger category, so the canonical isomorphism between representing objects from Remark 2.9.6 need not be unitary. We remedy this in §3.6 below.

It will helpful there to have the following lemma.

Lemma 3.4.3. *Suppose $\mathcal{C}, \mathcal{D}$ are unitary categories with $\mathcal{C}$ semisimple. Suppose $F, G : \mathcal{C} \to \mathcal{D}$ are dagger functors and $\rho : F \Rightarrow G$ is a natural transformation. Then ρ is unitary if and only if ρ_s is unitary for all $s \in \mathrm{Irr}(\mathcal{C})$.*

Proof. The forward direction is immediate. Suppose ρ_s is unitary for all $s \in \mathrm{Irr}(\mathcal{C})$. Write $c \in \mathcal{C}$ as an orthogonal direct sum $c \cong \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} s^{\oplus m_s}$ by Corollary 3.4.2 using isometries $\{v_j^s\}_{j=1}^{m_s} \subset \mathcal{C}(s \to c)$ satisfying $\sum_{s\in\mathrm{Irr}(\mathcal{C})} \sum_{j=1}^{m_s} v_j^s (v_j^s)^\dagger = \mathrm{id}_c$. By naturality,

$$\rho_c = \sum_{s\in\mathrm{Irr}(\mathcal{C})} \sum_{j=1}^{m_s} G(v_j^s) \rho_s F(v_j^s)^\dagger,$$

which is visibly unitary. □

3.5. **H*-algebras and 2-Hilbert spaces.** Recall that given a faithful positive linear functional φ on a (finite dimensional) unitary algebra, we get the GNS Hilbert space $L^2(A, \varphi) = A$ with inner product $\langle a|b\rangle := \varphi(a^\dagger b)$. If φ is also tracial, and we remember the algebra structure of A, we get the notion of an H*-algebra.

Definition 3.5.1. An H*-*algebra* is a unitary algebra A equipped with a faithful positive trace $\mathrm{Tr}_A : A \to \mathbb{C}$ (which is not necessarily a state). Observe that the GNS inner product satisfies

$$\langle b|a^*c\rangle = \langle ab|c\rangle = \langle a|cb^*\rangle \qquad \forall\, a, b, c \in A.$$

Since $L^2(A, \mathrm{Tr}_A) = A$, A is simultaneously a unitary algebra and a Hilbert space.[1] Since $M_n(\mathbb{C})$ has a unique tracial state, we get a complete classification of H*-algebras.

Lemma 3.5.2. *Every* H*-*algebra is a direct sum of* H*-*algebras if the form* $(M_n(\mathbb{C}), k\,\mathrm{Tr}_n)$.

Given an H*-algebra (A, Tr_A) and a right A-action on a Hilbert space H (which must satisfy $\langle \xi a|\eta\rangle_H = \langle \xi|\eta a^*\rangle_H$ for all $\eta, \xi \in H$ and $a \in A$), every $\eta \in H$ gives a left creation operator which is manifestly right A-linear:

$$L_\eta = |\eta\rangle : A_A \to H_A \qquad \text{is given by} \qquad a \mapsto \xi a.$$

[1] For those readers who know operator algebras, it is worth pointing out that if A is a unital C*-algebra equipped with a faithful positive trace Tr such that A is complete with respect to $\|a\|_2 := \sqrt{\langle a|a\rangle_{\mathrm{Tr}}}$, then A is finite dimensional. Indeed, since $\mathrm{End}(A_A) = A$, A is a von Neumann algebra. If A is an infinite dimensional von Neumann algebra, then there is an infinite sequence of mutually orthogonal projections (p_n) which sum to 1 SOT. We can then create an ℓ^2 sum with positive coefficients which is not bounded in the C* norm.

Since A is also a Hilbert space, we have an adjoint map, which is also right A-linear as for all $a, b \in A$,

$$\begin{aligned}\langle L_\eta^\dagger(\xi a)|b\rangle_A &= \langle \xi a|L_\eta b\rangle_H = \langle \xi a|\eta b\rangle_H = \langle \xi|\eta b a^\dagger\rangle_H \\ &= \langle \xi|L_\eta b a^\dagger\rangle_H = \langle L_\eta^\dagger \xi|b a^\dagger\rangle_A = \langle (L_\eta^\dagger \xi)a|b\rangle_A.\end{aligned} \tag{3.5.3}$$

We see that the A-*compact* operator $|\xi\rangle_A\langle\eta| := L_\xi L_\eta^\dagger \in \mathrm{End}(H_A)$, and $\langle\eta|\xi\rangle_A := L_\eta^\dagger L_\xi \in \mathrm{End}(A_A) = A$ gives an A-*valued inner product* on H which satisfies

$$\mathrm{Tr}_A(\langle\eta|\xi\rangle_A) = \langle 1_A|L_\eta^* L_\xi 1_A\rangle_{L^2(A,\mathrm{Tr}_A)} = \langle L_\eta 1_A|L_\xi 1_A\rangle_H = \langle\eta|\xi\rangle_H.$$

Exercise 3.5.4. Prove that $\langle\cdot|\cdot\rangle_A$ on H satisfies:

- (right A-linear) $\langle\eta|\xi_1 a + \xi_2\rangle_A = \langle\eta|\xi_1\rangle_A a + \langle\eta|\xi_2\rangle_A$ for all $\eta, \xi_1, \xi_2 \in H$ and $a \in A$,
- (anti-symmetric) $\langle\eta|\xi\rangle_A^* = \langle\xi|\eta\rangle_A$ for all $\eta, \xi \in H$, and
- (positive definite) for all $\eta \in H$, $\langle\eta|\xi\rangle_A \geq 0$ with equality if and only if $\eta = 0$.

Definition 3.5.5. A *right module* of an H*-algebra (A, Tr_A) is a Hilbert space H_A with a right A-action equipped with a faithful positive trace $\mathrm{Tr}_H : \mathrm{End}(H_A) \to \mathbb{C}$ such that[2]

$$\mathrm{Tr}_H(|\xi\rangle_A\langle\eta|) = \mathrm{Tr}_A(\langle\eta|\xi\rangle_A) = \langle\eta|\xi\rangle_H \qquad \forall\, \eta, \xi \in H. \tag{3.5.6}$$

We write $\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)$ for the category of right H*-modules over (A, Tr_A) with A-module maps with dagger structure given by the adjoint of linear maps, which is well-defined by (3.5.3).

Example 3.5.7. Let $d > 0$, and consider the H*-algebra $(\mathbb{C}, d)$ whose trace is determined by $\mathrm{Tr}(1_\mathbb{C}) = d$. We compute that

$$\langle\eta|\xi\rangle_H = \langle 1_\mathbb{C}|L_\eta^* L_\xi 1_\mathbb{C}\rangle_{L^2(\mathbb{C},d)} = d \cdot \langle\eta|\xi\rangle_{(\mathbb{C},d)} \qquad \Longrightarrow \qquad \langle\eta|\xi\rangle_{(\mathbb{C},d)} = \frac{1}{d}\cdot\langle\eta|\xi\rangle_H.$$

This means that $|\xi\rangle_{(\mathbb{C},d)}\langle\eta|$ is d^{-1} times the usual rank 1 operator $|\xi\rangle\langle\eta| : H \to H$ given by $\zeta \mapsto \langle\eta, \zeta\rangle\xi$. The unique trace Tr_H which endows H with the structure of an H* $(\mathbb{C}, d)$-module satisfies the equality

$$\frac{1}{d}\mathrm{Tr}_H(|\xi\rangle\langle\eta|) = \mathrm{Tr}_H(|\xi\rangle_{(\mathbb{C},d)}\langle\eta|) = \mathrm{Tr}_{(\mathbb{C},d)}(\langle\eta|\xi\rangle_{(\mathbb{C},d)}) = \langle\eta|\xi\rangle_H,$$

which implies Tr_H is the usual trace on $B(H)$ scaled by d.

Lemma 3.5.8. *$\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)$ is a finite semisimple unitary category.*

Proof. Clearly $\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)$ admits orthogonal direct sums, so unitarity follows from the fact that

$$\mathrm{End}(H_A) = \{x \in B(H) | x(\eta a) = (x\eta)a \text{ for all } \eta \in H \text{ and } a \in A\} = (A^{\mathrm{op}})' \cap B(H),$$

which is a unital $\dagger$-closed subalgebra of $B(H)$ (or a von Neumann algebra) for each object H_A. Since $\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)^\natural \cong \mathsf{Mod}(A)$ as categories, $\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)$ is finite semisimple. □

[2]Since H is finite dimensional, it follows that

$$\mathrm{End}(H_A) = \mathrm{span}_\mathbb{C}\{|\xi\rangle_A\langle\eta| | \eta, \xi \in H\}$$

Hence the trace Tr_H is completely determined by Tr_A and $\langle\cdot|\cdot\rangle_H$. However, this phenomenon will fail one categorical level higher.

Definition 3.5.9. A *pre-2-Hilbert space* is a finite unitary category $\mathcal{C}$ equipped with a family of faithful positive traces $\mathrm{Tr}_c^{\mathcal{C}} : \mathcal{C}(c \to c) \to \mathbb{C}$ for all $c \in \mathcal{C}$ satisfying

$$\mathrm{Tr}_b^{\mathcal{C}}(f \circ g) = \mathrm{Tr}_a^{\mathcal{C}}(g \circ f) \qquad\qquad \forall\, f : a \to b,\, g : b \to a.$$

Observe that every endomorphism algebra of a pre-2-Hilbert space is canonically an H*-algebra. Moreover, every hom space $\mathcal{C}(a \to b)$ comes equipped with an inner product $\langle f|g\rangle_{a\to b} := \mathrm{Tr}_a^{\mathcal{C}}(f^\dagger g)$, and these inner products satisfy

$$\langle g|hf^\dagger\rangle_{b\to c} = \langle gf|h\rangle_{a\to c} = \langle f|g^\dagger h\rangle_{a\to b} \qquad\qquad \forall\, f : a \to b,\, g : b \to c,\, h : a \to c. \tag{3.5.10}$$

Two pre-2-Hilbert spaces $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}}), (\mathcal{D}, \mathrm{Tr}^{\mathcal{D}})$ are called *equivalent* if there is a dagger equivalence $F : \mathcal{C} \to \mathcal{D}$ such that $\mathrm{Tr}^{\mathcal{D}}_{F(c)}(F(f)) = \mathrm{Tr}^{\mathcal{C}}_c(f)$ for all $f : c \to c$.

A pre-2-Hilbert space is called a *2-Hilbert space* if it is unitarily Cauchy complete.

Example 3.5.11. Equipping $\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)$ with the family of traces Tr_H satisfying (3.5.6), we see $\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)$ is a 2-Hilbert space. Indeed, for all $\eta_1, \eta_2 \in H_A$ and $\xi_1, \xi_2 \in K_A$,

$$\begin{aligned}\mathrm{Tr}_H(|\eta_1\rangle_A\langle\xi_1| \cdot |\xi_2\rangle_A\langle\eta_2|) &= \mathrm{Tr}_A(\langle\eta_2|\eta_1\rangle_A\langle\xi_1|\xi_2\rangle_A)\\ &= \mathrm{Tr}_A(\langle\xi_1|\xi_2\rangle_A\langle\eta_2|\eta_1\rangle_A)\\ &= \mathrm{Tr}_K(|\xi_2\rangle_A\langle\eta_2| \cdot |\eta_1\rangle_A\langle\xi_1|).\end{aligned}$$

Example 3.5.12. If (A, Tr_A) is an H*-algebra, then $\mathrm{B}(A, \mathrm{Tr}_A) := (\mathrm{B}A, \mathrm{Tr}_A)$ is a pre-2-Hilbert space.

Example 3.5.13. If $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$ is a pre-2-Hilbert space, then so is $\mathsf{Add}^\dagger(\mathcal{C})$ when we equip it with the faithful positive trace given by $\mathrm{Tr}^{\mathsf{Add}^\dagger(\mathcal{C})}_{(c_j)_{j=1}^n}(x_{ij}) := \sum_{j=1} \mathrm{Tr}^{\mathcal{C}}_{c_j}(x_{jj})$ for $(x_{ij}) \in \mathrm{End}((c_j)_{j=1}^n)$. Indeed, for all $(y_{ij}) : (a_j)_{j=1}^n \to (b_i)_{i=1}^m$ and $(z_{ji}) :\to (b_i)_{i=1}^m \to (a_j)_{j=1}^n$,

$$\mathrm{Tr}^{\mathsf{Add}^\dagger(\mathcal{C})}_{(b_i)_{i=1}^m}((y_{ij})(z_{ji})) = \sum_{i=1}^m \mathrm{Tr}^{\mathcal{C}}_{b_i}\left(\sum_{j=1}^n y_{ij}z_{ji}\right) = \sum_{j=1}^n \mathrm{Tr}^{\mathcal{C}}_{a_j}\left(\sum_{i=1}^m z_{ji}y_{ij}\right) = \mathrm{Tr}^{\mathsf{Add}^\dagger(\mathcal{C})}_{(a_j)_{j=1}^n}((z_{ji})(y_{ij})),$$

and faithfullness and positivity are also easy to check.

Example 3.5.14. If $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$ is a pre-2-Hilbert space, then so is $\mathsf{Proj}(\mathcal{C})$ when we equip it with the faithful positive trace given by $\mathrm{Tr}^{\mathsf{Proj}(\mathcal{C})}_{(c,p)}(x) := \mathrm{Tr}^{\mathcal{C}}_c(x)$. Indeed, for all $y : (c,p) \to (d,q)$ and $z : (d,q) \to (c,p)$, we have

$$\mathrm{Tr}^{\mathsf{Proj}(\mathcal{C})}_{(d,q)}(yz) = \mathrm{Tr}^{\mathcal{C}}_d(yz) = \mathrm{Tr}^{\mathcal{C}}_c(zy) = \mathrm{Tr}^{\mathsf{Proj}(\mathcal{C})}_{(c,p)}(zy),$$

and again faithfullness and positivity are also easy to check.

Example 3.5.15. Combining the previous two examples, if $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$ is a pre-2-Hilbert space, then $¢^\dagger(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$ is a 2-Hilbert space. In particular, for every H*-algebra (A, Tr_A), $¢\mathrm{B}(A, \mathrm{Tr}_A)$ is a 2-Hilbert space.

Definition 3.5.16. Given a simple object s in a 2-Hilbert space $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$, we define the *quantum dimension* of s by $d_s := \mathrm{Tr}^{\mathcal{C}}_s(\mathrm{id}_s)$.

Remark 3.5.17. Since $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$ is equivalent to $\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)$ for some H*-algebra (A, Tr_A) as a 2-Hilbert space, it is important to not confuse the quantum dimension d_s with the dimension of the corresponding Hilbert space in $\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)$.

Example 3.5.18. Suppose $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$ and $(\mathcal{D}, \mathrm{Tr}^{\mathcal{D}})$ are 2-Hilbert spaces. Then $\mathsf{Fun}^\dagger(\mathcal{C} \to \mathcal{D})$ (which is again finite unitary) is a pre-2-Hilbert space when equipped with the trace

$$\mathrm{Tr}_F(\rho : F \Rightarrow F) := \sum_{s \in \mathrm{Irr}(\mathcal{C})} d_s \cdot \mathrm{Tr}^{\mathcal{D}}_{F(s)}(\rho_s).$$

Indeed, if $F, G : \mathcal{C} \to \mathcal{D}$ and $\rho : F \Rightarrow G$ and $\sigma : G \Rightarrow F$, then

$$\begin{aligned}\mathrm{Tr}_F(\sigma \cdot \rho) &= \sum_{s \in \mathrm{Irr}(\mathcal{C})} d_s \cdot \mathrm{Tr}^{\mathcal{D}}_{F(s)}((\sigma \cdot \rho)_s) = \sum_{s \in \mathrm{Irr}(\mathcal{C})} d_s \cdot \mathrm{Tr}^{\mathcal{D}}_{F(s)}(\sigma_s \circ \rho_s) \\ &= \sum_{s \in \mathrm{Irr}(\mathcal{C})} d_s \cdot \mathrm{Tr}^{\mathcal{D}}_{G(s)}(\rho_s \circ \sigma_s) = \sum_{s \in \mathrm{Irr}(\mathcal{C})} d_s \cdot \mathrm{Tr}^{\mathcal{D}}_{G(s)}((\rho \cdot \sigma)_s) = \mathrm{Tr}_G(\rho \cdot \sigma).\end{aligned}$$

The reason for introducing this scaling above will become apparent in Lemma 3.6.5 below.

Proposition 3.5.19. *Suppose $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$ is a 2-Hilbert space. The trace $\mathrm{Tr}^{\mathcal{C}}$ is completely determined by $(d_s)_{s \in \mathrm{Irr}(\mathcal{C})}$.*

Proof. By Theorem 3.4.1, every $c \in \mathcal{C}$ can be written as an orthogonal direct sum of simples $c \cong \bigoplus_{s \in \mathrm{Irr}(\mathcal{C})} s^{m_s}$. Let $\left\{v^s_j : s \to c \middle| s \in \mathrm{Irr}(\mathcal{C}),\ j = 1, \ldots, m_s\right\}$ be a family of isometries satisfying $\sum_{s \in \mathrm{Irr}(\mathcal{C})} \sum_{j=1}^{m_s} v^s_j (v^s_j)^\dagger = \mathrm{id}_c$. Under the canonical isomorphism $\mathrm{End}_{\mathcal{C}}(c) \cong \bigoplus_{s \in \mathrm{Irr}(\mathcal{C})} M_{m_s}(\mathbb{C})$ from (2.3.2), we get matrix units for $f : c \to c$ given by

$$((f^s_{ij} = (v^s_i)^\dagger f v^s_j)_{i,j=1}^{m_s})_{s \in \mathrm{Irr}(\mathcal{C})}.$$

We calculate

$$\begin{aligned}\mathrm{Tr}^{\mathcal{C}}_c(f) &= \sum_{s,t \in \mathrm{Irr}(\mathcal{C})} \sum_{i,j=1}^{m_s} \sum_{i,j=1}^{m_t} \mathrm{Tr}^{\mathcal{C}}_c(v^s_i (v^s_i)^\dagger f v^t_j (v^t_j)^\dagger) = \sum_{s,t \in \mathrm{Irr}(\mathcal{C})} \sum_{i,j=1}^{m_s} \sum_{i,j=1}^{m_t} \mathrm{Tr}^{\mathcal{C}}_t((v^t_j)^\dagger v^s_i (v^s_i)^\dagger f v^t_j) \\ &= \sum_{s \in \mathrm{Irr}(\mathcal{C})} \sum_{j=1}^{m_s} \mathrm{Tr}^{\mathcal{C}}_s((v^s_j)^\dagger f v^s_j) = \sum_{s \in \mathrm{Irr}(\mathcal{C})} \sum_{j=1}^{m_s} \mathrm{Tr}^{\mathcal{C}}_s(f^s_{jj}\, \mathrm{id}_s) \\ &= \sum_{s \in \mathrm{Irr}(\mathcal{C})} d_s \sum_{j=1}^{m_s} f^s_{jj} = \sum_{s \in \mathrm{Irr}(\mathcal{C})} d_s \, \mathrm{Tr}(f^s_{ij})_{i,j=1}^{m_s}.\end{aligned}$$

Since the non-normalized trace on a simple summand of a multimatrix algebra does not depend on coordinates, this last expression is completely determined by the scalars d_s. □

Corollary 3.5.20. *Suppose $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}}), (\mathcal{D}, \mathrm{Tr}^{\mathcal{D}})$ are 2-Hilbert spaces and $F : (\mathcal{C}, \mathrm{Tr}^{\mathcal{C}}) \to (\mathcal{D}, \mathrm{Tr}^{\mathcal{D}})$ is an equivalence the underlying dagger categories. Then F is an equivalence of 2-Hilbert spaces if and only if $d_s = d_{F(s)}$ for all simples $s \in \mathrm{Irr}(\mathcal{C})$.*

Proof. It suffices to prove the reverse direction. If $d_s = d_{F(s)}$ for all simples $s \in \mathrm{Irr}(\mathcal{C})$, then for all $f : \mathcal{C}(c \to c)$, as in the previous proof,

$$\mathrm{Tr}^{\mathcal{D}}_{F(c)}(F(f)) = \sum_{s \in \mathrm{Irr}(\mathcal{C})} d_{F(s)} \, \mathrm{Tr}(f^s_{ij})_{i,j=1}^{m_s} = \sum_{s \in \mathrm{Irr}(\mathcal{C})} d_s \, \mathrm{Tr}(f^s_{ij})_{i,j=1}^{m_s} = \mathrm{Tr}^{\mathcal{C}}_c(f). \qquad \square$$

Proposition 3.5.21. *The map which sends a projection $p \in M_n(A) = \mathrm{End}(A^{\oplus n})$ to the right (A, Tr)-module $pA^{\oplus n}$ with trace satisfying (3.5.6) is an equivalence of 2-Hilbert spaces from $\text{¢}^\dagger(\mathrm{B}A, \mathrm{Tr}_A)$ to $\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)$.*

Proof. Since every right A-module is projective, the above functor is (unitarily) essentially surjective. Moreover, we have identifications

$$\mathrm{Hom}_{\mathsf{Mod}^\dagger(A,\mathrm{Tr}_A)}(qA^{\oplus n} \to pA^{\oplus m})) = pM_{m\times n}(A)q = \mathrm{Hom}_{\text{¢}^\dagger(\mathrm{BA})}((A^{\oplus n}, q) \to (A^{\oplus m}, p)),$$

which preserve the $\dagger$, so the above functor is a fully faithful dagger functor.

It remains to prove that the equivalence from Lemma 3.5.8 preserves the traces. If S_A is a simple right (A, Tr_A)-module and $v : S_A \to A_A$ is an isometry, we see that

$$\begin{aligned} d_S^{\text{¢}^\dagger(\mathrm{BA},\mathrm{Tr}_A)} &= \mathrm{Tr}_S^{\text{¢}^\dagger(\mathrm{BA},\mathrm{Tr}_A)}(\mathrm{id}_S) = \mathrm{Tr}_S^{\text{¢}^\dagger(\mathrm{BA},\mathrm{Tr}_A)}(v^\dagger v) = \mathrm{Tr}_A^{\text{¢}^\dagger(\mathrm{BA},\mathrm{Tr}_A)}(vv^\dagger) \\ &= \mathrm{Tr}_A^{(\mathrm{BA},\mathrm{Tr}_A)}(vv^\dagger) = \mathrm{Tr}_A(vv^\dagger) = \mathrm{Tr}_A(vv^\dagger) = \mathrm{Tr}_S(v^\dagger v) = d_S^{\mathsf{Mod}^\dagger(A,\mathrm{Tr}_A)}. \end{aligned}$$

We are now finished by Corollary 3.5.20. □

Theorem 3.5.22. *Every 2-Hilbert space is of the form* $\mathsf{Mod}^\dagger(A, \mathrm{Tr}_A)$ *for some* H^**-algebra* (A, Tr_A).

Proof. Let $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$ be a 2-Hilbert space. By Theorem 3.4.1, $\mathcal{C} \cong \mathsf{Hilb}(S)$ where $S = \mathrm{Irr}(\mathcal{C})$. Moreover, $\mathrm{Tr}^{\mathcal{C}}$ is completely determined by the positive scalars $d_s = \mathrm{Tr}^{\mathcal{C}}(\mathrm{id}_s)$ by Proposition 3.5.19.

Consider the H^*-algebra $(A, \mathrm{Tr}_A) = \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})}(\mathbb{C}, d_s)$ where $(\mathbb{C}, d_s)$ is the H^*-algebra whose modules were computed in Example 3.5.7. We see that

$$\mathsf{Mod}^\dagger\left(\bigoplus_{s\in\mathrm{Irr}(\mathcal{C})}(\mathbb{C}, d_s)\right) \cong \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathsf{Mod}^\dagger(\mathbb{C}, d_s)) \cong \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})}(\mathsf{Hilb}, d_s)$$

where (Hilb, d_s) means equip Hilb with the trace $d_s\,\mathrm{Tr}$. This 2-Hilbert space is exactly the one we are looking for. □

3.6. **Unitary Yoneda.** Using the notion of 2-Hilbert space, we can now provide a unitary version of the Yoneda Lemma. Suppose $(\mathcal{C}, \mathrm{Tr}^{\mathcal{C}})$ is a pre-2-Hilbert space, and recall that each hom space $\mathcal{C}(a \to b)$ carries the inner product $\langle f|g\rangle_{a\to b} := \mathrm{Tr}_a^{\mathcal{C}}(f^\dagger g)$, and these inner products satisfy (3.5.10). Observe now that for every $c \in \mathcal{C}$, we get a representable functor

$$\mathcal{C}(- \to c) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Hilb}.$$

Lemma 3.6.1 (Unitary Yoneda, part 1)**.** *Each functor* $\mathcal{C}(- \to c)$ *is a dagger functor, and the Yoneda embedding* $よ^\dagger : \mathcal{C} \to \mathsf{Fun}^\dagger(\mathcal{C}^{\mathrm{op}} \to \mathsf{Hilb})$ *given by* $c \mapsto \mathcal{C}(- \to c)$ *is a dagger functor.*

Proof. The first claim follows from the first equality in (3.5.10); when $f : a \to b$,

$$(- \circ f) : \mathcal{C}(b \to c) \to \mathcal{C}(a \to c) \qquad \text{and} \qquad \langle g|hf^\dagger\rangle_{b\to c} = \langle gf|h\rangle_{a\to c}$$

for all $g : b \to c$ and $h : a \to c$. The second claim follows from the second equality in (3.5.10); when $g : b \to c$,

$$(g \circ -) : \mathcal{C}(- \to b) \Rightarrow \mathcal{C}(- \to c),$$

and for each $f : a \to b$ and $h : a \to c$,

$$\langle gf|h\rangle_{a\to c} = \langle f|g^\dagger h\rangle_{a\to b}.$$

Hence $(g \circ -)^\dagger = (g^\dagger \circ -) : \mathcal{C}(- \to c) \Rightarrow \mathcal{C}(- \to b)$. □

We now make an observation on unitarily representing dagger functors.

Definition 3.6.2. Suppose $\mathcal{C}$ is a pre-2-Hilbert space. A dagger functor $F : \mathcal{C}^{\text{op}} \to \mathsf{Hilb}$ is called *unitarily representable* if there is an $a \in \mathcal{C}$ and a unitary natural isomorphism $\alpha : F \Rightarrow \mathcal{C}(- \to a)$. We call (a, α) a unitary representing pair for F.

Remark 3.6.3. Suppose $(a, \alpha), (b, \beta)$ are two unitary representing pairs for $F : \mathcal{C}^{\text{op}} \to \mathsf{Hilb}$. Since $\beta \circ \alpha^{-1} : \mathcal{C}(- \to a) \Rightarrow \mathcal{C}(- \to b)$ is a unitary natural isomorphism, and since the unitary Yoneda embedding from Lemma 3.6.1 is a dagger functor, $\beta \circ \alpha^{-1}$ must be post-composition with a unitary isomorphism $a \to b$; i.e., the canonical isomorphism $(\beta_a \circ \alpha_a^{-1})(\text{id}_a) \in \mathcal{C}(a \to b)$ from Remark 2.9.6 is unitary. Thus the object unitarily representing a unitarily representable dagger functor is uniquely determined (up to a contractible space).

Now suppose $(\mathcal{C}, \text{Tr}^{\mathcal{C}})$ is a 2-Hilbert space. Equipping Hilb with its usual trace, the finite unitary category $\mathsf{Fun}^\dagger(\mathcal{C}^{\text{op}} \to \mathsf{Hilb})$ inherits a trace from Example 3.5.18.

Theorem 3.6.4 (2-Riesz Representation)**.** *If $(\mathcal{C}, \text{Tr}^{\mathcal{C}})$ is a 2-Hilbert space, the unitary Yoneda embedding $よ^\dagger : \mathcal{C} \to \mathsf{Fun}^\dagger(\mathcal{C}^{\text{op}} \to \mathsf{Hilb})$ is an equivalence of 2-Hilbert spaces.*

Proof. By Lemma 3.6.1, $よ^\dagger$ is a dagger functor, and the underlying linear functor of $よ^\dagger$ is $よ : \mathcal{C}^\natural \to \mathsf{Fun}((\mathcal{C}^\natural)^{\text{op}} \to \mathsf{Vec})$, which is an equivalence by Corollary 2.10.13. By Corollary 3.1.15, $よ^\dagger$ is a dagger equivalence. By Corollary 3.5.20, it remains to prove quantum dimensions of simples agree. For $s \in \text{Irr}(\mathcal{C})$,

$$\text{Tr}^{\mathsf{Fun}^\dagger}_{\mathcal{C}(-\to s)}(\text{id}_{\mathcal{C}(-\to s)}) = \sum_{t \in \text{Irr}(\mathcal{C})} d_s \cdot \text{Tr}^{\mathsf{Hilb}}_{\mathcal{C}(t \to s)}((\text{id}_{\mathcal{C}(-\to s)})_t) = d_s \cdot \text{Tr}^{\mathsf{Hilb}}_{\mathcal{C}(s\to s)}(\text{id}_s) = d_s. \qquad \square$$

Lemma 3.6.5 (Unitary Yoneda, part 2)**.** *For every dagger functor $F : \mathcal{C}^{\text{op}} \to \mathsf{Hilb}$, the Yoneda isomorphism*

$$\begin{aligned} よ(a, F) : \text{Hom}(\mathcal{C}(- \to a) \Rightarrow F) &\longrightarrow F(a) \\ \rho &\longmapsto \rho_a(\text{id}_a) \end{aligned}$$

is unitary.

Proof. Observe that for one fixed dagger functor $F : \mathcal{C}^{\text{op}} \to \mathsf{Hilb}$, $a \mapsto \text{Hom}(\mathcal{C}(- \to a) \Rightarrow F)$ is also a second dagger functor $\mathcal{C}^{\text{op}} \to \mathsf{Hilb}$, and $よ(-, F)$ is a natural isomorphism between them. To check unitarity, it suffices to check each $よ(s, F)$ is unitary by Lemma 3.4.3.

For a simple $s \in \text{Irr}(\mathcal{C})$ and $\rho : \text{Hom}(\mathcal{C}(- \to a) \Rightarrow F)$,

$$\|\rho\|^2_{\mathcal{C}(-\to s) \Rightarrow F} = \text{Tr}^{\mathsf{Fun}^\dagger}_{\mathcal{C}(-\to s)}(\rho^\dagger \circ \rho) = \sum_{t \in \text{Irr}(\mathcal{C})} d_t \, \text{Tr}^{\mathsf{Hilb}}_{\mathcal{C}(t \to s)}(\rho_t^\dagger \circ \rho_t) = d_s \, \text{Tr}^{\mathsf{Hilb}}_{\mathcal{C}(s \to s)}(\rho_s^\dagger \circ \rho_s)$$

Since $\{d_s^{-1/2} \, \text{id}_s\}$ is an orthonormal basis for $\mathcal{C}(s \to s)$, we can continue our calculation using the formula for $\text{Tr}^{\mathsf{Hilb}}$ as

$$d_s \langle d_s^{-1/2} \, \text{id}_s \, | (\rho_s^\dagger \circ \rho_s)(d_s^{-1/2} \, \text{id}_s) \rangle_{s \to s} = \langle \rho_s(\text{id}_s) | \rho_s(d_s^{-1/2} \, \text{id}_s) \rangle_{F(s)} = \|\rho_s(\text{id}_s)\|_{F(s)}.$$

We conclude each $よ(s, F)$ is unitary, so we are finished. $\square$

We now work to construct the canonical unitary isomorphism exhibiting c as an orthogonal direct sum.

Lemma 3.6.6. *Suppose $s \in \mathrm{Irr}(\mathcal{C})$. For every $f : a \to s$ and $g : s \to b$,*

$$f \circ f^\dagger = d_s^{-1} \cdot \mathrm{Tr}_s^{\mathcal{C}}(f \circ f^\dagger) \cdot \mathrm{id}_s \qquad \textit{and} \qquad g^\dagger \circ g = d_s^{-1} \cdot \mathrm{Tr}_s^{\mathcal{C}}(g^\dagger \circ g) \cdot \mathrm{id}_s .$$

Moreover, we have

$$\|g \circ f\|_{a \to b} = d_s^{-1/2} \cdot \|g\|_{s \to b} \cdot \|f\|_{a \to s}.$$

Proof. Since $s \in \mathrm{Irr}(\mathcal{C})$, there are real numbers $r_f, r_g \geq 0$ such that $f \circ f^\dagger = r_f \,\mathrm{id}_s$ and $g^\dagger \circ g = r_g \,\mathrm{id}_s$. Taking $\mathrm{Tr}_s^{\mathcal{C}}$, we see that

$$\mathrm{Tr}_s^{\mathcal{C}}(f \circ f^\dagger) = r_f d_s \qquad \Longleftrightarrow \qquad r_f = d_s^{-1} \cdot \mathrm{Tr}_s^{\mathcal{C}}(f \circ f^\dagger)$$

and similarly, $r_g = d_s^{-1} \cdot \mathrm{Tr}_s^{\mathcal{C}}(g^\dagger \circ g)$. This proves the first claim.

For the second claim, we calculate that

$$\begin{aligned} \|g \circ f\|_{a \to b}^2 &= \mathrm{Tr}_a^{\mathcal{C}}(f^\dagger \circ g^\dagger \circ g \circ f) = \mathrm{Tr}_s^{\mathcal{C}}((f \circ f^\dagger) \circ (g^\dagger \circ g)) \\ &= d_s^{-2}\, \mathrm{Tr}_s^{\mathcal{C}}(f \circ f^\dagger)\, \mathrm{Tr}_s^{\mathcal{C}}(g^\dagger \circ g)\, \mathrm{Tr}_s^{\mathcal{C}}(\mathrm{id}_s) = d_s^{-1} \cdot \|g\|_{s \to b} \cdot \|f\|_{a \to s}. \end{aligned}$$ □

Due to the previous lemma, we introduce the following notation.

Notation 3.6.7. Given $H \in \mathsf{Hilb}$ and $\lambda > 0$, we write λH for the Hilbert space obtained from H by scaling the inner product by λ, i.e., $\langle \eta | \xi \rangle_{\lambda H} = \lambda \cdot \langle \eta | \xi \rangle_H$ for all $\eta, \xi \in H$.

Proposition 3.6.8. *The composition isomorphism*

$$\bigoplus_{s \in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \to s) \otimes_{\mathbb{C}} d_s^{-1} \mathcal{C}(s \to b) \longrightarrow \mathcal{C}(a \to b)$$

is unitary.

Proof. We know that the composition map is invertible by semisimplicity, and we know that the subspaces $\mathcal{C}(a \to s) \otimes_{\mathbb{C}} d_s^{-1}\mathcal{C}(s \to b)$ are mutually orthogonal for distinct simples $s \in \mathrm{Irr}(\mathcal{C})$. Hence by the Pythagorean Theorem, it suffices to check that composition restricted to each subspace

$$\mathcal{C}(a \to s) \otimes_{\mathbb{C}} d_s^{-1} \mathcal{C}(s \to b) \longrightarrow \mathcal{C}(a \to b)$$

is an isometry. This is immediate from Lemma 3.6.6. □

Note that in Proposition 3.6.8 above, we could also have chosen to scale the inner product on $\mathcal{C}(a \to s)$. The purpose of choosing to rescale the inner product on $\mathcal{C}(s \to b)$ is the next result.

Corollary 3.6.9. *For each $c \in \mathcal{C}$, we have a canonical unitary isomorphism $\upsilon_c : c \cong \bigoplus_{s \in \mathrm{Irr}(\mathcal{C})} d_s^{-1} \mathcal{C}(s \to c) \otimes s$ satisfying (2.10.8) for all morphisms $f : a \to b$.*

Proof. The canonical isomorphism from Lemma 2.10.7 was constructed by writing down a natural isomorphism of representable functors. With the appropriately scaled multiplicity spaces $d_s^{-1}\mathcal{C}(s \to c)$, we have the following unitary natural isomorphism between unitarily representable functors:

$$\begin{aligned}
\mathcal{C}\left(a \to \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})} d_s^{-1}\mathcal{C}(s\to c)\otimes s\right) &\cong \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})} \mathcal{C}\left(a\to d_s^{-1}\mathcal{C}(s\to c)\otimes s\right) && \text{(Ex. 3.2.2)}\\
&\cong \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})} \mathcal{C}(a\to s)\otimes d_s^{-1}\mathcal{C}(s\to c) && \text{(3.2.9)}\\
&\cong \mathcal{C}(a\to c) && \text{Prop. (3.6.8).}
\end{aligned}$$

Hence the canonical isomorphism $\upsilon_c : c \cong \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})} d_s^{-1}\mathcal{C}(s \to c)\otimes s$ is unitary by Remark 3.6.3. The final claim follows as in Lemma 2.10.7. □

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