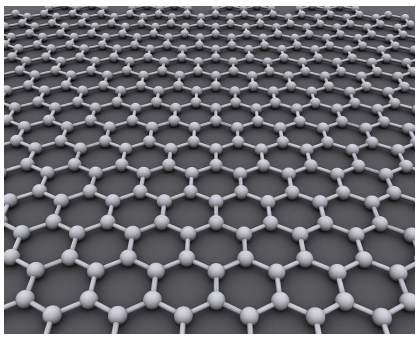
2. Lattice models

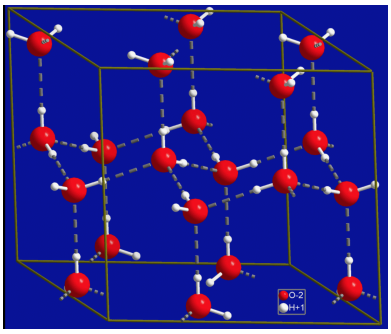
For a many body system of distinguishable particles, the Hilbert space of a many body system is the tensor product of the Hilbert spaces of the single-body subsystems. When the particles are bosons, the total wave function is invariant under exchanging two bosons; in fact, for any permutation $s \in S_N$, the amplitudes ($|c|^2$) of $|\psi_1 \cdots \psi_N\rangle$ and $|\psi_{s(1)} \cdots \psi_{s(N)}\rangle$ should agree, leading to the symmetric tensor product. For fermions, we take the antisymmetric tensor product. Unfortunately, this does not let us change the number of particles unless we work in a 'Fock space' where we take a direct sum over N for the tensor products for N-particles. This description is called the *particle basis representation*.

In the *occupation basis representation*, single particles can occupy certain *modes*, which can be labelled by position, momentum, or any other physical observable of the particle. For bosons, any number can occupy the same mode, giving an infinite dimensional Hilbert space, whereas for fermions, only 0 or 1 particle can occupy the same mode, giving $\mathbb{C}^2$. We work with some upper bound d for the number of particles that can inhabit any mode, making our Hilbert space $\mathbb{C}^d$ for a single mode; a state in $\mathbb{C}^d$ will be called a *qudit*. For multiple sites, we again take a tensor product. So the properties of boson and fermion are no longer properties of the Hilbert space, but rather of the operators that act. We will work in this second occupation basis representation.

A lattice model is meant to represent a piece of (quantum) matter which is arranged in a spatially symmetric way. For example, graphene forms a hexagonal 2D lattice, and ice forms a 3D lattice of alternating sheets of crinkled 2D hexagonal lattices.



https://en.wikipedia.org/wiki/Graphene



https://en.wikipedia.org/wiki/Hydrogen_bond (2.0.1)

2.1. Classical statsitical mechanical lattice models.

In actual matter, there are far too many particles to compute the Hamiltonian or its energy eigenstates. We can never know the complexities of a system with 10^{23} particles [ZCZW19, §5.2.2]. Statistical mechanics essentially uses statistical and probabilistic techniques to describe physical interactions of these many many many body systems.

In statistical mechanics, there is a set (not a space) of states $\mathcal{S}$ that the system can potentially be in, and the probability the system is in state $\sigma \in \mathcal{S}$ is given by the *Boltzmann Law*:

$$\mathbb{P}(\sigma) := \frac{e^{-\beta E(\sigma)}}{Z_\beta} \qquad Z_\beta := \sum_\sigma e^{-\beta E(\sigma)}.$$

where $E(\sigma)$ is the energy of state σ, $\beta = (kT)^{-1}$ is the *inverse temperature* where k is *Boltzmann's constant*, and Z_β is the *partition function*. Observe Z_β is defined so that $\mathbb{P}$ is a probability measure on $\mathcal{S}$.

Definition 2.1.1. An *equilibrium distribution* with respect to an observable H is a probability distribution which maximizes the entropy amongst all states with the same expected value $\mathbb{E}(H)$.

Once the system reaches the equilibrium distribution, it remains constant.

Exercise 2.1.2. Consider the entropy function for a distribution over the set $\mathcal{S}$ of states.

$$H(\{p_\sigma\}) = -\sum_\sigma p_\sigma \log(p_\sigma)$$

subject to the constraints $\sum p_\sigma = 1$ and $\mathbb{E}(E) = \sum p_\sigma E(\sigma)$ where $E(\sigma)$ denotes the energy of state σ. Use the method of Lagrange multipliers using the multiplier β for $\mathbb{E}(E) - \sum p_\sigma E(\sigma)$ to show that the Boltzmann Law maximizes the entropy of $\{p_\sigma\}$.

Example 2.1.3 (Classical Ising model)**.** The Ising model is a classical statistical mechanical model of ferromagnetism (a material is *ferromagnetic* if it retains magnetism outside an external field). A *spin configuration* σ is an assignment σ_v of a spin ± 1 to each vertex, i.e., $\sigma : V(L) \to \{\pm 1\}$. (This assignment of a spin to each site makes the system classical.) For each pair of neighbors u, v, there is an interaction term $J_{u,v} > 0$, and for each vertex v, there is an external magnetic field term h_v. The Hamiltonioan function value at state σ is given by

$$E_{\text{Is}}(\sigma) := -\sum_{u \sim v} J_{u,v}\sigma_u\sigma_v - B\sum_v h_v\sigma_v$$

where B is called the *magnetic moment*. Here, one can think of h_v as pointing the direction of the external field in comparison with a given electron's spin via the inner product.

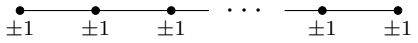
The d-dimensional *translation invariant ferromagnetic zero field* Ising model is on the lattice $\mathbb{Z}^d$ and has $J_{u,v} = 1$ independent of $u \sim v$ and $B = 0$. Observe that in this case, when spins are aligned so that $\sigma_u\sigma_v = 1$, this gives a lower energy value.

Remark 2.1.4. When $B = 0$, observe that E_{Is} is invariant under the $\mathbb{Z}/2\mathbb{Z}$ symmetry of flipping all σ_v.

Exercise 2.1.5. Assume $B = 0$. Show that minimizing $E_{\text{Is}}(\sigma)$ can be viewed as a min-cut graph theory problem by partitioning the vertices into those spin up and those spin down.

Remark 2.1.6 (No phase transitions in finite volume)**.** There is no phase transition for a finite lattice. Since the partition function is entire as a finite sum of entire functions, there are no singularities. This no longer holds in the *thermodynamic limit*, in which the lattice size goes to infinity.

Example 2.1.7. The 1D translation invariant ferromagnetic zero field Ising model is somewhat boring and has no phase transition as β varies [Isi25].



In fact, this model is *exactly solvable*, i.e., there is a closed form for the partition function Z_β. See Exercise 2.1.8 below for a walkthrough for the 1D model.

In higher dimensions, there is a phase transition [Pei36]; when β is sufficiently large, the correlations have a lower bound:

$$\langle \sigma_u \sigma_v \rangle_\beta \geq c(\beta) > 0.$$

The 2D model was solved analytically by [Ons44] using the transfer matrix method.

Exercise 2.1.8. Consider the 1D translation invariant ferromagnetic zero field Ising model.

(1) Show that under free boundary conditions, the partition function is given by

$$Z_\beta = 2\left(e^{\beta J} + e^{-\beta J}\right)^{L-1}.$$

Hint: One method is to perform the change of variables $\rho_j = \sigma_{j-1}\sigma_j$ and sum over ρ.

(2) Show that the correlations are given by

$$\langle \sigma_i \sigma_{i+n} \rangle = \left(\frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{-\beta J}}\right)^N$$

so that

$$\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle = C(\beta) e^{-c(\beta)|i-j|}$$

where $C(\beta), c(\beta)$ are positive functions of β for $\beta = T^{-1} > 0$.

(3) Show that $c(\beta) \to 0$ as $\beta \to \infty$ $(T \to 0)$.

Remark 2.1.9. Observe that we may view the classical Ising model as a Markov chain, as the probability $P_\beta(\sigma')$ of transitioning to state σ' only depends on the current state σ.

Exercise 2.1.10. Write a Monte Carlo simulation of the 2D translation invariant ferromagnetic zero field Ising model. Simulate the phase transition from disordered to ordered as β increases.

Remark 2.1.11. The Potts model is defined similarly to the Ising model, but with d spin values for each site instead of the 2 values ± 1, which are usually taken to be $\exp(k2\pi i/d)$ for $k = 0, 1, \ldots, d-1$.

2.2. **Local Hamiltonians and quantum spin chains.** Suppose we have an N particle system of *qudits* $\mathcal{H} = (\mathbb{C}^d)^{\otimes N}$. (One should think that these N particles are fixed spatially, and so the degrees of freedom correspond to some quantum observable, like spin.) A Hamiltonian H on the system $\mathcal{H}$ is called *local* if it is a sum of subsystem Hamiltonians $H = \sum H_j$ where each H_j as the identity except on a few number of tensorands, which are usually close together in some arrangement of the N qudits. We call H *k-local* if each H_j involves at most k-body local interactions.

Definition 2.2.1. The *ground state space* of H is the eigenspace corresponding to the smallest eigenvalue of H. We say that H has *ground state degeneracy* if the dimension of the ground state space is greater than 1.

Remark 2.2.2. Sometimes a local Hamiltonian H has some kind of symmetry, like flipping or translation invariance, which is easily seen to lead to ground state degeneracy. In this setting, we say the ground state *breaks* the symmetry, since a choice of ground state $|\psi\rangle$ of H must ignore this symmetry. This phenomenon is called *spontaneous symmetry breaking*.

Exercise 2.2.3. Suppose $|\psi\rangle$ is a ground state for the k-local Hamiltonian H, and $|\phi\rangle$ has the same k-RDMs as $|\psi\rangle$.

(1) Prove that $\langle\psi|H|\psi\rangle = \langle\phi|H|\phi\rangle$.
Hint: $\langle\psi|H|\psi\rangle = \mathrm{Tr}(H|\psi\rangle\langle\psi|)$.
(2) Deduce that $|\phi\rangle$ is also a ground state of H.

Definition 2.2.4. A local Hamiltonian $H = \sum H_j$ is

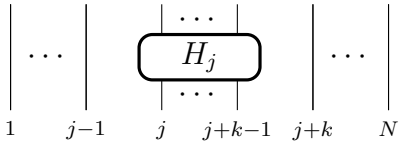
- *frustration free* if every ground state $|\psi\rangle$ of H is also a ground state of each H_j, and
- *commuting projector* if all the H_j pairwise commute.

Observe that a commuting projector local Hamiltonian is necessarily frustration free.

Remark 2.2.5. Given a commuting projector local Hamiltonian $H = \sum H_j$, we do not require that each H_j is actually an orthogonal projection. The term 'commuting projector' here indicates that the spectral projections of the H_j will all commute as the H_j generate an abelian unital $*$-subalgebra of $B(\mathcal{H})$ (i.e., an abelian von Neumann algebra).

Definition 2.2.6. A *quantum spin chain* is an array of *spins*, which are states $|\sigma_i\rangle \in \mathbb{C}^d$. If the array has N sites, the total Hilbert space is $\mathcal{H} = (\mathbb{C}^d)^{\otimes N}$. A Hamiltonian H on $\mathcal{H}$ is *k-local* if $H = \sum_j H_j$ where each H_j is a Hermetian/self-adjoint operator which only acts nontrivially on k adjacent sites. (The adjacency of the k-body local interactions is what makes these models 1D.) In the simplest examples, $d = 2$ and $k = 2$.

For example, if each H_j only acts on k adjacent sites, we can represent H_j graphically by

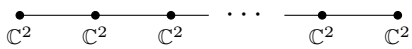
$H_j =$ 

(2.2.7)

where the lines represent qudits, horizontal juxtaposition denotes tensor product, through strings denote identity operators, and the coupon H_j denotes the local operator acting on the k adjacent sites j through $j + k - 1$. (We will revisit these diagrams when we discuss graphical calculus for tensor categories; above is in the graphical calculus for tensor category of finite dimensional Hilbert spaces.)

Remark 2.2.8. Later in the semester, we will try to build a symmetric monoidal 2-category of quantum spin chains and their defects following [Hen21, 5:00-14:30].

Example 2.2.9 (1D Transverse-field Ising (tIs) model)**.** The 1D transverse-field Ising model is a quantum statistical mechanical model with a quantum phase transition. Its local Hamiltonian is given by



$$H_{\mathrm{tIs}} := -J\sum_j Z_jZ_{j+1} - B\sum_j X_j,$$

where we write Z_j to mean apply Pauli Z at site j and I everywhere else, and similarly for X_j. The Hamiltonian H_{tIs} is a nearest neighbor model, and involves only 2-body local interactions, so H_{tIs} is 2-local. Without loss of generality, we may assume $J = 1$.

Exercise 2.2.10. Observe that when $B = 0$, H_{tIs} is commuting projector and therefore frustration free. Compute its ground state space. Does it have degeneracy?

Remark 2.2.11. *Solving* a model means finding all the eigenstates of H and their energies. To solve the 1D transverse Ising model, one uses the *transfer matrix method*, which maps

between this model and the classical 2D Ising model. Other such strategies involve the *Bethe Ansatz* and *matrix product states*.

When $|B| < 1$, the system is in the *ordered phase*, and when $|B| > 1$, the system is in the *disordered phase*. Both of these phases are *gapped*, meaning the change ΔE between the ground state of H_{tIs} and the next lowest energy state remains strictly positive in the thermodynamic limit. When $B = 0$, the ground state is spanned by $\{|0\cdots 0\rangle, |1\cdots 1\rangle\}$, and when $J = 0$, the ground space is spanned by $\{(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle)^{\otimes N}\}$.

At $|B| = 1$, the system undergoes a *quantum phase transition*. At this value, the system is *gapless* (not gapped), and low-energy behavior is described by the 2D Ising conformal field theory. This is seen by matching up the phase transition with the 1D quantum Ising model with the 2D classical model under the transfer matrix method.

[[More on all this later.]]

Remark 2.2.12. There is also an d-state *quantum Potts model* which replaces $\mathbb{C}^2$ with $\mathbb{C}^d$.

Example 2.2.13 (Heisenberg model)**.** The local Hilbert space is $\mathbb{C}^2$; taking N sites gives the total space $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$. Given *coupling constants* $J_x, J_y, J_z \in \mathbb{R}$, the local Hamiltonian is given by

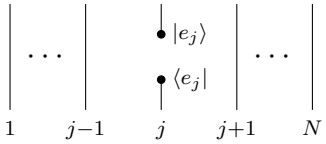
$$H_{\text{Heis}} := -\frac{1}{2}\sum_j J_x X_j X_{j+1} + J_y Y_j Y_{j+1} + J_z Z_j Z_{j+1} + B Z_j.$$

This Hamiltonian is clearly not commuting projector. Under various relationships amongst J_x, J_y, J_z, this model has other names:

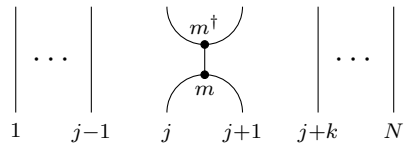
$J_y = J_z = 0$	Transverse-field Ising model
$J_x \neq J_y \neq J_z \neq J_x$	XYZ model
$J_x = J_y \neq J_z$	XXZ model
$J_x = J_y = J_z$	XXX model

We remark that when $B = 0$, this system is *gapless* [ZCZW19, p114].

Example 2.2.14 ([CGHP], 1D transverse-field nearest neighbor group algebra model)**.** Let G be a finite group, and consider the group algebra $\mathbb{C}[G]$ whose standard basis are unitaries u_g satisfying the multiplication rule $u_g u_h = u_{gh}$. We write $|g\rangle$ for u_g. The total space is $\mathcal{H} = (\mathbb{C}^G)^{\otimes N}$ We define a 2-local Hamiltonian H on $\mathcal{H}$ as follows. The first term is the 1-local *unit term* $U_j := |e_j\rangle\langle e_j|$ which projects to the unit $e \in G$ on site j. We denote the unit term pictorially by

$U_j =$ 

The second term M_j is the composite of multiplication on G and its adjoint. That is, the multiplication $m : \mathbb{C}[G] \otimes \mathbb{C}[G] \to \mathbb{C}[G]$ given by $|g_1 h_2\rangle \mapsto |gh\rangle$, and its adjoint $m^\dagger : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ is given by $|g\rangle \mapsto \sum |(gh)_1 h_2^{-1}\rangle$. Pictorially, we denote M_j by

$M_j =$ 

Our 2-local Hamiltonian is then defined by

$$H := -\sum_j U_j - B\sum_j M_j$$

[[more on this]]

Exercise 2.2.15. Suppose we have N sites which can be occupied by fermions (so $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$), and define for each pair $1 \leq i, j \leq N$, a 2-body Hamiltonian

$$H_{ij} := |1_i 1_j\rangle\langle 1_i 1_j| + |EPR_{ij}\rangle\langle EPR_{ij}|$$

where $|EPR_{ij}\rangle = \frac{1}{\sqrt{2}}(|0_i 1_j\rangle - |1_i 0_j\rangle)$ is the singlet state for particles i and j. Consider the Hamiltonian $H := \sum_{i<j} H_{ij}$.

(1) Prove that the ground state is spanned by $|0\rangle^{\otimes N}$ and

$$|W_N\rangle := \frac{1}{\sqrt{N}}(|10\cdots 0\rangle + |010\cdots 0\rangle + \cdots + |0\cdots 01\rangle).$$

(2) Prove that H is frustration free.
(3) Compute $\Lambda_{\max}(|W_N\rangle)$ and show the geometric measure of entanglement $E_G(|W_N\rangle)$ increases with N.

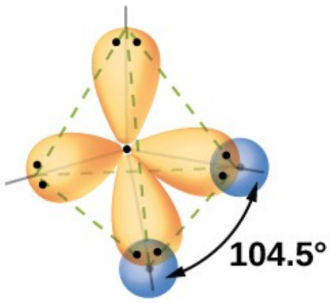
Exercise 2.2.16 (Adapted from [ZCZW19, p103])**.**

(1) Use the Schmidt decomposition to prove that up to an invertible operator $T = T_A \otimes T_B$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$, any state $|\psi\rangle$ is a product state (e.g., $|00\rangle$) or essentially a singlet state (e.g., $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$).
(2) Prove that up to an invertible operator $T = T_A \otimes T_B \otimes T_C$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, any state $|\psi\rangle$ is a product state (e.g., $|000\rangle$), a tensor product of a pure state and a singlet state (e.g., $|0\rangle \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$), a GHZ state (e.g., $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$), or a W state (e.g., $|W_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |001\rangle)$),

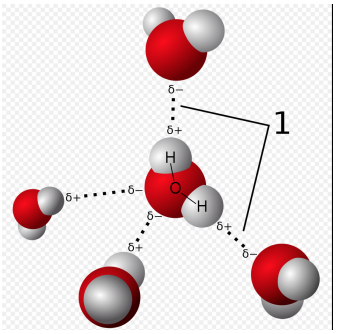
Example 2.2.17 (Hulthén 1D symmetric spin chain [1938])**. TODO:**

2.3. **Temperley-Lieb ice type quantum spin chain.** A water molecule H_2O consists of one oxygen atom O ($1s^2 2s^2 2p^4$ - two full s orbitals and 2 lone pairs of electrons in a p orbital) and two hydrogen atoms (each $1s^1$). Each H atom shares an electron with O giving a tetrahedral shape. Water molecules also share *hydrogen bonds* with other water molecules, which is an *electrostatic force* between an H atom of one molecule and a lone pair of electrons

of an O atom of another molecule



https://courses.lumenlearning.com/sanjacinto-atdcoursereview-chemistry1-1/chapter/hybrid-atomic-orbitals/



https://en.wikipedia.org/wiki/Hydrogen_bond

In fact, the reason water boils at such a high temperature for its molecular mass ($100°C$) is the presence of these hydrogen bonds. At lower temperatures, hydrogen bonding leads to the hexagonal lattice structure of ice (2.0.1), where each water molecule forms roughly 4 hydrogen bonds.

In [TL71], Temperley and Lieb studied various 2D planar *ice type* or *hydrogen bond* lattice models building on work of Lieb [Lie67a, Lie67b]. In these models, single H_2O molecules are treated as vertices on a $\mathbb{Z}^2$ lattice, and each molecule is connected to 4 other molecules via oriented edges. The lone pairs of electrons have a negative charge whereas the H atoms have a positive charge, so a hydrogen bond edge is oriented from negative to positive. Hence Lieb studied 2D oriented lattice models where exactly two edges point in to each vertex; there are exactly $\binom{4}{2} = 6$ of these:

We will see these 4-valent vertices again when we discuss braidings in tensor categories.

The now famous *Temperley-Lieb-Jones* algebras arise from analyzing an 'ice-type' quantum spin chain, where instead of looking at the total Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$, one passes to a subspace of allowed states which correspond to admissible configurations. Here, the configurations correspond to non-crossing partitions, which are meant to represent hydrogen bonds (which cannot intersect!) between N molecules arranged on a circle.

Example 2.3.1 (1D Temperley-Lieb (TL) quantum spin chain)**.** The Hilbert space for a single site is $\mathbb{C}^2$, and the local Hamiltonian is built from Pauli X and Z operators. The 2-local Hamiltonian is given by the $H = \sum_i U_i$

$$U_i = U_{i,i+1} := \left(\frac{r}{s} + X_iX_{i+1}\right)\frac{(1+Z_i)(1-Z_{i+1})}{4} + \left(\frac{s}{r} + X_iX_{i+1}\right)\frac{(1-Z_i)(1+Z_{i+1})}{4}$$

acts on sites i and $i+1$. Simplifying using the substitution $q := r/s$ and writing instead $U_i = U_{i,i+1}$, we have

$$U_i = \begin{pmatrix} q & 0 & 0 & 1 \\ 0 & q & 1 & 0 \\ 0 & 1 & q & 0 \\ 1 & 0 & 0 & q \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} q^{-1} & 0 & 0 & 1 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 1 & 0 & 0 & q^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.3.2}$$

This 4×4 matrix should be viewed as an element of $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ acting on sites i and $i+1$.

While the Hamiltonian $H = \sum_j U_j$ acts on the total space $\mathcal{H} = (\mathbb{C}^2)^{\otimes M}$, not all states in $\mathcal{H}$ are admissible. $2M$ electrons are numbered $1, \ldots, 2M$ on a circle, and 'Hulthén type' states are represented as non-crossing pairings of $2M$ points on a line with periodic boundary conditions, a.k.a. a circle. We only consider the space spanned by non-crossing partitions which have total spin zero, since crossings would corresond to crossings of bonds [TL71, (c) on p. 279]; On [TL71, p. 263] it is stated that there is a canonical bijection between:

- "the quantum states of an assembly of $2M$ electrons, with total spin zero (which implies a magnetic moment of zero)," and
- "The number of even-odd pairings of $2M$ points round a circle that can be realized without any crossing of the straight lines joining the M pairs."

Under this bijection, Temperley and Lieb give the following graphical representation of a basis of the admissible Hilbert space:

[12][34][56] [14][23][56] [12][36][45] [16][23][45] [16][25][34]

$M = 3.$

They use the notation μ_j is used for spin -1 in site j, so

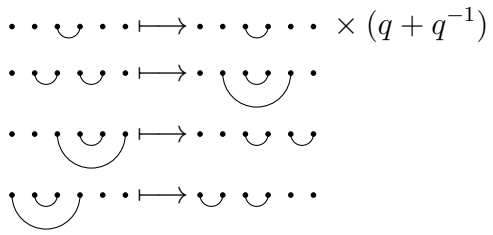
$$\frac{1+\mu_j}{2} = |0_j\rangle \qquad \text{and} \qquad \frac{1-\mu_j}{2} = |1_j\rangle.$$

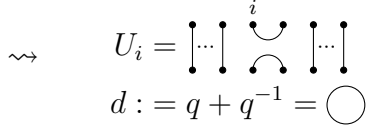
They then define

$$\begin{aligned} [ij] &:= r\frac{(1+\mu_i)(1-\mu_j)}{4} + s\frac{(1-\mu_i)(1+\mu_j)}{4} \\ &\propto q\frac{(1+\mu_i)(1-\mu_j)}{4} + 1\frac{(1-\mu_i)(1+\mu_j)}{4} \\ &= |0_i 1_j\rangle + q|1_i 0_j\rangle. \end{aligned}$$

The action of the local operators U_i on these states can also be represented diagrammatically. For example, in [TL71, Fig. 4], the local action of U_3 on states involving [34], [23][45], [36][45],

[14][23] is represented by:





(2.3.3)

We will see in Definition 2.4.4 below how this diagrammatic action can be interpreted as stacking a local string diagram operator on top, smoothing strings, and trading closed loops for a multiplicative factor of $d = q + q^{-1}$ as depicted on the right hand side of (2.3.3).

Exercise 2.3.4. Show that H_{TL} is not commuting projector.

Exercise 2.3.5. Find a singlet state vector $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ such that the matrix from (2.3.2) is of the form $d|\psi\rangle\langle\psi|$ where $d = q + q^{-1}$.
Hint: Under the isomorphism $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$*, the standard basis for* $\mathbb{C}^4$ *is given by lexicographic ordering:* $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

Exercise 2.3.6. Verify the relations on the left hand side of (2.3.3).

2.4. **Temperley-Lieb-Jones algebras and Kauffman diagrams.**

Definition 2.4.1 ([Jon83])**.** The n-th Temperley-Lieb-Jones algebra of modulus $d = q+q^{-1}$, denoted $TLJ_n(d)$, is the universal unital complex $*$-algebra generated by $1, E_1, \ldots, E_{n-1}$ subject to the relations

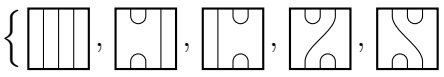
(TLJ1) $E_i E_j = E_j E_i$ whenever $|i - j| > 1$,
(TLJ2) $E_i E_{i\pm1} E_i = E_i$, and
(TLJ3) $E_i^2 = dE_i = dE_i^\dagger$.

Exercise 2.4.2. Prove that the operators U_i with modulus $d := q+q^{-1}$ satisfy the Temperley-Lieb-Jones relations (TLJ1) – (TLJ3).

Exercise 2.4.3.

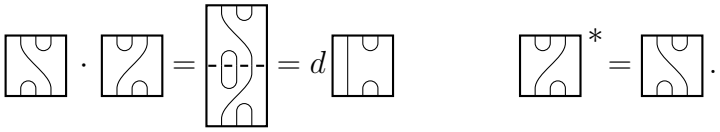
(1) Use the relations (TLJ1) – (TLJ3) to prove that any word in $E_1, \ldots, E_n$ is equal to a word with at most one E_n.
(2) Prove that $\dim(TLJ_n(d)) \leq \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.

Definition 2.4.4. For $d = q+q^{-1}$, we define $TLK_n(d)$ to be the complex vector space whose standard basis is the set of *Kauffman diagrams* [Kau87], which are non-intersecting string diagrams (up to isotopy) on a rectangle with n boundary points on the top and bottom. For example, the standard basis for $TLK_3(d)$ is given by



On $TLK_n(d)$, we define a multiplication by (the bilinear extension of) stacking boxes, removing the middle line segment, and smoothing the strings, and removing any closed loops and multiplying by a factor of d. We define an involution by (the anti linear extension of)

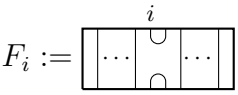
reflection about a horizontal line.



The multiplication and the adjoint make $TLK_n(d)$ a complex $*$-algebra.

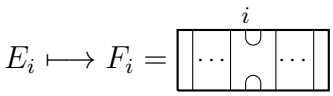
Exercise 2.4.5. Prove that $\dim(TLK_n(d)) = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.

Exercise 2.4.6. Prove that for $i = 1, \ldots, n-1$, the elements



satisfy the relations (TLJ1) – (TLJ3).

Exercise 2.4.7. Prove that the $*$-algebra map $\Phi_n : TLJ_n(d) \to TLK_n(d)$ given by



is a unital $*$-algebra isomorphism.
Hint: By Exercises 2.4.3 and 2.4.5, it suffices to prove Φ_n is surjective.

Exercise 2.4.8. Let T_i denote the transposition acting on sites i and $i+1$ on $(\mathbb{C}^2)^{\otimes N}$.
(1) Write the local 4×4 part of T_i in the computational basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.
(2) Show that at $q = -1$, $U_i = T_i - I$.

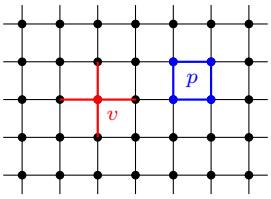
[[at $d = -2$, $TLJ(-2)$ is Rep$(SU(2))$, and $\mathbb{C}[S_n]$ have relationship...]]

2.5. Transfer matrices and matrix product states. TODO: Fill this in later

2.6. Toric code.

In this section, we define Kitaev's toric code lattice model [Kit03] and its application as a *stabilizer* error correction code for quantum computation.

Consider a square lattice L on a 2-torus; below we have periodic boundary conditions:



$$A_v = \bigotimes_{\ell \sim v} Z_\ell \quad B_p = \bigotimes_{\ell \sim p} X_\ell. \tag{2.6.1}$$

To each edge/link ℓ in the lattice, we assign the local Hilbert space $\mathcal{H}_\ell := \mathbb{C}^2 = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle$; the total space is then $\mathcal{H} = \bigotimes_{\ell \in L} \mathcal{H}_\ell$. There is a *vertex term* $A_v = \bigotimes_{v \sim \ell} Z_\ell$ for each vertex $v \in L$, where we write $\ell \sim v$ to denote ℓ is attached to v, and a *face/plaquette term* $B_p = \bigotimes_{\ell \sim p} X_\ell$, where we write $\ell \sim p$ to denote ℓ bounds p. The 4-local Hamiltonian is then given by

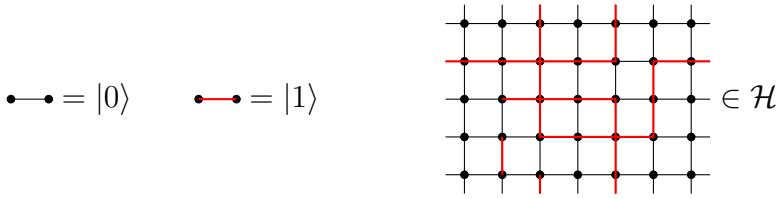
$$H_{\mathrm{TC}} := -\sum_v A_v - \sum_p B_p.$$

We remark that our definition here agrees with the one given in [ZCZW19, §3.5] and is dual to Kitaev's under swapping Pauli X with Pauli Z.

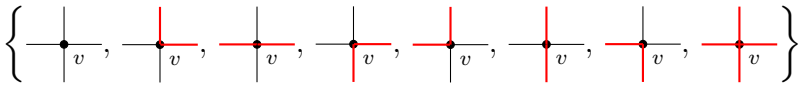
Exercise 2.6.2. Show that H_{TC} is commuting projector.

We now give a pictorial description of the toric code local Hamiltonian. I first heard this description from a colleague at the 2015 Oberwolfach meeting on Subfactors and Conformal Field Theory. It reminds me of the childhood game 'Lights Out' (see https://en.wikipedia.org/wiki/Lights_Out_(game); you can play here: https://www.neok12.com/games/lights-out/lights-out.htm).

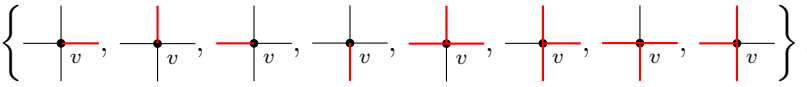
We view $|0\rangle$ as *off* and $|1\rangle$ as *on*. A computational basis vector in $\mathcal{H}$ can be pictorially represented by whether the corresponding edge is on or off:



For each vertex $v \in L$, the *vertex term* A_v ensures that an even number of edges are on at every vertex. Observe that A_v is a diagonal self-adjoint/Hermetian operator with 2 eigenvalues ± 1; the space E_1 is spanned by (amplifications of)



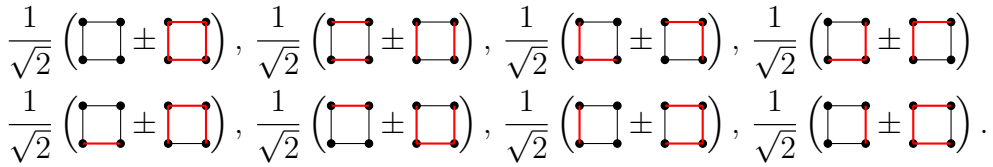
and the space E_{-1} is spanned by (amplifications of)



We thus take $-A_v$ in the Hamiltonian to energetically favor the $+1$ eigenspace over the -1 eigenspace

Exercise 2.6.3. Verify that $A_v = \bigotimes_{\ell \sim v} Z_\ell$ has the eigenspaces described above.

For each face/plaquette $p \in L$, the *plaquette term* B_p swaps which edges are on and off around the face/plaquette p; this is the 'Lights Out' operator. This means that B_p also has two eigenspaces corresponding to eigenvalues ± 1; $E_{\pm 1}$ is spanned by (amplifications of)

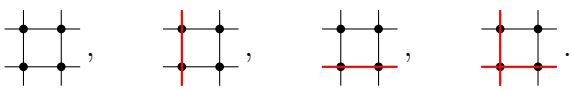


We thus take $-B_p$ in the Hamiltonian to energetically favor the $+1$ eigenspace over the -1 eigenspace.

We now compute the ground state space on the following lattice with 4 vertices and 8 edges, assuming periodic boundary conditions.

$$\mathcal{H} = (\mathbb{C}^2)^{\otimes 8}$$

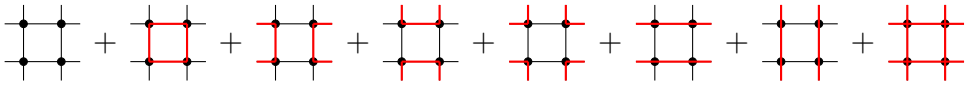
Since H is commuting projector, we may pass to the ground state space of $-\sum A_v$, so that we only consider the 'admissible' subspace where an even number of edges are on at every vertex. Consider the following four computational basis elements:

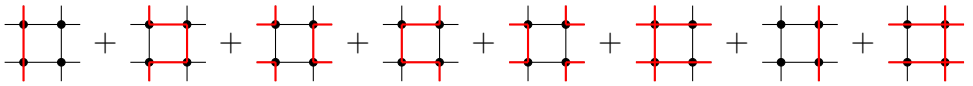


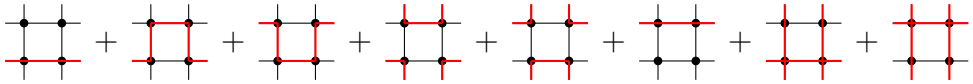
(2.6.4)

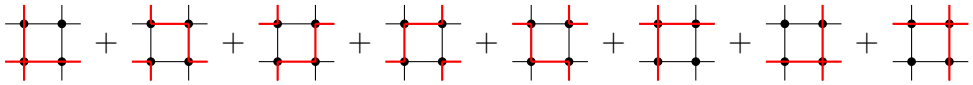
Exercise 2.6.5. Explain how these four diagrams can be viewed as representatives for the four elements of $H_1(\mathbb{T}^2, \mathbb{Z}/2)$.

We can now compute four ground states by 'averaging' over the four computational basis vectors in (2.6.4) by applying the operators $(I + B_p)/2$ for each plaquette p. This results in the following four ground states:

$|\psi_{00}\rangle =$ 

$|\psi_{01}\rangle =$ 

$|\psi_{10}\rangle =$ 

$|\psi_{11}\rangle =$ 

Exercise 2.6.6. Prove that these four states span the ground state space.

Exercise 2.6.7. Adapt the above procedure (or use Exercise 2.7.3 below) to prove that regardless of the lattice size, the dimension of the ground state space is 4.

Exercise 2.6.8 ($\star$)**.** Consider the toric code Hamiltonian H_{TC} on a lattice on a compact orientable surface Σ. Show that the dimension of the ground state space is $|H_1(\Sigma, \mathbb{Z}/2)|$. *Hint: Pick some representative computational basis diagram for each element of $H_1(\Sigma, \mathbb{Z}/2)$; call the space spanned by these diagrams $\mathcal{G}$. Construct injections both ways between the ground state space and $\mathcal{G}$. One way uses the averaging procedure above. For the other direction, take an arbitrary ground state and partition it into summands which represent the same element of $H_1(\Sigma, \mathbb{Z}/2)$. Show that one can pass between every representative of a cycle in $H_1(\Sigma, \mathbb{Z}/2)$ by applying the operators B_p. Then deduce each representative for a cycle appears in the sum with the same coefficient.*

2.7. **Toric code: stabilizer quantum error correction code.** The toric code is a quantum error correction code in the class of *stabilizer codes*.

Example 2.7.1 (Stabilizer codes)**.** Suppose $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$. Let $\mathcal{P}_N$ denote the group of Pauli operators generated by the X_i, Y_j, Z_k on $\mathcal{H}$. Let $\mathcal{S} \subset \mathcal{P}_N$ be any abelian subgroup (so $\mathcal{S}$ can be simultaneously diagonalized of $\mathcal{H}$), and set

$$\mathcal{H}^{\mathcal{S}} := \{|\psi\rangle \in \mathcal{H} |\, S|\psi\rangle = |\psi\rangle \text{ for all } S \in \mathcal{S}\}\,.$$

That is, $\mathcal{H}^{\mathcal{S}}$ is the *stabilizer* of the $\mathcal{S}$-action on $\mathcal{H}$.

[[more on this: $\mathcal{S}^{\perp}$, code distance]]

Exercise 2.7.2. Verify the following properties of the Pauli group $\mathcal{P}_N$.

(1) Every element of $\mathcal{P}_N$ can be written uniquely as ± 1 or $\pm i$ times an elementary tensor of Pauli matrices X_i, Y_j, Z_k.
(2) Deduce $|\mathcal{P}_N| = 2^{2n+2}$, so $\mathcal{P}_N$ is non-abelian of order 16. Which group is it.
(3) For every $u \in \mathcal{P}_N$, $u^2 = \pm I$.
(4) Any pair $u, v \in \mathcal{P}_N$ either commute or anti-commute.

Note: This exercise is adapted from the presentation of [Pre, §7.9.1] *which focuses on the 'real' subgroup of $\mathcal{P}_N$ generated by the X_i, iY_j, Z_k, which has index 2 in $\mathcal{P}_N$. So if one of the statements doesn't hold for $\mathcal{P}_N$, show it holds for this index 2 subgroup.*

Exercise 2.7.3 ([Pre, §7.9.1])**.** Prove that $\dim(\mathcal{H}^{\mathcal{S}}) = 2^{N-m(\mathcal{S})}$ where

$$m(\mathcal{S}) := \min\{|S| | S \text{ generates } \mathcal{S}\}. \tag{2.7.4}$$

Hint: One could proceed as follows.

(1) Show every $s \in \mathcal{S}$ satisfies $s^2 = I$.
(2) Show that the ± 1 eigenspaces of each $s \in \mathcal{S}$ have the same dimension, i.e., $\dim(E_{-1}) = \dim(E_1)$.
(3) Show by induction that for every minimal generating set $s_1, \ldots, s_m$ of $\mathcal{S}$, for each $1 \leq j \leq m-1$, there is a $u \in \mathcal{P}_N$ such that $us_i = s_i u$ for $1 \leq i \leq j$, but $us_{i+1} = -s_{i+1}u$.
(4) Deduce that the intersection of the $+1$ eigenspaces for $s_1, \ldots, s_j$ has dimension exactly twice that of the intersection of the $+1$ eigenspaces for $s_1, \ldots, s_{j+1}$.

Exercise 2.7.5. Show $H_1(\Sigma_g, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{2g}$ for a genus g compact orientable surface Σ_g.

Exercise 2.7.6. Suppose L is a square lattice on a compact orientable genus g surface Σ_g, where we assign a copy of $\mathbb{C}^2$ to each edge/link of L. Consider the subgroup $\mathcal{S} \subset \mathcal{P}_N$ generated by the vertex terms A_v and the plaquette terms B_p as in (2.6.1).

(1) Show that the only relations in $\mathcal{S}$ are $\prod_v A_v = I$ and $\prod_p B_p = I$.
(2) Use an Euler characteristic argument and Exercise 2.7.3 to show $\dim(\mathcal{H}^{\mathcal{S}}) = 4^g$. We thus say that the toric stabilizer code on Σ_g encodes $2g$ qubits.

The quantum code of the toric code as the ground state space. We now consider the error operators X_ℓ. Applying an X_ℓ at a single link lifts us out of the ground state space and into the first excited state of H_{TC}. If we now measure the vertex terms A_v for v and endpoint of ℓ, we will get the eigenvalue -1 instead of $+1$. If we constantly measure the A_v for our system, we can easily detect when an error occurs when we measure -1 for some pair of vertices. If they are adjacent, we can correct the error by applying X_ℓ on the edge connecting these vertices. If they are not adjacent, we may pick *any* path π connecting the vertices (e.g., a *shortest* path) and correct the error along this path (apply X_ℓ for every $\ell \in \pi$. If we chosen any other path π' connecting these vertices, observe that concatenating π and $(\pi')^{-1}$ gives a closed loop.

Exercise 2.7.7. Show that applying X_ℓ for every edge ℓ in a closed loop is exactly the product of the plaquette operators B_p such that p is in the interior of the closed loop. (Since the loop is on $\mathbb{T}^2$, the choice of which side is the interior is arbitrary.)

Now an error could go undetected is if the X_ℓ were applied along a closed loop which was homologically non-trivial; this could change our ground state to another linearly independent ground state. But if the torus is large enough/the lattice size is fine enough, the probability of this event would be low.

Remark 2.7.8. There are also error operators Y_ℓ, Z_ℓ; we will omit discussing these errors and instead discuss the lowest energy excitations. After the next section, it should be clear how to repeat the above discussion for the Z_ℓ. Since $Y = iXZ$, and our error correction protocol can remove both X and Z errors (in a way that does not create further errors!), we will also correct for Y errors along the way.

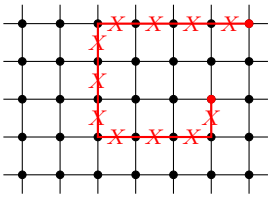
2.8. **Toric code: lowest energy excitations.** In the previous section, we say that $|\psi\rangle$ is in the ground state of the toric code local Hamiltonian H_{TC} if and only if

$$A_v|\psi\rangle = |\psi\rangle = B_p|\psi\rangle \qquad \forall\, v, p \in L.$$

A *lowest energy excitation* is an eigenvector $|\psi\rangle$ for H_{TC} corresponding to the energy level just above the ground state.

Exercise 2.8.1. Show that the lowest energy excitations violate exactly 2 of the conditions $A_v|\psi\rangle = |\psi\rangle$ or exactly 2 of the conditions $B_p|\psi\rangle = |\psi\rangle$.

We now describe *string operators* which produce these lowest energy excitations. Consider a path π in the lattice L:



$$S^X_\pi := \prod_{\ell\in\pi} X_\ell$$

We define the *string operator* $S^X_\pi := \prod_{\ell\in\pi} X_\ell$.

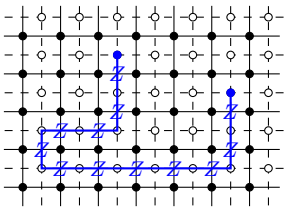
Exercise 2.8.2. Suppose π is a path on L with two distinct endpoints. (Note that π may self-intersect.)

(1) Show that S^X_π commutes with every B_p and every A_v *except* when v is one of the two endpoints of π.
(2) Suppose $|\psi\rangle$ is a ground state of H_{TC}. Show that $S^X_\pi|\psi\rangle$ is a lowest energy excitation. What is its energy eigenvalue?
(3) Show that whenever π and π' share the same endpoints, $S^X_\pi|\psi\rangle = S^X_{\pi'}|\psi\rangle$.

Remark 2.8.3. The string operator S^X_π is akin to a *ladder operator* for the quantum harmonic oscillator. **TODO: discuss quantum harmonic oscillator in basic QM section.**

We view the excited state $S^X_\pi|\psi\rangle$ as having two 'electric charge' *quasi-particle excitations* e living on the two endpoints of π. These quasi-particle excitations are *topological*, as the string operator which moves the locations of these e excitations does not change the energy of the system. That is, if we can concatenate paths π and π' to get a path $\pi + \pi'$ (the end of π' is the start of π, reading right to left), $S^X_\pi S^X_{\pi'} = S^X_{\pi+\pi'}$, and thus $S^X_\pi(S^X_{\pi'}|\psi\rangle) = S^X_{\pi+\pi'}|\psi\rangle$. We can thus topologically move e excitations at zero energy cost.

Now consider a path π^* on the *dual lattice* L^*:



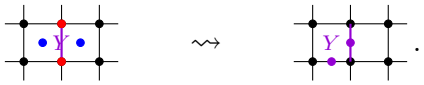
$$S^Z_{\pi^*} := \prod_{\ell\perp\pi^*} Z_\ell$$

We define the string operator $S^Z_{\pi^*} := \prod_{\ell\perp\pi^*} Z_\ell$, where we write $\ell \perp \pi^*$ to denote the edges/links $\ell \in L$ which are are orthogonal to the path π^*.

Exercise 2.8.4. Repeat Exercise 2.8.2 for $S^Z_{\pi^*}$.

We view the excited state $S^Z_{\pi^*}|\psi\rangle$ as having two 'magnetic flux' *quasi-particle excitations* m living on the two endpoints of π^*.

Remark 2.8.5. A composite of an e particle and an m particle is called an ϵ particle; a pair of such particles is created by applying a string operator $Y = iXZ$ along an edge/link:



We also have a *vacuum* (or empty) particle 1, which represents the lack of any lowest energy excitation. This gives four quasi-particle excitations: $1, e, m, \epsilon$; we will see later in [[]] that this four is the same four as the dimension of the ground state space on the 2-torus.

2.9. **Toric code: braiding statistics.** We can now calculate the *braiding statistics* of the e and m particles. Consider the following operators performed on a ground state. Pick two paths π on L and π^* on L^*. Applying the operator S^X_π twice creates and the annihilates the two electric charge e particles at the endpoints of π. Similarly, applying the operator $S^Z_{\pi^*}$ twice creates and the annihilates the two magnetic flux m particles at the endpoints of π^*. Observe that $S^X_\pi S^X_\pi = I$ and $S^Z_{\pi^*} S^Z_{\pi^*} = I$, so these operations are not very interesting separately.

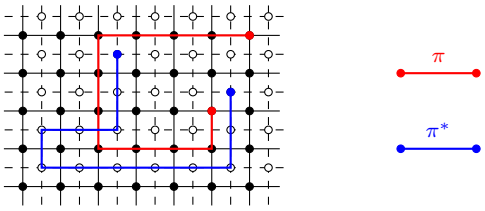
These observations, together with Remark 2.8.5, gives the following *fusion rules* amongst the particles e, m, ϵ, where we omit 1 which acts as the identity:

$\otimes$	e	m	ϵ
e	1	ϵ	m
m	ϵ	1	e
ϵ	m	e	1

,

which is the multiplication table for the group $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Now suppose that π, π^* intersect at exactly one edge/link ℓ. We create our pair of electric charge e particles along π, then create our pair of magnetic charge m particles along π^*, then annihilate our e particles along π, and then annihilate our m particles along π^*.

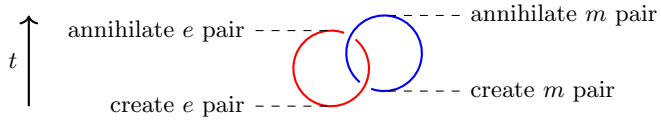


The result is the operator $S^Z_{\pi^*} S^X_\pi S^Z_{\pi^*} S^X_\pi$, which is no longer the identity. In fact, we get $-I$, as we get exactly one tensorand of the form $ZXZX = -I$. We thus say the e and m particles have *braiding statistic* -1.

Remark 2.9.1. Really, the toric code is a (2+1)D lattice model, in that operators are applied sequentially in time. We can view the time evolution of the above procedure as a movie in the 2D time slices as follows:

If we think of a particle moving in time, it looks like a string; the above operation can then be represented by the Hopf link where the e and m particles braid around one another, thus

leading to the name 'braiding statistics.'



Observe that for any two paths π, π' on L, the operators S^X_π and $S^X_{\pi'}$ commute, and thus we say the e particle has *self-braiding statistic* $+1$. Similarly, the m particle has self-braiding statistic $+1$.

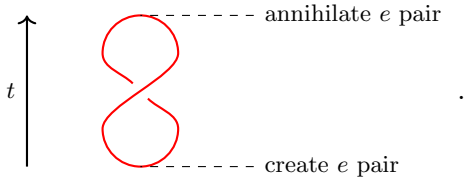
Exercise 2.9.2. Show that the self-braiding statistic of the ϵ particle is 1.

The matrix of braiding statistics of the system is called the *S-matrix* of the toric code and is given by

$$S_{\text{TC}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

where we have ordered the columns as $1, e, m, \epsilon$.

For a quasi-particle p of the toric code, *self-exchange* or *twist statistic* θ_p is the number we get by creating a pair of particles, exchanging them, and then annihilating them. For the e particle, this process is represented by the following string diagram:



.

The *T-matrix* is the diagonal matrix whose entries are the twist statistics.

Exercise 2.9.3. Verify that the T-matrix of the toric code is given by

$$T_{\text{TC}} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

where we again order the columns as $1, e, m, \epsilon$.

Definition 2.9.4. A quasi-particle p is called *invertible* or an *abelian anyon* if there is another quasi-particle q such that $p \otimes q = 1$. An abelian anyon is called:

- a *boson* if it has twist statistic $+1$,
- a *fermion* if it has twist statistic -1, and
- a *semion* if it has twist statistic $\pm i$.

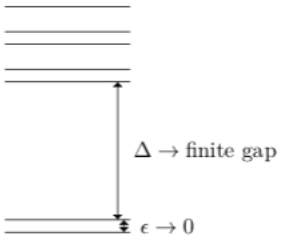
Later, we will see examples of *non-abelian anyons*, which have more exotic braiding and twist statistics.

Remark 2.9.5. The e and m particles are bosons, while the ϵ particle is a fermion. However, e and m show neither bosonic nor fermionic mutual exchange statistics, since $S_{em} = -1$. They actually show *semionic* mutual exchange statistics.

Remark 2.9.6. In [Kit03], Kitaev also produces a model based on any finite group G, which leads to the so-called *quantum double models* for finite groups. We will not go into detail here; rather, we will save this for the Levin-Wen string net model for the corresponding unitary fusion category $\mathsf{Hilb}_{\mathsf{fd}}(G)$ of G-graded finite dimensional Hilbert spaces.

2.10. **Gapped topological systems.** Suppose we have a quantum many-body system of size N, which varies as we take the thermodynamic limit $N \to \infty$. [[Yes, this is a bit vague.]] Denote by $\mathcal{H}_N$ the total Hilbert space and by H_N the local Hamiltonian on the system of size N. (Here, size may refer to the length of a quantum spin chain, or the length of a square lattice on $\mathbb{T}^2$, etc.) The system is called *gapped* if as $N \to \infty$, either:

- the ground state degeneracy has an upper bound, and the gap Δ_N between the energy of the ground state space and the energy of the space of lowest energy excitations is bounded below by some $\Delta > 0$.
- the dimension of the space of lowest energy excitations is finite, the energy gap ε_N between this space and the ground state space is exponentially small in N, and the energy gap Δ_N to the next space of excited states is bounded below by some $\Delta > 0$.



[arXiv:MR3929747, p113]

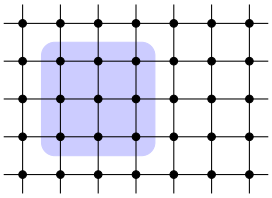
Suppose A is a contractible (simply connected) region in our 2D system. As $N \to \infty$ so the lattice gets finer, the number of sites in A grows as N^2. Recall that the *entanglement entropy* for the region A is given by $S_A := S(\rho_A^{\text{red}}) = -\operatorname{Tr}(\rho_A^{\text{red}} \log(\rho_A^{\text{red}}))$, where ρ_A^{red} is the reduced density matrix of some ground state $|\psi\rangle$.

Entanglement area law for 2D gapped quantum systems. The entanglement entropy S_A is proportional to N; that is, the *perimeter* of the region A.[a] Hence a gapped ground state contains far less entanglement than a 'generic quantum many-body entangled state' [ZCZW19, p118].
If the system is *topological* [[to be discussed below!]], then $S_A \sim \alpha N - \gamma$ where $\gamma > 0$ is called the *topological entanglement entropy*, which indicates the existence of *long-range entanglement* that arises from topological order.

[a] The term *area law* makes better sense for 3D gapped systems, where entanglement entropy is proportional to surface area rather than volume. In 2D, we could reasonably call this the entanglement perimeter law.

In the previous section, we saw that the toric code is a gapped (2+1)D quantum system. This system is in fact topological, and we will now calculate its topological entanglement entropy. Consider some contractible square A on $\mathbb{T}^2$ with sides of length N which plays

nicely with the lattice, e.g.,



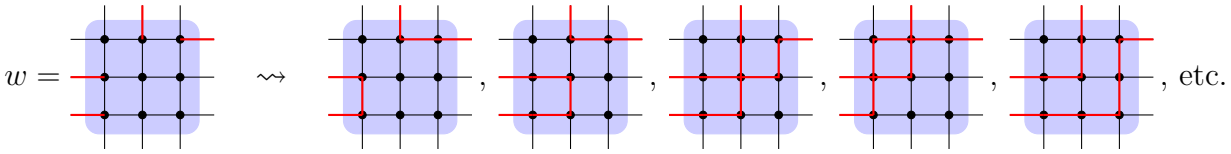
By convention, the edges which meet the boundary are outside of A.

A given ground state $|\psi\rangle$ is a linear combination of closed loops of 'on' edges of L. Observe that the number of intersections of these loops with ∂A, the boundary of A, is always even, and there are 2^{4N-1} ways w for half the edges $\ell_1, \ldots, \ell_{4N}$ to be 'on.'

Definition 2.10.1. Let $|\psi_A(w)\rangle$ be state vector given by the uniform sum of computational basis vectors on A such that:

- an even number of edges meeting every vertex v in A are 'on,' except when the vertex v meets boundary edge ℓ_i which happens to be 'on' in w, in which case there are an odd number of A-edges 'on.'

For example, each of the following terms would appear with the same coefficient $+1$ (up to normalization) in $|\psi_A(w)\rangle$ for the same w:



(Even though the boundary edges are not inside A, it is important to draw them when we represent terms of $|\psi_A(w)\rangle$.)

Now consider a toric code ground state, which is a linear combination of the four ground states from Exercise 2.7.6, each representing a different element of $H^1(\mathbb{T}^2, \mathbb{Z}/2)$.

$$|\psi\rangle = \lambda_{00}|\psi^{00}\rangle + \lambda_{01}|\psi^{01}\rangle + \lambda_{10}|\psi^{10}\rangle + \lambda_{11}|\psi^{11}\rangle \qquad |\lambda_{00}|^2 + |\lambda_{01}|^2 + |\lambda_{10}|^2 + |\lambda_{11}|^2 = 1$$

For $i, j \in \{0, 1\}$, let $|\psi^{ij}(w)\rangle$ denote the state vector obtained by projecting $|\psi^{ij}\rangle$ to the subspace spanned by the computational basis vectors which have boundary pattern w on ∂A and renormalizing so that

$$|\psi\rangle = \sum_{i,j=0}^{1} \lambda_{ij}|\psi^{ij}\rangle = \sum_{i,j=0}^{1} \frac{\lambda_{ij}}{\sqrt{2^{4N-1}}} \sum_w |\psi^{ij}(w)\rangle = \sum_{i,j=0}^{1} \frac{\lambda_{ij}}{2^{4N}} \sum_w |\psi^{ij}(w)\rangle. \tag{2.10.2}$$

Since A is contractible and small relative to the lattice size, we may write each

$$|\psi^{ij}(w)\rangle = \sum_w |\psi_A(w)\rangle \otimes |\psi^{ij}_{A^c}(w)\rangle \tag{2.10.3}$$

where $|\psi_A(w)\rangle$ was defined *independently* of $i, j \in \{0, 1\}$. Indeed, each term in $|\psi_A(w)\rangle$ appears with the same coefficient as we may pass between them by applying the operators B_p internal to A and remain in the state vector $|\psi^{ij}\rangle$.

Now in order to calculate ρ_A^{red}, we combine (2.10.2) and (2.10.3) to write

$$|\psi\rangle = \frac{1}{2^{4N}} \sum_w |\psi_A(w)\rangle \otimes \underbrace{\sum_{i,j=0}^{1} \lambda_{ij} |\psi_{A^c}^{ij}(w)\rangle.}_{=:\sigma_{A^c}(w)} = \frac{1}{2^{4N}} \sum_w |\psi_A(w)\rangle \otimes |\sigma_{A^c}(w)\rangle$$

where the $|\sigma_{A^c}(w)\rangle$ are orthonormal. Hence, we see

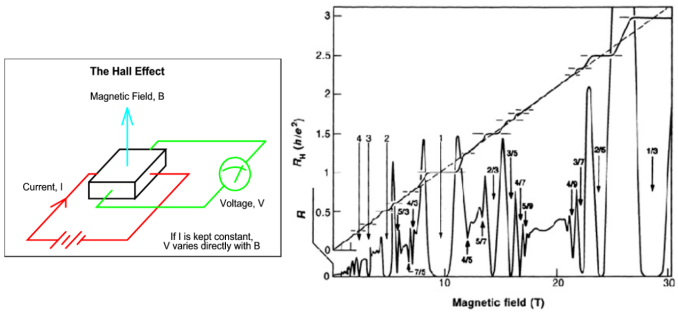
$$\rho_A^{\text{red}} = \mathbb{E}_A(|\psi\rangle\langle\psi|) = \frac{1}{2^{4N-1}} \sum_w |\psi_A(w)\rangle\langle\psi_A(w)|,$$

i.e., the reduced density matrix ρ_A^{red} is the uniform mixture of all the 'admissible' state vectors $|\psi_A(w)\rangle$ in the subsystem $\mathcal{H}_A$. We now compute that the von Neumann entropy is given by

$$S_A = -\operatorname{Tr}_A(\rho_A^{\text{red}} \log(\rho_A^{\text{red}})) = (4N-1)\log(2) \sim \alpha N - \log(2).$$

Thus $\gamma = \log(2)$.

2.11. **Topological order.** Until the 1980s, it was believed that Landau's theory of symmetry breaking described phases of matter. For example, the symmetry groups of H_2O in water and ice phases differ from continuous to lattice symmetry. The discovery of Chern-Simons topological quantum field theory (TQFT) and the Fractional Quantum Hall (FQH) Effect changed this. FQH states arise from a gapped quantum system where a 2D electron gas is subjected to a large magnetic field; the current density j_y then induces a transverse electric field E_x subject to the relation $E_x = R_H j_x$, where R_H is the *Hall coefficient*. Surprisingly, the Hall coefficient R_H is *quantized* and *rational* when measured in the units h/e^2 [**?**][1] (see [ZCZW19, p157]).



[ZCZW19, p4]

While the notion of topological order of a quantum many-body system remains beyond the scope of this class, we will give a list of ways that topological order can be observed. That is, when one sees several of the following characteristics of a quantum many-body system, one should suspect the presence of topological order.

(TO1) topological ground state degeneracy, i.e., ground state degeneracy that depends on the topology of the system. We have seen an example of this in the toric code.

(TO2) lowest energy quasi-particle excitations (*anyons*) can carry fractional and non-abelian statistics and fractional charges.

[1]Two of the coauthors Tsui and Störmer were awarded the Nobel Prize in 1998 along with Laughlin; recently, the 2016 Nobel Prize was awarded to Thouless, Haldane, and Kosterlitz for phase transitions between topological phases.

(TO3) *non-abelian geometric phase*, a unitary from a one parameter family of gapped Hamiltonians H_t, $t \in \mathbb{T}$. If d is the dimension of the ground state degeneracy on $\mathbb{T}^2$, then there are unitary matrices $S, T \in M_d(\mathbb{C})$ which arise from looking at 'squeezing' and 'shear' deformations of $\mathbb{T}^2$ respectively. It can be shown that the S and T matrices are invertible and satisfy the relations

$$S^2 = \alpha C \qquad (ST)^3 = \beta C \qquad \alpha\beta \neq 0,\ C = \pm I,$$

and thus give rise to a *projective representation* of the *modular group*

$$SL(2,\mathbb{Z}) := \{\text{determinant 1 integer matrices}\}.$$

Exercise 2.11.1. Prove the following facts about $SL(2,\mathbb{Z})$.

(1) Show that the center of $SL(2,\mathbb{Z})$ is $\{\pm I\}$.

(2) Prove that $SL(2,\mathbb{Z})$ has the following presentation:

$$\langle s, t | s^2 = (st)^3,\ s^4 = I \rangle. \tag{2.11.2}$$

Hint: Show that $\mathfrak{s} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $\mathfrak{t} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *satisfy the relations in* (2.11.2).

(3) We define $PSL(2,\mathbb{Z})$ as the quotient of $SL(2,\mathbb{Z})$ modulo its center. Prove that $PSL(2,\mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$.
Some authors refer to $PSL(2,\mathbb{Z})$ *as the modular group.*

(4) Prove that $PSL(2,\mathbb{Z})$ contains a free group of index 12.

These 'squeeze' and 'shear' deformations of $\mathbb{T}^2$ generate its *mapping class group*. In more detail, the mapping class group of a compact orientable surface Σ is the countable discrete group

$$\mathrm{MCG}(\Sigma) := \frac{\mathrm{Homeo}^+(\Sigma)}{\mathrm{Homeo}_0(\Sigma)}$$

where $\mathrm{Homeo}^+(\Sigma)$ is the group of orientation preserving homeomorphisms of Σ, and $\mathrm{Homeo}_0(\Sigma)$ is the connected component of the identity, which is a normal subgroup. For $\Sigma = \mathbb{T}^2$, $\mathrm{MCG}(\mathbb{T}^2) \cong SL(2,\mathbb{Z})$; one easily constructs a homomorphism $SL(2,\mathbb{Z}) \to \mathrm{MCG}(\mathbb{T}^2)$ via

$$SL(2,\mathbb{Z}) \ni A \longmapsto \left(x + \mathbb{Z}^2 \mapsto Ax + \mathbb{Z}^2\right)$$

after identifying $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

It was even believed by many researchers (e.g., see [ZCZW19, Box 6.5]) that the S and T matrices provide a *complete characterization* and definition of (2+1)D topological order. This has since been proven false by the existence of non-equivalent modular categories with the same modular date [MS17].

(TO4) The ground state space of the gapped quantum many-body system should be robust/stable to any local perturbation in the thermodynamic limit. It gives a quantum error correcting code with macroscopic distance. That is, for any orthonormal basis $\{|\gamma_i\rangle\}$ of the ground state space and any local operator M, we have

$$\langle \gamma_i | M | \gamma_j \rangle = c(M)\delta_{i=j},$$

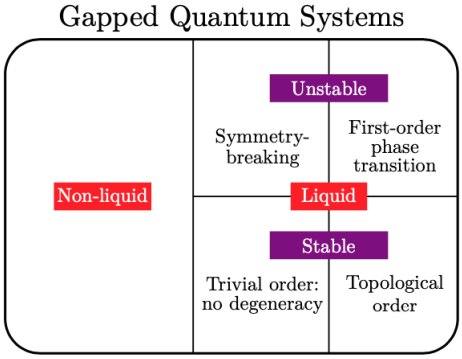
where $c(M)$ is a constant depending only on M.

(TO5) The low energy *effective theory* is a TQFT. That is, the ground state spaces on compact orientable surfaces forms a (2+1)D TQFT.

Quantum phases are equivalence classes of quantum many-body systems which can be 'smoothly deformed' into one another without inducing a singularity in any local observable. For gapped systems, one manifestation of 'non-smooth' deformation or *phase transition* is closure of the energy gap. The idea is that if we can parametrize the state of our gapped quantum many-body system as $|\psi(t)\rangle$, then measurement of any local observable M is given by $\langle\psi(t)|M|\psi(t)\rangle$, which will vary smoothly in t provided the energy gap $\Delta_t > 0$. This is shown via *perturbation theory*.

Gapped quantum phases. Two gapped quantum many-body systems $(\mathcal{H}_t, H_t)$ for $t = 0, 1$ are in the same phase if and only if there is a smooth path H_t of local Hamiltonians connecting H_0 and H_1 which has energy gap $\Delta_t > 0$ for all $t \in [0, 1]$. Such a path is called an *adiabatic evolution* of the Hamiltonian.

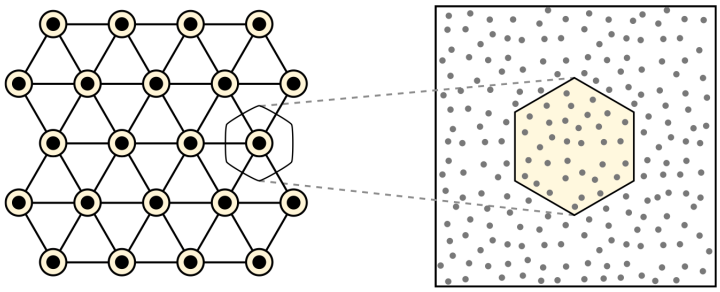
[[Something more about trading gapped Hamiltonians for gapped ground states and *local unitary* evolution. One such example of local unitary evolution is given by conjugating the local Hamiltonian H by a *finite-depth quantum circuit* U. Every finite time unitary evolution of a local Hamiltonian can be simulated by a constant depth quantum circuit.]] [[For future: RG flow]]



[ZCZW19, Fig. 7.12, p204]

2.12. **Fusion rules from entanglement** [SKK20]. **TODO: edit this section below**

Next week, we will introduce lattice models with local Hamiltonians. Basically, a 2D lattice model is an assignment of Hilbert spaces to vertices and edges of some graph on a 2D surface, together with a Hamiltonian built from locally acting operators. One can simplify to nearest-neighbor models by *coarse graining*.



(a) (b)

[SKK20, p4]

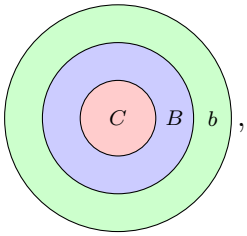
We will assume there is some global *reference state* σ, which can be taken to be pure $|\psi\rangle\langle\psi|$ if we want. Denote by $\mathcal{B}_r$ the set of balls of radius at most r. For $b \in \mathcal{B}_r$, we denote by σ_b the reduced density matrix of σ on b obtained by conditionally expecting to only the subsystem contained in b. We define

$$\mathbf{RDM}(\sigma, r) := \{\sigma_b | b \in \mathcal{B}_r\},$$

and for a certain $r > 0$, we set $\mathbf{R} = \mathbf{RDM}(\sigma, r)$.

We assume the following axioms for all $\sigma_b \in \mathbf{R}$.

(A0) For any $\sigma_b \in \mathbf{R}$ and any subsystems $B, C \subset b$ which topologically look like



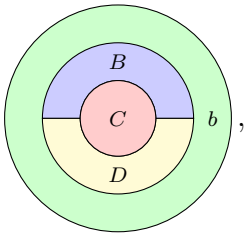
,

we have the following identity of von Neumann entropy:

$$S(\sigma_{BC}) + S(\sigma_C) - S(\sigma_B) = 0.$$

This says that the correlation between two sufficiently separated subsystems is zero.

(A1) For any $\sigma_b \in \mathbf{R}$ and any subsystems $B, C \subset b$ which topologically look like



,

we have the following identity of von Neumann entropy:

$$S(\sigma_{BC}) + S(\sigma_{CD}) - S(\sigma_B) - S(\sigma_D) = 0.$$

This says that although one can merge the reduced density matrices σ_{BC} and σ_{CD} into a reduced density matrix σ_{BCD} which recovers these reduced densities. This is not always possible in general (eg., if σ_{BC}, σ_{CD} are both maximally entangled states).

Remark 2.12.1. It is claimed these axioms may be derived from the *entanglement area law*

$$S(A) = \alpha\ell - \gamma$$

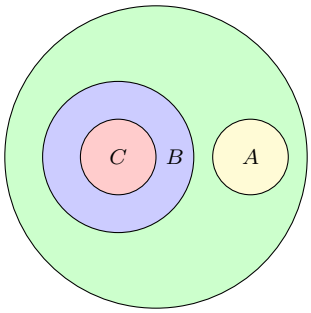
which we will discuss after we talk about lattice models next week.

Exercise 2.12.2. Assume (A0) and (A1).

(1) Prove that (A0) and strong subadditivity of von Neumann entropy implies

$$I(A : C) = S(\sigma_A) + S(\sigma_C) - S(\sigma_{AC}) = 0.$$

for any A contained in the complement of BC.



(2) Prove that (A1) implies

$$I(A : C|B) = S(\sigma_{AB}) + S(\sigma_{BC}) - S(\sigma_B) - S(\sigma_{ABC}) = 0$$

for any A contained in the complement of BCD.

Using these axioms, the authors show that the *information convex set* of reduced densities on a certain subsystem recovers modular data of supposed unitary modular tensor category (UMTC) describing the *topological order* of the system! The main results are as follows:

(1) If A and B may be topologically deformed to one another, their information convex sets are isomorphic, i.e., there is an isometric bijection Φ which preserves ΔS:

$$S(\rho) - S(\rho') = S(\Phi(\rho)) - S(\Phi(\rho')).$$

(2) When A is the annulus, the information convex set is a finite dimensional simplex, and the extreme points $\{\sigma^a\}$ of this simplex are shown to satisfy certain fusion rules with non-negative integer coefficients.

(3) The information convex set of a 2-hole disk/'pair of pants' is also a simplex, and the number of extreme points gives the dimension of certain *fusion channels* in the supposed UMTC.

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