7. 3-categories

7.1. **k-tuply monoidal n-categories.** Just as there is a notion of a monoidal structure on a category, we can define a monoidal structure on an n-category. In fact, it is possible to define multiple levels of monoidality on an n-category. For example, we will see that a braided monoidal category is a 2-tuply monoidal 1-category, as the braiding should be viewed as a higher monoidal structure on the category. Just as a monoidal category is exactly the same data as a 2-category with one object, a braided monoidal category is exactly the same data as a 3-category with one object. The *Delooping Hypothesis* [BS10, Hyp. 22] makes the correspondence between k-tuply monoidal n-categories and $(k-i)$-fold degenerate $(n+i)$-categories precise.

Hypothesis 7.1.1 (Delooping [BS10, Hyp. 22])**.** *There is an adjoint pair* $\mathrm{B}^i \dashv \Omega^i$ *between*

$$\{k\text{-monoidal } n\text{-categories}\} \underset{\Omega^i}{\overset{\mathrm{B}^i}{\rightleftarrows}} \{\text{pointed } (k-i)\text{-monoidal } (n+i)\text{-categories}\}$$

Moreover, B^i, Ω^i *restrict to an equivalence when we restrict the right hand side to i-fold degenerate pointed $(k-i)$-monoidal $(n+i)$-categories.*

Here, the term *pointed* means equipped with a distinguished i-functor from the terminal i-category B^i*; moreover, all higher morphisms between pointed $(k-i)$-monoidal $(n+i)$-categories must preserve these pointings. The following hypothesis further refines the Delooping Hypothesis 7.1.1. Indeed, one can justify this hypothesis by assuming [BS10, Hyp. 17]; we will not comment further on this here.

Hypothesis 7.1.2 ([JPR20, Hyp. 1.2])**.** *Let $\mathcal{G}$ be a n-category. The full $(n+1)$-subcategory of the under-$(n+1)$-category $n\mathsf{Cat}_{\mathcal{G}/}$ on the k-surjective functors out of $\mathcal{G}$ is an $(n-k)$-category, i.e., all hom $(k+1)$-categories between parallel $(n-k)$-morphisms are contractible.*

Here, we call an n-functor $F : \mathcal{C} \to \mathcal{D}$ of n-categories *k-surjective*[1] if it is essentially surjective on objects and parallel r-morphisms for $r \leq k$. By convention, any functor is (-1)-surjective. The under-$(n+1)$-category is the $(n+1)$-category of n-categories equipped with n-functors out of $\mathcal{G}$, and higher morphisms must be compatible with these n-functors from $\mathcal{G}$. For $k = 1, \ldots, n-1$ morphisms, this compatibility is a structure on the k-morphism, and at the top level, compatibility is a property.

In fact, the Delooping Hypothesis 7.1.1 is often used to *define* the notion of k-tuply monoidal n-category. The following chart of Baez-Dolan-Shulman shows how multiple levels of monoidality and higher categories interact, and in fact *stabilize*.

[1]This notion of k-surjectivity does not coincide with the one used in [BS10], where a functor is said to be k-surjective if it is essentially surjective on k-morphisms.

k-tuply monoidal n-categories [BD95, BS10]**.** For a k-tuply monoidal n-category, being trivial at level k corresponds to extra structure on an n-category, except at level $n-1$, which is a property of an $(n+2)$-tuply monoidal n-category.

	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$
$k=0$	$*=T$	$\{T,F\}$	set	category	2-category
$k=1$	"	[[?]]	monoid	monoidal	monoidal
$k=2$	"	"	commutative	braided	braided
$k=3$	"	"	"	symmetric	sylleptic
$k=4$	"	"	"	"	symmetric
$k=5$	"	"	"	"	"

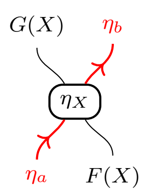
In the chart above, we included columns for $n=-2,-1,0$, when strictly speaking, these values of n donot give categories. It is helpful to think of these levels as 'lower' categories. **TODO: negative categorical thinking** [BS10] **using the homotopy hypothesis.**

7.2. **The 3-category of 2-categories.** In this section, we sketch the definition of the 3-category of 2-categories [Gur13, §5].

TODO: think about conventions below; do they match with those from fusion categories and MTCs?

Definition 7.2.1. The 3-category $2\mathsf{Cat}$ has:

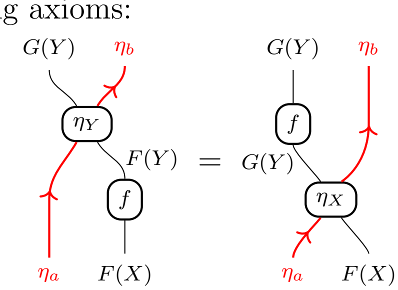
- objects 2-categories,
- 1-morphisms $F:\mathcal{C}\to\mathcal{D}$ are 2-functors,
- 2-morphisms $\eta : F \Rightarrow G$ are 2-transformations. That is, η assigns to each object $c\in\mathcal{C}$ a 1-morphism $\eta_c\in\mathcal{D}(F(c)\to G(c))$ and to each 1-morphism $X\in\mathcal{C}(a\to b)$, an invertible 2-morphism



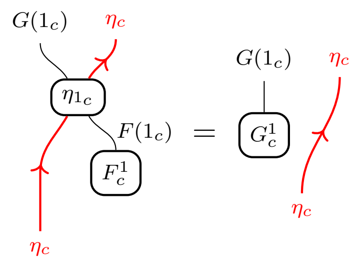
$$\in \mathcal{C}(\eta_a\otimes_a F(X)\otimes_b \to G(X)\otimes\eta_b).$$

The 2-transformation satisfies the following axioms:

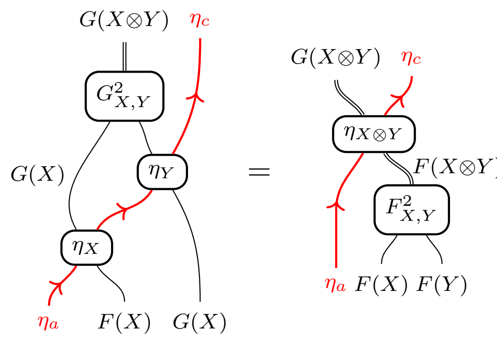
– (naturality) for all $f\in\mathcal{C}({}_aX_b\to {}_aY_b)$,



– (unitality) For all $c \in \mathcal{C}$,

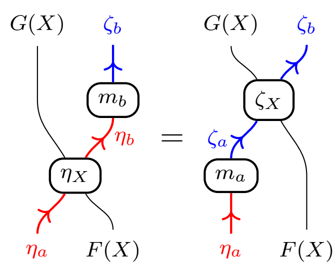


– (monoidality) for all ${}_aX_b, {}_bY_c \in \mathcal{C}$,



.

- 3-morphisms $m : \eta \Rrightarrow \zeta$ are 2-modifications. That is, m assigns to each object $c \in \mathcal{C}$ a 2-morphism $m_c \in \mathcal{C}(\eta_c \Rightarrow \zeta_c)$ which satisfies the following axiom:



$\forall\, {}_aX_b \in \mathcal{C}(a \to b)$.

In order to see how the above data compiles into a 3-category, we observe that by an exercise from the 2-categories module, for all 2-categories, $\mathcal{C}, \mathcal{D}$ $\mathrm{Hom}(\mathcal{C} \to \mathcal{D})$ itself forms a 2-category. The remaining data to define a 3-category is the 1-composition 2-functor

$$\boxtimes : \mathrm{Hom}(\mathcal{D} \to \mathcal{E}) \boxtimes \mathrm{Hom}(\mathcal{C} \to \mathcal{D}) \longrightarrow \mathrm{Hom}(\mathcal{C} \to \mathcal{E})$$

together with higher coherences. For now, we refer the reader to [Gur13, §5].

7.3. **Aside: monoidal categories are 2-categories with one object.** In this section, we prove the Delooping Hypothesis 7.1.1 for $k = 1, n = 2, i = 1$ to show that *monoidal categories are 2-categories with one object*. This section is intended to be self-contained, and the notation may clash with other sections.

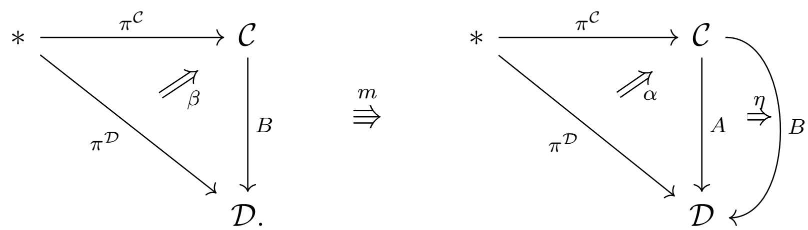
We repeat the following exercise from the 2-categories module:

Exercise 7.3.1. Show that if a 1-morphism ${}_aX_b$ is invertible in a 2-category $\mathcal{C}$, then there is an inverse ${}_bY_a$ such that the isomorphisms $1_a \cong {}_aX \otimes_b Y_a$ and $1_b \cong {}_bY \otimes_a X_b$ also satisfy the zig-zag/snake relations. The 1-morphism ${}_aX_b$ equipped with such an inverse is called an *adjoint equivalence* between $a, b \in \mathcal{C}$.

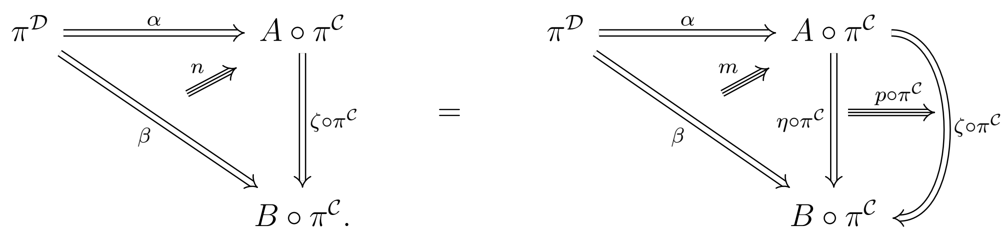
Definition 7.3.2. Let $* := B\{e\}$ be the trivial 2-category. Consider the 3-category $2\mathsf{Cat}_*$:

- objects: pairs $(\mathcal{C}, \pi^{\mathcal{C}})$ where $\mathcal{C}$ is a 2-category and $\pi^{\mathcal{C}} : * \to \mathcal{C}$ is a 2-functor which is essentially surjective on objects. We write $1_{\mathcal{C}} := \pi^{\mathcal{C}}(\mathrm{id}_*)$.

- 1-morphisms: $(A,\alpha) : (\mathcal{C},\pi^{\mathcal{C}}) \to (\mathcal{D},\pi^{\mathcal{D}})$ consists of a 2-functor $(A, A^1, A^2) : \mathcal{C} \to \mathcal{D}$ together with a natural isomorphism $\alpha : \pi^{\mathcal{D}} \Rightarrow A \circ \pi^{\mathcal{C}}$.
- 2-morphisms: $(\eta, m) : (A,\alpha) \Rightarrow (B,\beta)$ consists of a natural transformation $\eta = (\eta_*, \eta_a) : A \Rightarrow B$ and an invertible modification



- 3-morphisms: $p : (\eta, m) \Rrightarrow (\zeta, n)$ consists of a modification $p : \eta \Rrightarrow \zeta$ such that the following diagram commutes, where we suppress whiskering from the notation:



(7.3.3)

Theorem 7.3.4. *The 3-category* $2\mathsf{Cat}_*$ *is equivalent to the full (at the 3-morphism level) 3-subcategory* $2\mathsf{Cat}_*^{\mathsf{pt}}$ *with*

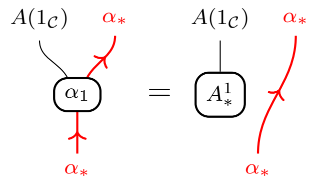
- *objects* $(\mathcal{C},\pi^{\mathcal{C}})$ *are those objects of* $2\mathsf{Cat}_*$ *for which* $\mathcal{C}$ *is a strict 2-category with exactly one object and* $\pi^{\mathcal{C}} : * \to \mathcal{C}$ *is a strict 2-functor. For such an object, we define* $1_{\mathcal{C}} := \mathrm{id}_{*_{\mathcal{C}}}$.
- *1-morphisms* $(A,\alpha) : (\mathcal{C},\pi^{\mathcal{C}}) \to (\mathcal{D},\pi^{\mathcal{D}})$ *satisfy* $A(*_{\mathcal{C}}) = *_{\mathcal{D}}$ *and* $\alpha_* = 1_{\mathcal{D}}$ *and* $\alpha_{1_{\mathcal{C}}} = A^1_*$,
- *2-morphisms* $(\eta, m) : (A,\alpha) \Rightarrow (B,\beta)$ *satisfy* $\eta_* = 1_{\mathcal{D}}$ *and* $m = \mathrm{id}_{1_{\mathcal{D}}}$, *and*
- *all 3-morphisms.*

Proof.

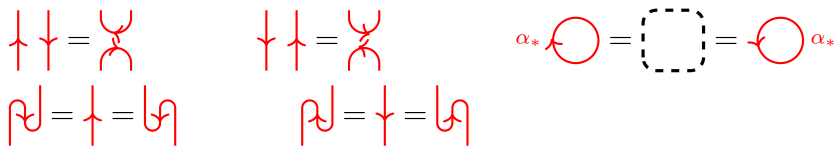
Strictifying objects: First, observe that every object $(\mathcal{C},\pi)$ of $2\mathsf{Cat}_*$ is equivalent to one of the form $(\mathcal{C}',\pi')$ where $\mathcal{C}'$ is a strict 2-category with exactly one object and $\pi' : * \to \mathcal{C}'$ is a strict 2-functor which is surjective on objects. Indeed, let $\mathcal{C}'$ be the full 2-category of $\mathcal{C}$ whose only object is $\pi(*)$. Set $\pi'(*) = \pi(*)$ so that $\pi'(e) = 1_{\pi(*)}$ and $\pi'(\mathrm{id}_e) = \mathrm{id}_{1_{\pi(*)}}$, and set all higher coherence data to be identities. One then checks $(\mathcal{C},\pi) \cong (\mathcal{C},\pi')$.

Strictifying 1-morphisms: Suppose now we have objects $(\mathcal{C},\pi^{\mathcal{C}}), (\mathcal{D},\pi^{\mathcal{D}}) \in 2\mathsf{Cat}_*^{\mathsf{pt}}$ and a 1-morphism $(A,\alpha) : (\mathcal{C},\pi^{\mathcal{C}}) \to (\mathcal{D},\pi^{\mathcal{D}})$ in $2\mathsf{Cat}_*$, so that $A = (A, A^1, A^2) : \mathcal{C} \to \mathcal{D}$ is a 2-functor and $\alpha = (\alpha_*, \alpha_1) : \pi^{\mathcal{D}} \Rightarrow A \circ \pi^{\mathcal{C}}$ is a 2-natural isomorphism. The 2-natural isomorphism α is comprised of an invertible object α_* which we depict by an red oriented strand:

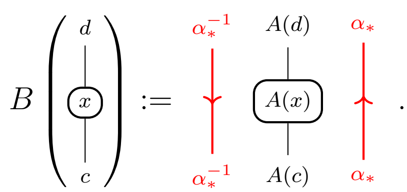
$\alpha_* =$  , and an invertible 2-morphism $\alpha_1 : \alpha_* \boxtimes 1_{\mathcal{D}} = \alpha_* \Rightarrow A(1_{\mathcal{C}}) \boxtimes \alpha_*$ which by the unitality axioms of a natural isomorphism, is equal to



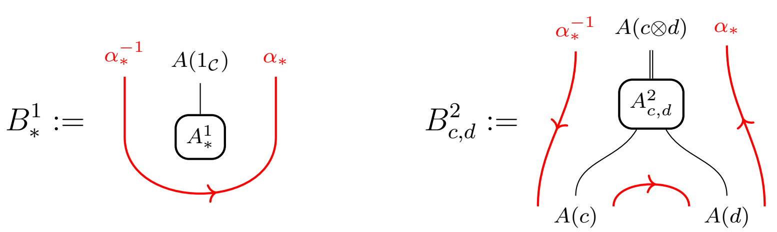
where $A^1_* : 1_{\mathcal{D}} \Rightarrow A(1_{\mathcal{C}})$ is the unitor of A. By Exercise 7.3.1, α_* is part of an adjoint equivalence, giving us the following relations:



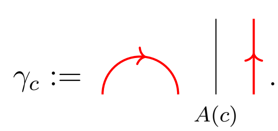
We claim there is an equivalent 1-morphism $(B, \beta) \in 2\mathsf{Cat}^{\mathsf{pt}}_*((\mathcal{C}, \pi^{\mathcal{C}}) \to (\mathcal{D}, \pi^{\mathcal{D}}))$. Indeed, we define $B(*) = A(*)$, $B(c) = \alpha_*^{-1} \otimes A(c) \otimes \alpha_*$ for all $c \in \mathcal{C}$, and for a 2-morphism $x : c \to d$ in $\mathcal{C}$, we define



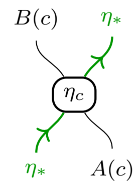
We define the unitor B^1 and tensorator B^2 by



It is straightforward to check that (B, B^1, B^2) is a 2-functor, and $\beta = (\beta_* := 1_{\mathcal{D}}, \beta_1 := B^1_*) : \pi^{\mathcal{D}} \Rightarrow B \circ \pi^{\mathcal{C}}$ is a 2-natural isomorphism. We now observe that $(A, \alpha) \cong (B, \beta)$ in $2\mathsf{Cat}_*$ via the 2-morphism $(\gamma, \mathrm{id}) : (B, \mathrm{id}) \Rightarrow (A, \alpha)$ where $\gamma = (\gamma_*, \gamma_c) : B \Rightarrow A$ is given by $\gamma_* = \alpha_*$ and $\gamma_c : \gamma_* \otimes B(c) \Rightarrow A(c) \otimes \gamma_*$ is given by



Strictifying 2-morphisms: Suppose we have two 1-morphisms $(A, \alpha), (B, \beta) \in 2\mathsf{Cat}^{\mathsf{pt}}_*((\mathcal{C}, \pi^{\mathcal{C}}) \to (\mathcal{D}, \pi^{\mathcal{D}}))$, which means that $\alpha_* = 1_{\mathcal{D}} = \beta_*$, $\alpha_1 = A^1_*$, $\beta_1 = B^1_*$. Suppose we have a 2-morphism $(\eta, m) \in 2\mathsf{Cat}_*((A, \alpha) \Rightarrow (B, \beta))$. This means $\eta = (\eta_*, \eta_c) : A \Rightarrow B$ is a 2-natural transformation where η_* is an invertible 1-morphism in $\mathcal{D}$ and $\eta_c : \eta_* \otimes A(c) \Rightarrow B(c) \otimes \eta_*$ is depicted graphically by



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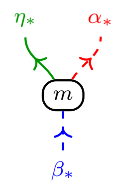
and the invertible modification $m : \beta_* \Rrightarrow \alpha_* \otimes \eta_*$ gives an isomorphism $1_{\mathcal{D}} \cong \eta_*$, which we denote graphically by



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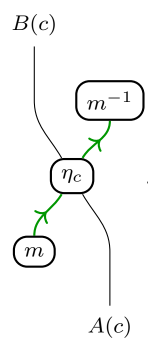
We claim there is an equivalent 2-morphism $(\zeta, \mathrm{id}) : (A, \alpha) \Rightarrow (B, \beta)$ such that $\zeta = (\zeta_*, \zeta_c)$ satisfies $\zeta_* = 1_{\mathcal{D}}$. Indeed, we just set $\zeta_* = 1_{\mathcal{D}}$ and for $c \in \mathcal{C}$, we define

$\zeta_c :=$



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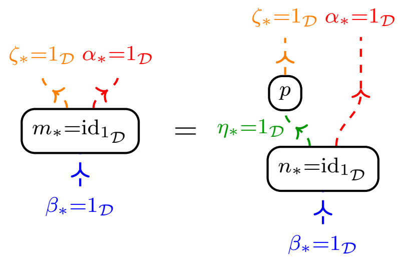
It is straightforward to verify that $\zeta : A \Rrightarrow B$ is a 2-natural transformation. Moreover, since $\alpha_* = 1_{\mathcal{D}} = \beta_*$, we may view m as an invertible 2-cell $m : 1_{\mathcal{D}} \Rightarrow \eta_*$. Finally, it is easily checked that m is exactly an invertible 3-morphism in $2\mathsf{Cat}_*((\zeta, \mathrm{id}) \Rrightarrow (\eta, m))$. □

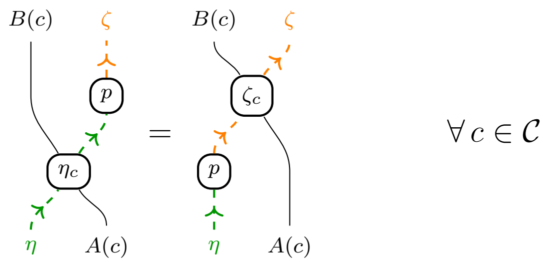
Remark 7.3.5. With a little more effort, we can arrange that 1-morphisms $(A, \alpha) \in 2\mathsf{Cat}_*^{\mathsf{pt}}(\mathcal{C} \to \mathcal{D})$ satisfy the additional strictness property that $A(1_{\mathcal{C}}) = 1_{\mathcal{D}}$ and $A_*^1 : 1_{\mathcal{D}} \to A(1_{\mathcal{C}})$ is equal to $\mathrm{id}_{1_{\mathcal{D}}}$. Indeed, in the construction of B in the proof above, one makes the following changes by hand: $B(1_{\mathcal{C}}) := 1_{\mathcal{D}}$ and $B^1_{1_{\mathcal{C}}} := \mathrm{id}_{1_{\mathcal{D}}}$, and $\beta = \mathrm{id}$. In the equivalence $(\eta, \mathrm{id}) : (B, \mathrm{id}) \Rightarrow (A, \alpha)$, one then defines $\eta_{1_{\mathcal{C}}} := \alpha_1$. Making B strictly unital has the advantage that $\pi^{\mathcal{D}} = B \circ \pi^{\mathcal{C}}$ on the nose, and we may choose $\beta = \mathrm{id}$.

Corollary 7.3.6. *The 3-category* $2\mathsf{Cat}_*^{\mathsf{pt}}$ *is 1-contractible, i.e., there are only identity 3-morphisms.*

Proof. Given two 2-morphisms $(\eta, \mathrm{id}), (\zeta, \mathrm{id}) \in 2\mathsf{Cat}_*^{\mathsf{pt}}((A, \mathrm{id}) \Rightarrow (B, \beta))$ such that $\eta_* = 1_{\mathcal{D}} = \zeta_*$, if there is a 3-morphism $p : (\eta, \mathrm{id}) \Rrightarrow (\zeta, \mathrm{id})$, then $\eta = \zeta$ and $p = \mathrm{id}$. Indeed, (7.3.3) in string diagrams is the equation



which implies $p = \mathrm{id}_{1_{\mathcal{D}}}$. Finally, the relation



together with $p = \mathrm{id}_{1_{\mathcal{D}}}$ implies that $\eta_c = \zeta_c$ for all $c \in \mathcal{C}$. □

Corollary 7.3.7. *The 3-category* $\mathsf{2Cat}_*^{\mathsf{pt}}$ *is isomorphic to the 2-category* $\mathsf{2Cat}_*^{\mathrm{st}}$ *with*

- *objects strict 2-categories with exactly one object,*
- *1-morphisms are 2-functors* $A : \mathcal{C} \to \mathcal{D}$*, and*
- *2-morphisms are natural transformations* $\eta = (\eta_*, \eta_a) : A \Rightarrow B$ *such that* $\eta_* = 1_{\mathcal{D}}$*.*

This 2-category is manifestly isomorphic to the 2-category $\mathsf{MonCat}^{\mathrm{st}}$ *of strict monoidal categories, strong monoidal functors, and monoidal natural transformations.*

Remark 7.3.8. Since every monoidal category is equivalent to a strict monoidal category [ML98], we have the following equivalences $\sim$ and isomorphisms $\cong$ of categories:

$$\mathsf{2Cat}_* \xleftrightarrow{\sim} \mathsf{2Cat}_*^{\mathsf{pt}} \xleftrightarrow{\cong} \mathsf{2Cat}_*^{\mathrm{st}} \xleftrightarrow{\cong} \mathsf{MonCat}^{\mathrm{st}} \xleftrightarrow{\sim} \mathsf{MonCat}$$

7.4. **3-categories.** The notion of weak 3-category known as a *tricategory* is due to [GPS95], which was later refined in [Gur06, Gur13] to the notion of *algebraic tricategory*. The main difference here is that to work with a 3-category, it is helpful to augment every 2-isomorphism with the structure of an adjoint equivalence (see Exercise 7.3.1), and every 1-isomorphism with the structure of a *biadjoint biequivalence* [Gur12].

To define these notions, it helps to work with a graphical calculus for 3-categories. Just as 2-categories admit a 2D graphical calculus, 3-categories admit a 3D graphical calculus. To prove that the 2D graphical calculus was well-defined, we appeal to the fact that every 2-category is equivalent to a strict 2-category. But really the proof is a bit more complicated.

Given a 2-category $\mathcal{C}$, we can replace it with the 2-category $\widehat{\mathcal{C}}$ with the same objects as $\mathcal{C}$, whose 1-morphisms are formal fully-parenthesized composites of 1-morphisms in $\mathcal{C}$, and whose 2-morphisms are generated by the 2-morphisms of $\mathcal{C}$. Sometimes this $\widehat{\mathcal{C}}$ is called a *cofibrant replacement* of $\mathcal{C}$, as it is a cofibrant object in some *model category structure* on a 1-category of 2-categories, but we will simply use the term *cofibrant* to mean there is no accidental equality of 1-morphisms. The 2-category $\widehat{\mathcal{C}}$ comes with a *strict* evaluation

2-equivalence $\mathrm{ev} : \widehat{\mathcal{C}} \to \mathcal{C}$, which evaluates every formal composite 1- and 2-morphism back into $\mathcal{C}$. Now there is a strict 2-category $\mathcal{C}^{\mathrm{st}}$ and an equivalence $\widehat{\mathcal{C}} \to \mathcal{C}^{\mathrm{st}}$. However, as $\widehat{\mathcal{C}}$ is cofibrant, this equivalence is itself isomorphic to a *strict equivalence* $\mathrm{st} : \widehat{\mathcal{C}} \to \mathcal{C}^{\mathrm{st}}$. We thus have the following zig-zag of strict equivalences:

$$\mathcal{C} \xleftarrow{\mathrm{ev}} \widehat{\mathcal{C}} \xrightarrow{\mathrm{st}} \mathcal{C}^{\mathrm{st}}. \tag{7.4.1}$$

Observe we also have a canonical inclusion $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ which splits ev by sending each object, 1-morphism, and 2-morphism of $\mathcal{C}$ to the same object, one word 1-morphism, and generating 2-morphism in $\widehat{\mathcal{C}}$.

These equivalences have the property that for every two 1-morphisms ${}_aX_b, {}_aY_b \in \widehat{\mathcal{C}}(a \to b)$ and every two 2-morphisms $f, g \in \widehat{\mathcal{C}}({}_aX_b \Rightarrow {}_aY_b)$, we have $\mathrm{ev}(f) = \mathrm{ev}(g)$ if and only if $\mathrm{st}(f) = \mathrm{st}(g)$. This property allows us to show the graphical calculus is well-defined for $\mathcal{C}$. The idea is that we have take two morphisms f, g in $\mathcal{C}$, lift them to $\widehat{\mathcal{C}}$, and compare them back down in $\mathcal{C}^{\mathrm{st}}$, where graphical calculus, dual to pasting diagrams, can be shown to be well-typed. Thus equalities of string diagrams labelled by objects, 1-morphisms, and 2-morphisms in $\mathcal{C}$ should be interpreted as an infinite family of equalities in $\widehat{\mathcal{C}}$, one for every way of fully parenthesizing the 1-morphisms corresponding to the source and target of f, g.

The same strategy works for 3-categories by work of [Gut19]. Every algebraic tricategory $\mathcal{C}$ has a cofibrant replacement of formal composites $\widehat{\mathcal{C}}$, which comes equipped with a strict 3-functor $\widehat{\mathcal{C}} \to \mathcal{C}$ and splitting $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$. By [Gur13], every 3-category is equivalent to a Gray-category, which is the correct notion of a strict 3-category, but defined as an *enriched* 1-category in she sense of [Kel05]. We thus strictify $\widehat{\mathcal{C}}$ to get a Gray-category $\mathcal{C}^{\mathrm{st}}$, and since $\widehat{\mathcal{C}}$ is *cofibrant*, we get a strict equivalence $\widehat{\mathcal{C}} \to \mathcal{C}^{\mathrm{st}}$, giving a zig-zag of strict equivalences as in (7.4.1).

We will thus only define here the notion of a Gray-category and the graphical calculus for Gray-categories [BMS12]. We assert that all graphical arguments that we apply here also hold for algebraic tricategories by the work of [Gut19]. In fact, to make things even simpler, we will actually only discuss Gray-*monoids* in detail, which are the strict versions of monoidal 2-categories, which are 3-categories with one object.

Definition 7.4.2. The symmetric monoidal category Gray is the 1-category of strict 2-categories and strict 2-functors equipped with the Gray monoidal structure [Gur06, §5], defined as follows. Given strict 2-categories $\mathcal{C}, \mathcal{D}$, we define $\mathcal{C} \boxtimes \mathcal{D}$ to be the strict 2-category with:

- objects order pairs (c, d) where $c \in \mathcal{C}$ and $d \in \mathcal{D}$,
- 1-morphisms generated by those of the form $(X, 1_d)$ for ${}_{c_1}X_{c_2} \in \mathcal{C}(c_1 \to c_2)$ and $d \in \mathcal{D}$, and $(1_c, Y)$ for $c \in \mathcal{C}$ and ${}_{d_1}Y_{d_2} \in \mathcal{D}(d_1 \to d_2)$, subject to the relations

$$({}_{c_1}X^1_{c_2}, 1_d) \otimes ({}_{c_2}X^2_{c_3}, 1_d) = ({}_{c_1}X^1 \otimes {}_{c_2}X^2_{c_3}, 1_d) \qquad (1_c, {}_{d_1}Y^1_{d_2}) \otimes (1_c, {}_{d_2}Y^2_{d_3}) = (1_c, {}_{d_1}Y^1 \otimes {}_{d_2}Y^2_{d_3})$$

  The identity 2-cell for (c, d) is $(1_c, 1_d)$.
- 2-morphisms generated by those of the form (f, id) where $f \in \mathcal{C}({}_{c_1}X^1_{c_2} \Rightarrow {}_{c_1}X^2_{c_2})$ and (id, g) where $g \in \mathcal{D}({}_{d_1}Y^1_{d_2} \Rightarrow {}_{d_1}Y^2_{d_2})$ together with formal *interchanger* 2-isomorphisms

$$\phi_{X,Y} : ({}_{c_1}X_{c_2}, 1_{d_2}) \circ (1_{c_1} \otimes {}_{d_1}Y_{d_2}) \Rightarrow (1_{c_2} \otimes {}_{d_1}Y_{d_2}) \circ ({}_{c_1}X_{c_2}, 1_{d_1}).$$

If either X or Y is a unit 1-morphism, then the corresponding interchanger is the identity.

For the generation process, we first take formal 1-compositions with relations similar to those above for 1-composition. We then allow for formal 2-composition of these 1-composites, subject to a number of additional relations. We refer the reader to [Gur13, §3.1] for the rest of the details.

A Gray-*category* is simply a Gray-enriched category in the sense of [Kel05]. A Gray-*monoid* is an algebra object in Gray. Given a Gray-monoid $\mathcal{C}$, its delooping $\mathrm{B}\mathcal{C}$ is the Gray-category with one object and endomorphisms $\mathcal{C}$.

Remark 7.4.3. Just as every 3-category (algebraic tricategory) is equivalent to a Gray-category, every monoidal 2-category (algebraic tricategory with one object) is equivalent to a Gray-monoid [Gur13, Cor. 9.16].

We now unpack the notion of Gray-monoid. Our exposition is taken directly from [JPR20, §2], which was itself adapted from [DR18].

Remark 7.4.4. Unpacking Definition 7.4.2, a Gray-*monoid* consists of the following data:

(D1) a strict 2-category $\mathcal{C}$, where composition of 1-morphisms is denoted by $\otimes$ and composition of 2-morphisms is denoted by $\circ$;

(D2) an *identity* 0-cell $\mathbf{1} \in \mathcal{C}$;

(D3) strict left and right *tensor product* 2-functors $L_a = a \boxtimes -$ and $R_a = - \boxtimes a$ for each object $a \in \mathcal{C}$:

$$L_a = a \boxtimes - : \mathcal{C} \to \mathcal{C}$$
$$R_a = - \boxtimes a : \mathcal{C} \to \mathcal{C},$$

(D4) an *interchanger* 2-isomorphism $\phi_{x,y}$ for each pair of 1-cells $x : a \to b$ and $g : c \to d$:

$$\phi_{x,y} : (x \boxtimes 1_d) \otimes (1_a \boxtimes y) \Rrightarrow (1_b \boxtimes y) \otimes (x \boxtimes 1_c)$$

subject to the following conditions:

(C1) left and right tensor product agree: for all objects $a, b \in \mathcal{C}$, $L_a b = R_b a = a \boxtimes b$;

(C2) tensor product is strictly unital and associative:

$$L_{\mathbf{1}} = \mathrm{id}_{\mathcal{C}} = R_{\mathbf{1}}$$
$$L_a L_b = L_{a\boxtimes b}$$
$$R_b R_a = R_{a\boxtimes b}$$
$$L_a R_b = R_b L_a;$$

(C3) the interchanger ϕ respects identities, i.e., for a 0-cell $A \in \mathcal{C}$ and a 1-cell $f : C \to D$,

$$\phi_{f,1_A} = \mathrm{id}_{f\boxtimes A}$$
$$\phi_{1_A,f} = \mathrm{id}_{A\boxtimes f}$$

(C4) the interchanger ϕ respects composition, i.e., for $x : a \to a'$, $x' : a' \to a''$, $y : b \to b'$ and $y' : b' \to b''$,

$$\phi_{x'\otimes x,y} = (\phi_{x',y} \otimes (x \boxtimes 1_b)) \circ ((x' \boxtimes 1_{b'}) \otimes \phi_{x,y})$$
$$\phi_{x,y'\otimes y} = ((1_{a'} \boxtimes y') \otimes \phi_{x,y}) \circ (\phi_{x,y'} \otimes (1_a \boxtimes y))$$

(C5) the interchanger ϕ is natural, i.e., for 1-cells $x, x' : a \to a', y, y' : b \to b'$ and 2-cells $\alpha : x \Rrightarrow x'$, $\beta : y \Rrightarrow y'$,

$$\phi_{x',y} \circ ((\alpha \boxtimes 1_{b'}) \otimes (1_a \boxtimes y)) = ((1_{a'} \boxtimes y) \otimes (\alpha \boxtimes 1_b)) \circ \phi_{x,y}$$
$$\phi_{x,y'} \circ ((x \boxtimes 1_{b'}) \otimes (1_a \boxtimes \beta)) = ((1_{a'} \boxtimes \beta) \otimes (x \boxtimes 1_b)) \circ \phi_{x,y}$$

(C6) the interchanger ϕ respects tensor product, i.e., for $x : a \to a'$, $y : b \to b'$ and $z : c \to c'$,

$$\phi_{1_a \boxtimes y, z} = 1_a \boxtimes \phi_{y,z}$$
$$\phi_{x \boxtimes 1_b, z} = \phi_{x, 1_b \boxtimes z}$$
$$\phi_{x, y \boxtimes 1_c} = \phi_{x,y} \boxtimes 1_c$$

A Gray-monoid is called *linear* if the underlying 2-category is linear and for all objects a the functors $a \boxtimes -$ and $- \boxtimes a$ are linear.

Exercise 7.4.5. Unpack the definition of a Gray-category.
Hint: Look at the graphical calculus below and 'add shadings.'

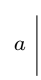
Warning 7.4.6 (Horizontal composition of 1-morphisms)**.** We warn the reader that the tensor product in a Gray-monoid does *not* provide a unique definition of the tensor product of two 1-cells. Given $x : a \to b$ and $y : c \to d$, we define

$$x \boxtimes y := (x \boxtimes 1_d) \otimes (1_a \boxtimes y)\,; \tag{7.4.7}$$

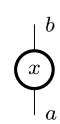
this convention is known as *nudging* [GPS95, §4.5]. We use a similar nudging convention for the tensor product of 2-cells. With this convention, the data of a Gray-monoid $\mathcal{C}$ as described in Remark 7.4.4 gives rise to an (*opcubical* cf. [Gur13, §8]) algebraic tricategory B$\mathcal{C}$ [Gur13, Thm. 8.12].

Gray-categories admit a graphical calculus of surfaces, lines, and vertices in three-dimensional space. We refer the reader to [BMS12, §2.6] for a rigorous discussion. Here, we will only ever work in a two-dimensional projection of this graphical calculus for Gray-monoids. Our exposition below follows [Bar14].

The 0-cells of our strict 2-category $\mathcal{C}$ (D1) are denoted by strands in the plane

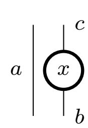


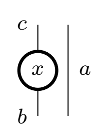
and the identity 0-cell $1_{\mathcal{C}}$ (D2) is denoted by the empty strand. The 1-cells are denoted by coupons between labelled strands

$x : a \to b$ 

The composition of 1-cells is denoted by vertical stacking of such diagrams.

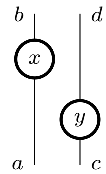
The strict tensor product $\boxtimes$ is denoted by horizontal juxtaposition. For example, the tensor product functors L_a and R_a (D3) are denoted by placing a strand labelled by a to the left or right respectively.

$L_a(x : b \to c) := \mathrm{id}_a \boxtimes x =$ 

$R_a(x : b \to c) := x \boxtimes \mathrm{id}_a =$ 

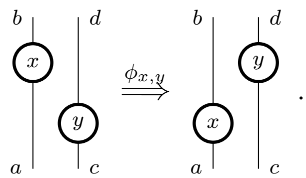
Given $x : a \to b$ and $y : c \to d$, we define their tensor product using the nudging convention from Warning 7.4.6.

$$x \boxtimes y := (x \boxtimes 1_d) \otimes (1_a \boxtimes y) =$$

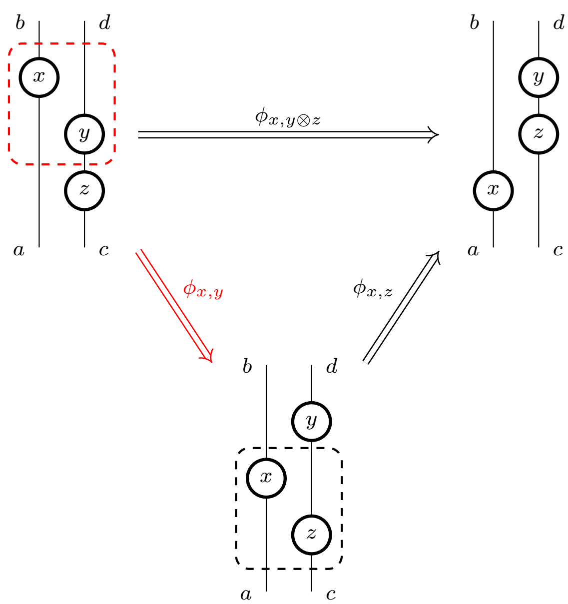


Observe that no two coupons ever share the same vertical height.

The 2-cells are inherently 3-dimensional, and can be thought of as 'movies' between our 2-dimensional string diagrams. Rather than drawing 2-cells, we denote them by arrows $\Rightarrow$ between diagrams corresponding to their source and target 1-cells. For example, the interchanger $\phi_{x,y}$ from (D4) is simply denoted by



Notation 7.4.8. When working with Gray-monoids, one often needs to whisker 2-cells between 1-cells, and the notation can quickly become cumbersome. Instead, we use the convention of a dashed box when we apply a 2-cell locally to a 1-cell, and we simply label the whiskered 2-cell by the name of the locally applied 2-cell. Later on, we will draw commutative diagrams whose vertices are 1-cells. When we want to apply two 2-cells locally in different places to the same 1-cell, we will use two dashed boxes with different colors, usually red and blue. When one of these two 2-cells is applied to the entire diagram, we do not use a dashed box, and we only use one dashed box of another color, usually red. As an explicit example, the second equation in (C4) in string diagrams is given by:



Example 7.4.9. The monoidal 2-category $2\mathsf{Vec}$ is the strict 2-category of finitely semisimple linear categories, linear functors, and natural transformations. The tensor product is given by the *Deligne product* $\mathcal{C} \boxtimes \mathcal{D}$ of linear categories, which we define as the Cauchy completion of the category whose objects are formal tensor products $c \boxtimes d$ of $c \in \mathcal{C}$ and $d \in \mathcal{D}$, and whose morphism spaces are tensor products: $(\mathcal{C}\boxtimes\mathcal{D})(c_1\boxtimes d_1 \to c_2\boxtimes d_2) := \mathcal{C}(c_1 \to c_2)\otimes\mathcal{D}(d_1 \to d_2)$.

While this model of $2\mathsf{Vec}$ is not a Gray-category as the Deligne product is not strictly associative, we will apply graphical calculus to this monoidal 2-category as discussed in the beginning of this section.

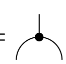
Exercise 7.4.10. Suppose $\mathcal{C}$ is a Gray-monoid with one object. Show that $\Omega_*\mathcal{C}$ is a strict braided monoidal category. Conversely, show that given a strict braided monoidal category $\mathcal{B}$, $\mathrm{B}\mathcal{B}$ is a Gray-monoid with one object.

Remark 7.4.11. Recall that braided monoidal categories form a (strict) 2-category, but 3-categories with one object and one 1-morphism form a 4-category. Similar to §7.3, the 4-category of 3-categories with one object and one 1-morphism equipped with a pointing from $* = \mathrm{B}\{e\}$ is 2-truncated, and thus equivalent to a 2-category [JPR20]. That is, the Delooping Hypothesis 7.1.1 holds for $k = n = 1$ and $i = 2$. Earlier work [CG11] used so called 'iconic natural transformations' rather than pointings to prove a similar result.

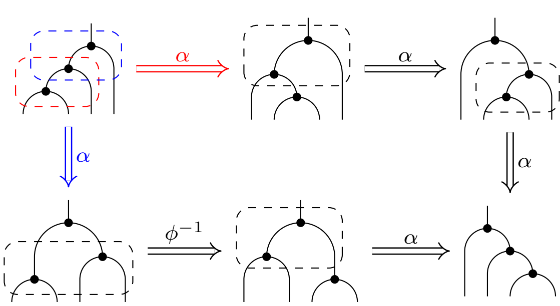
7.5. **Rigid 2-algebras in** $2\mathsf{Vec}$**.** This section is taken from work in progress with Corey Jones and David Reutter.

Definition 7.5.1. A *monoidal category object* or *2-algebra* (A, μ, α) in a Gray-monoid $\mathcal{C}$[2] consists of:

- An object $A \in \mathcal{C}$,
- A *monoidal product* 1-morphism $\mu : A \boxtimes A \to A$ denoted graphically by a trivalent vertex

$\mu =$ 

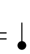
- An *associator* 2-isomorphism $\alpha : \mu\otimes(\mu\boxtimes\mathrm{id}_M) \Rightarrow \mu\otimes(\mathrm{id}_M \boxtimes\mu)$ such that the following diagram commutes:



(7.5.2)

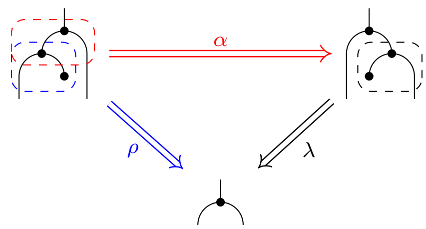
We call a 2-algebra *unital* if if there exist

- A *unit* 1-morphism $\iota : \mathbf{1}_\mathcal{C} \to A$ denoted graphically by a univalent vertex

$\iota =$ 

[2]One can use the results of [Gut19] to apply graphical calculus to any weak monoidal 2-category.

- and *unitor* 2-morphisms $\lambda : \mu \otimes (\iota \boxtimes \mathrm{id}_A) \Rightarrow \mathrm{id}_A$ and $\rho : \mu \otimes (\mathrm{id}_A \boxtimes \iota) \Rightarrow \mathrm{id}_A$ such that the following diagram commutes:



(7.5.3)

Observe that unitality of a 2-algebra A is a property and not additional structure; the space of choices of units and unitors is either empty or contractible.

We call a 2-algebra *rigid* if it is unital and μ admits a right adjoint $\mu^R : A \to A \boxtimes A$ as an $A - A$ bimodule map. We call the counit of this adjuntion $\varepsilon : \mu \otimes \mu^R \Rightarrow \mathrm{id}_A$. Observe that rigidity of a 2-algebra is also a property and not additional structure; the space of choices of μ^R is contractible.

A rigid 2-algebra is called *separable* if for any choice of the 2-unit μ^R, the counit ε admits a splitting as an $A - A$ bimodule natural transformation, i.e., there exists $\delta : \mathrm{id}_A \Rightarrow \mu \otimes \mu^R$ such that $\varepsilon \circ \delta = \mathrm{id}_{\mathrm{id}_A}$. Observe that separability is *independent* of the choice of μ^R.

A *multifusion 2-algebra* is a separable 2-algebra (A, μ, α) whose underlying object A is 2-dualizable. Observe that all separable 2-algebras in fusion 2-categories are multifusion.

Remark 7.5.4. The reader should notice that unitality, rigidity, separability, and multifusion for a 2-algebra are all properties, and not additional structure. If we pick such structure, we can define canonical separators, which will endow our 2-algebra with the structure of a *condensation 2-algebra* [GJF19]. We will discuss this further in §7.6 below.

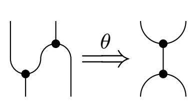
Exercise 7.5.5. Prove that a unital 2-algebra $(\mathcal{A}, \mu, \alpha, 1_{\mathcal{A}}, \lambda, \rho) \in 2\mathsf{Vec}$ is exactly a finitely semisimple linear monoidal category.

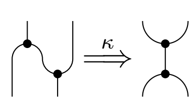
The main result of this section is the classification of rigid 2-algebras in 2Vec, the monoidal 2-category of finitely semisimple categories, linear functors, and natural transformations. In the graphical calculus, this means that strings are labelled by finitely semisimple categories, coupons are labelled by linear functors, and 2-morphisms between 2D string diagrams are labelled by natural transformations. We proceed in slightly more generality at this point, with some ideas adapted from [BDSPV15, §4].

Assumption 7.5.6. For the rest of this section, we assume $\mathcal{C}$ is a Gray-monoid in which all hom categories are finitely semisimple linear categories, and all 1-morphisms admit right adjoints.

Example 7.5.7. The *fusion 2-categories* of [DR18] satisfy Assumption 7.5.6; indeed, these are the main examples of interest. In particular, 2Vec satisfies Assumption 7.5.6.

First, given a rigid 2-algebra (A, μ, α) in $\mathcal{C}$, we claim the *Frobeniusator* natural isomorphisms

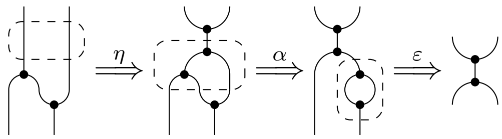


and


,

which endow μ^R with the structure of an $A - A$ bimodule functor such that η, ε witness an adjunction of $A - A$ bimodule functors, are over-determined.

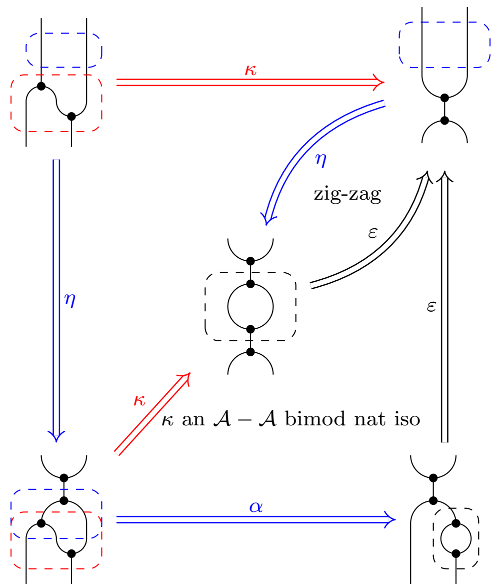
Lemma 7.5.8. *The Frobeniusator κ is given by the following composite:*



(7.5.9)

A similar statement holds for θ.

Proof. Indeed, κ and (7.5.9) both fit in the following pasting diagram:

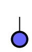
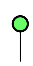
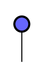


That the lower right triangle commutes is best seen by inverting the bottom associator arrow. □

We now specialize to the case $\mathcal{C}$ is a Gray-monoid in which all hom categories are finitely semisimple linear categories, and all 1-morphisms admit right adjoints. Clearly 2Vec satisfies this condition, as do the *fusion 2-categories* of [DR18].

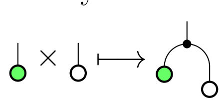
Many diagrams in the following construction come from [JPR20].

Construction 7.5.10. Suppose $A \in \mathcal{C}$ is a unital 2-algebra, and define $\mathcal{T} := \mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{F}} \to A)$. In the graphical calculus, we represent A by a black strand, and we represent objects $s, t, r \in \mathcal{T}$ by shaded disks with an A-strand emanating from the top:

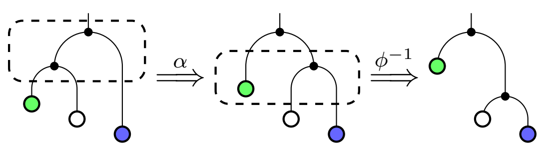
 $:= s$  $:= t$  $:= r$  $:= s^R$,  $:= t^R$,  $:= r^R$

We represent adjoints by disks with A-strands emanating from the bottom.

We endow $\mathcal{T}$ with a monoidal product by

,

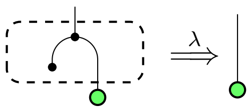
and we define an associator by



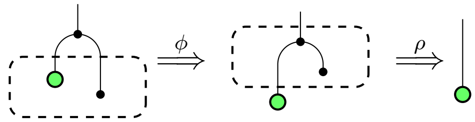
. (7.5.11)

Observe that (7.5.2) implies that the associators (7.5.11) satisfy the pentagon axiom.

We define the unit object $1_{\mathcal{T}} := \iota \in \mathcal{C}(\mathbf{1} \to A)$. The unitors are given by

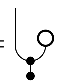
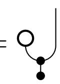


and

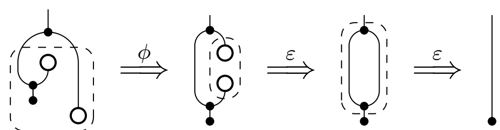


Observe that (7.5.3) implies that the unitors satisfy the triangle axiom. Hence $\mathcal{T}$ is a semisimple monoidal category.

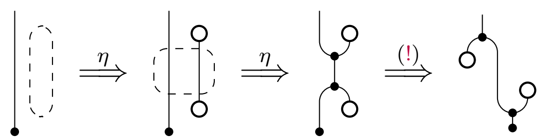
Now suppose that $A \in \mathcal{C}$ is a rigid 2-algebra. Since every $t \in \mathcal{T} = \mathcal{C}(\mathbf{1} \to A)$ has a right adjoint $t^R \in \mathcal{C}(A \to \mathbf{1})$, using the graphical calculus for Gray-monoids, we define left and right duals $t^\vee$ and ${}^\vee t$ of t in $\mathcal{T}$ by

$t^\vee :=$  and ${}^\vee t :=$ .

The adjunction $t \dashv t^R$ allows the following evaluation and coevaluation morphisms to exhibit $t^\vee$ and ${}^\vee t$ as left and right duals of t respectively:

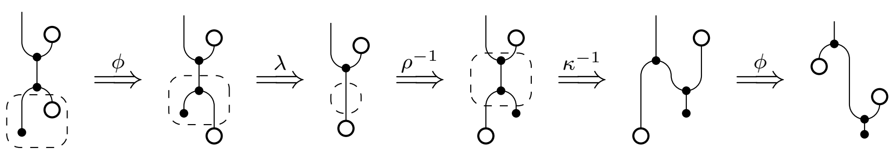
$\mathrm{ev}^L_t : t^\vee \otimes t =$ 

$= 1_{\mathcal{T}}$

$\mathrm{coev}^L_t : 1_{\mathcal{T}} =$ 

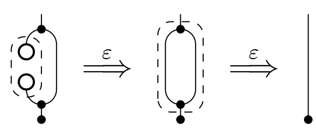
$= t \otimes t^\vee$

Here, the exclamation point (!) in coev^L_t denotes the composite of coheretors

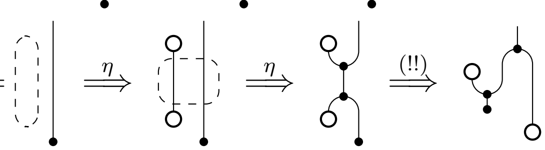


(!)

Similarly, we can define evaluation and coevaluation for the right dual ${}^\vee t$, where a similar composite of coheretors is needed for coev^R_t:

$\mathrm{ev}^R_t : t \otimes {}^\vee t =$ 

$= 1_{\mathcal{T}}$

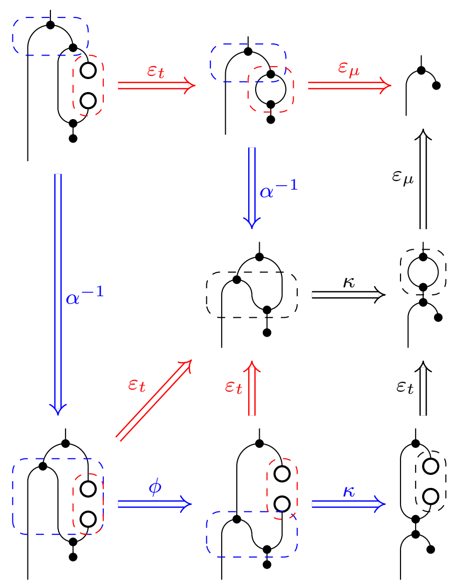
$\mathrm{coev}^R_t : 1_{\mathcal{T}} =$ 

$= {}^\vee t \otimes t$

for another similarly defined composite of coheretors (!!). In Theorem 7.5.12 below, we prove that the zig-zag axiom is satisfied. Thus $\mathcal{T}$ is a rigid finitely semisimple monoidal category, i.e., a multifusion category.

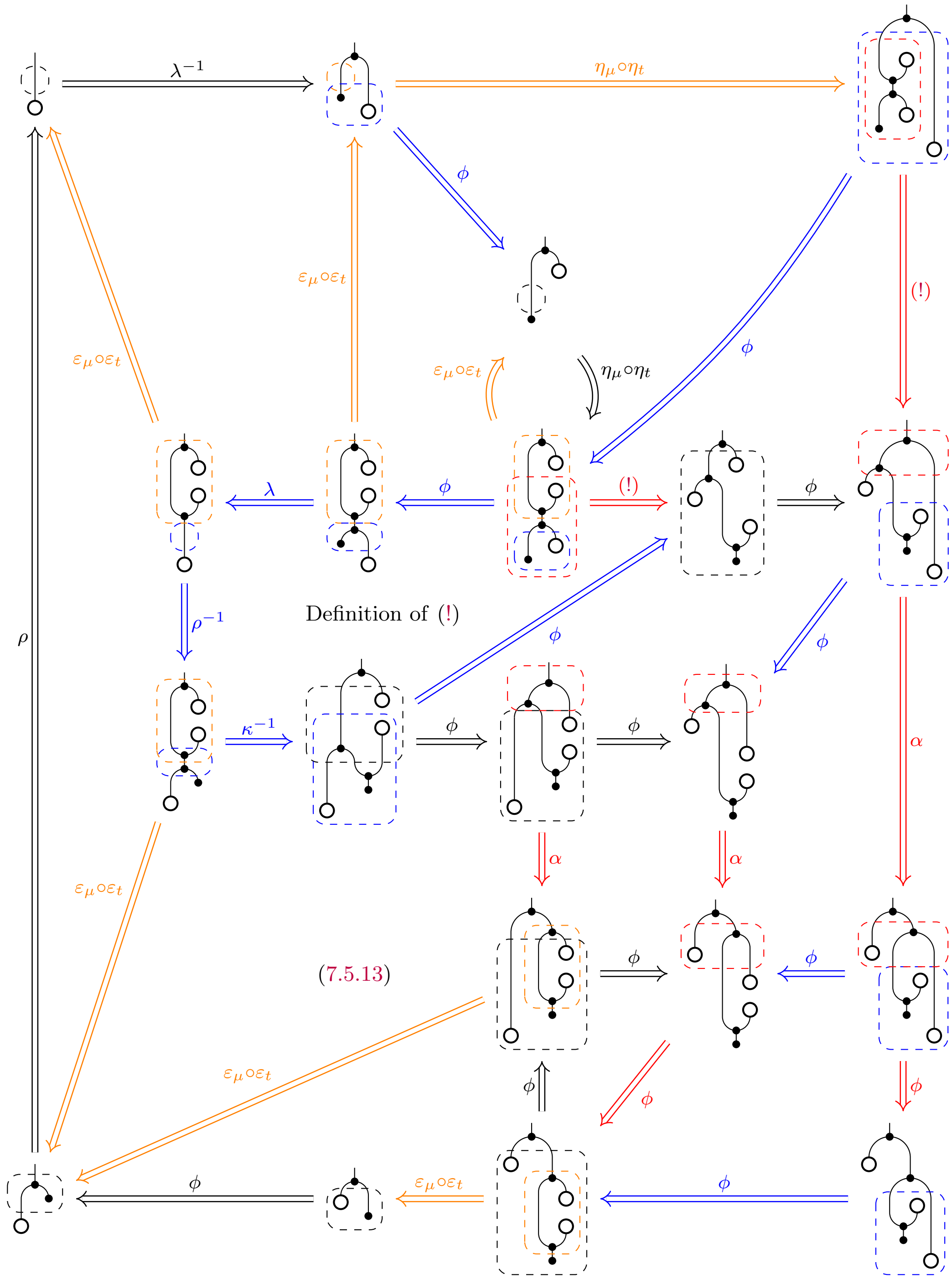
Theorem 7.5.12. *Suppose A is a rigid 2-algebra in a Gray-monoid $\mathcal{C}$ in which all hom categories are finitely semisimple linear categories, and all 1-morphisms admit right adjoints. The semisimple tensor category $\mathcal{T} = \mathcal{C}(\mathbf{1} \to A)$ from Construction 7.5.10 above is a multifusion category. When A is* connected *(the unit $\iota \in \mathcal{C}(\mathbf{1} \to A)$ is simple), $\mathcal{T}$ is fusion.*

Proof. All that remains is to verify the zig-zag axioms. We explicitly prove the relation $(\mathrm{id}_t \otimes \mathrm{ev}_t^L) \circ (\mathrm{coev}_t^L \otimes \mathrm{id}_t) = \mathrm{id}_t$; the other 3 relations are left to the reader. First, we observe that $\varepsilon = \varepsilon_\mu \circ \varepsilon_t$ is left A-modular, i.e., the following diagram commutes: **TODO: explain top right square using Lemma 7.5.8**



(7.5.13)

The zig-zag relation follows from commutativity of the diagram below; the composite map along the outside of the diagram is the zig-zag formula, and each 2-cell commutes, meaning we obtain the identity.



This concludes the proof. ☐

To connect Construction 7.5.10 with the classification of rigid 2-algebras in 2Vec, we have the following exercise.

Exercise 7.5.14. Show that for any $\mathcal{V} \in \mathsf{2Vec}$, $\mathcal{V} \cong \mathrm{Hom}_{\mathsf{2Vec}}(\mathbf{1} \to \mathcal{V})$, where $\mathbf{1} = \mathsf{Vec}$. Moreover, this equivalence is a monoidal equivalence between the monoidal structure μ and the monoidal structure from Construction 7.5.10.

Corollary 7.5.15. *Rigid 2-algebras* $(\mathcal{A}, \mu, \alpha, 1_{\mathcal{A}}, \lambda, \rho) \in \mathsf{2Vec}$ *are exactly multifusion categories.*

Proof. Every multifusion category gives a rigid 2-algebra in $\mathsf{2Vec}$ as **TODO: justify why the 2-category of $\mathcal{A} - \mathcal{A}$ bimodules, bimodule functors, and bimodule natural transformations admits adjoints**

The converse direction is exactly Construction 7.5.10 above. □

7.6. **Karoubi completion of** $\mathrm{B}\mathsf{2Vec}$. As discussed in the 2-categories module, there is a notion of a condensation 2-algebra in a 3-category $\mathcal{C}$, which can be described as a 3-functor from a certain 3-category $\clubsuit_3 \subset \spadesuit_3$ into $\mathcal{C}$ following [GJF19]. Moreover, we can ask when a condensation 2-algebra is unital, and forgetting the separator structures for a unital condensation 2-algebra, we get the notion of a separable 2-algebra.

There is an organic notion of a condensation bimodule between (condensation) 2-algebras, where one of the A strings is replaced by another 1-morphism M in our 3-category $\mathcal{C}$. There are many relations one can prove amongst the defining axioms; in particular, unital bimodules between separable 2-algebras admit a canonical separating structure.

The 3-category $\mathsf{Kar}(\mathcal{C})$ has objects condensation 2-algebras, 1-morphisms condensation bimodules, 2-morphisms condensation intertwiners, and 3-morphisms bimodule modifications. There is an obvious 3-subcategory of unital condensation 2-algebras $\mathsf{Kar}_{\mathsf{u}}(\mathcal{C})$, and forgetting the separator structures, we get the 3-category $\mathsf{2Alg}_{\mathsf{u}}$ of separable 2-algebras, separable bimodules, separable intertwiners, and bimodule modifications. Similar to our discussion for 2-categories, by [GJF19, Prop. 3.3], we have a zig-zag of equivalences:

$$\mathsf{Kar}(\mathcal{C}) \overset{\sim}{\hookleftarrow} \mathsf{Kar}_{\mathsf{u}}(\mathcal{C}) \overset{\sim}{\to} \mathsf{2Alg}_{\mathsf{u}}. \tag{7.6.1}$$

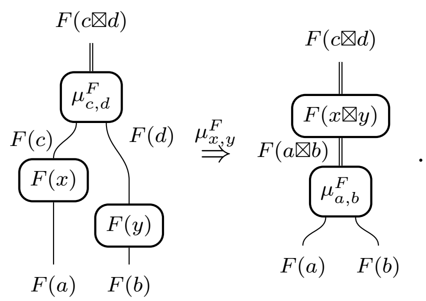
Theorem 7.6.2. $\mathsf{Kar}(\mathsf{2Vec}) \cong \mathsf{MultFusCat}$.

Proof. We know that $\mathsf{Kar}(\mathsf{2Vec}) \cong \mathsf{2Alg}_{\mathsf{u}}(\mathsf{2Vec})$ by (7.6.1), and we saw that separable 2-algebras in $\mathsf{2Vec}$ are (separable) multifusion categories in Corollary 7.5.15. It is straightforward to prove that a separable bimodule between separable 2-algebras in $\mathsf{2Vec}$ is a (separable) finitely semisimple bimodule category, separable intertwiners are bimodule functors, and modifications are bimodule natural transformations. Finally, to see the 3-category structures match up, we appeal to [Hau17],[JFS17, Ex. 8.10], and [DSPS20, DSPS19], where the 3-category structure on $\mathsf{MultFusCat}$ arises as a sub 3-category of algebra objects in the 2-category of small finitely-cocomplete linear categories, finitely cocontinuous linear functors, and natural transformations. □

7.7. **Separable lax 2-functors.** For the definition of a *monoidal structure* on a 2-functor $F : \mathcal{C} \to \mathcal{D}$ between Gray-monoids, we refer the reader to [JPR20, App. A]. Below, we edit this notion to give the definition of a lax 2-functor between Gray-monoids. We then define the notion of separability.

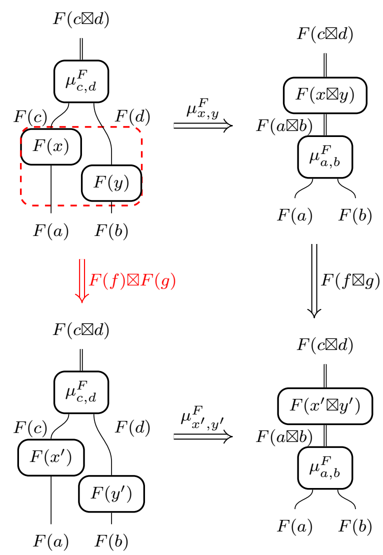
Definition 7.7.1. Suppose $\mathcal{C}, \mathcal{D}$ are Gray-monoids and $A = (F, F^1, F^2) : \mathcal{C} \to \mathcal{D}$ is a strict 2-functor. A *lax monoidal structure* on F consists of: **TODO: check this definition is correct for lax**

(Lax1) A 2-transformation $\mu^F : \boxtimes_{\mathcal{D}} \circ (F \times F) \Rightarrow F \circ \boxtimes_{\mathcal{C}}$ in the 2-category of 2-functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$. Explicitly, this is given by, for each pair of 0-cells $(a, b) \in \mathcal{C} \times \mathcal{C}$, a 1-cell $\mu^F_{a,b} : F(a) \boxtimes F(b) \to F(a \boxtimes b)$ and for each pair of 1-cells $(x, y) : (a, b) \to (c, d)$, an invertible 2-cell

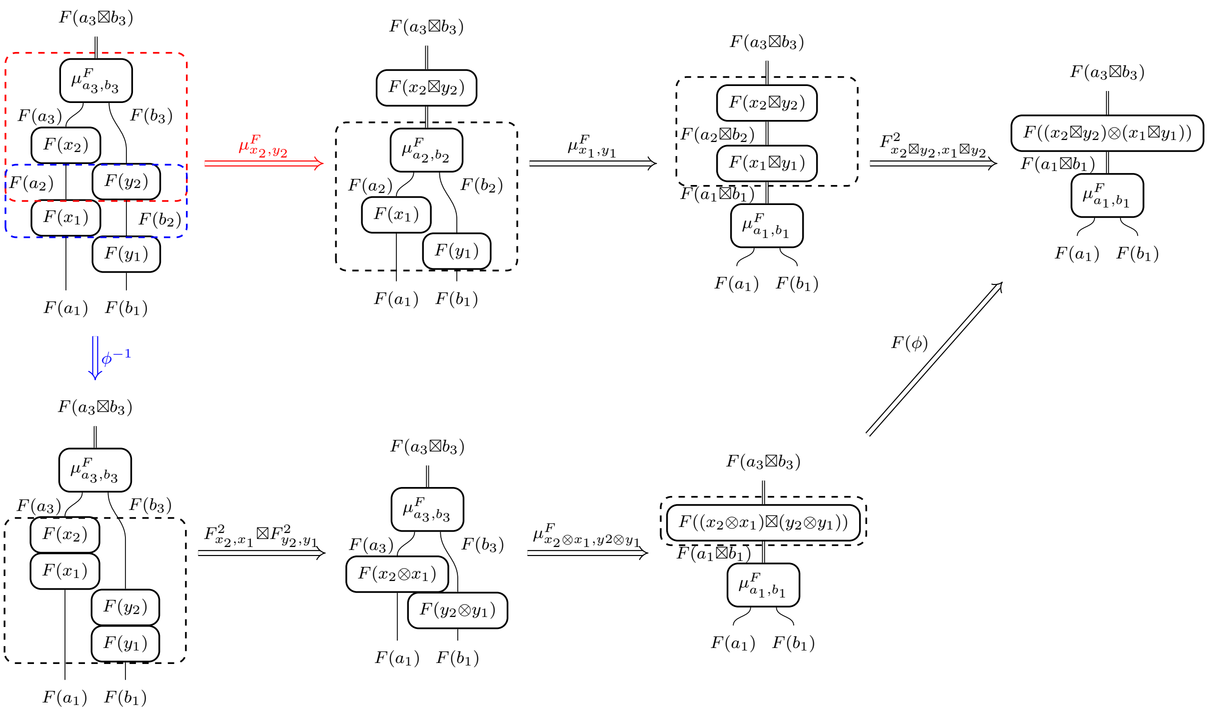


That μ^F is a 2-transformation means we have the following cohrences.

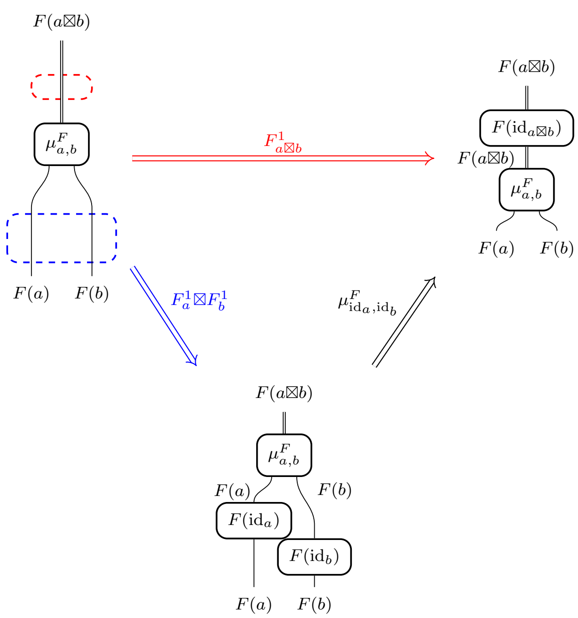
(Lax1).i For all $x, x' : a \to c$ and $y, y' : b \to d$ and all $f : x \Rightarrow x'$ and $g : y \Rightarrow y$, the following square commutes:



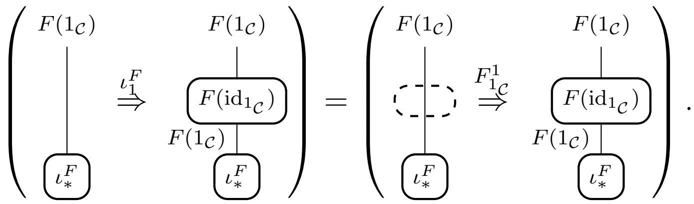
(Lax1).ii For all 1-cells $x_1 \in \mathcal{C}(a_1 \to a_2)$, $x_2 \in \mathcal{C}(a_2 \to a_3)$, $y_1 \in \mathcal{C}(b_1 \to b_2)$, and $y_2 \in \mathcal{C}(b_2 \to b_3)$,



(Lax1).iii For all 0-cells $a, b \in \mathcal{C}$, the following diagram commutes:

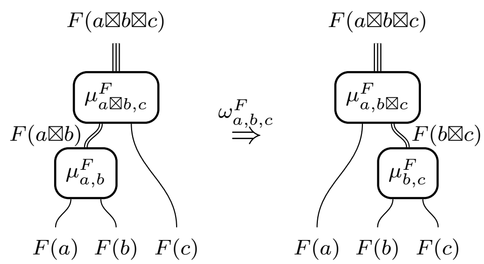


(Lax2) A 2-transformation $\iota^F : I_{\mathcal{D}} \Rightarrow F \circ I_{\mathcal{C}}$ (in the 2-category of 2-functors $* \to \mathcal{D}$) where $I_{\mathcal{C}} : * \to \mathcal{C}$ is the inclusion of the trivial 2-category into $\mathcal{C}$ which picks out $1_{\mathcal{C}}, \mathrm{id}_{1_{\mathcal{C}}}, \mathrm{id}_{\mathrm{id}_{1_{\mathcal{C}}}}$, and similarly for $\mathcal{D}$. Explicitly, this is given by a 1-cell $\iota^F_* : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ and an invertible 2-cell

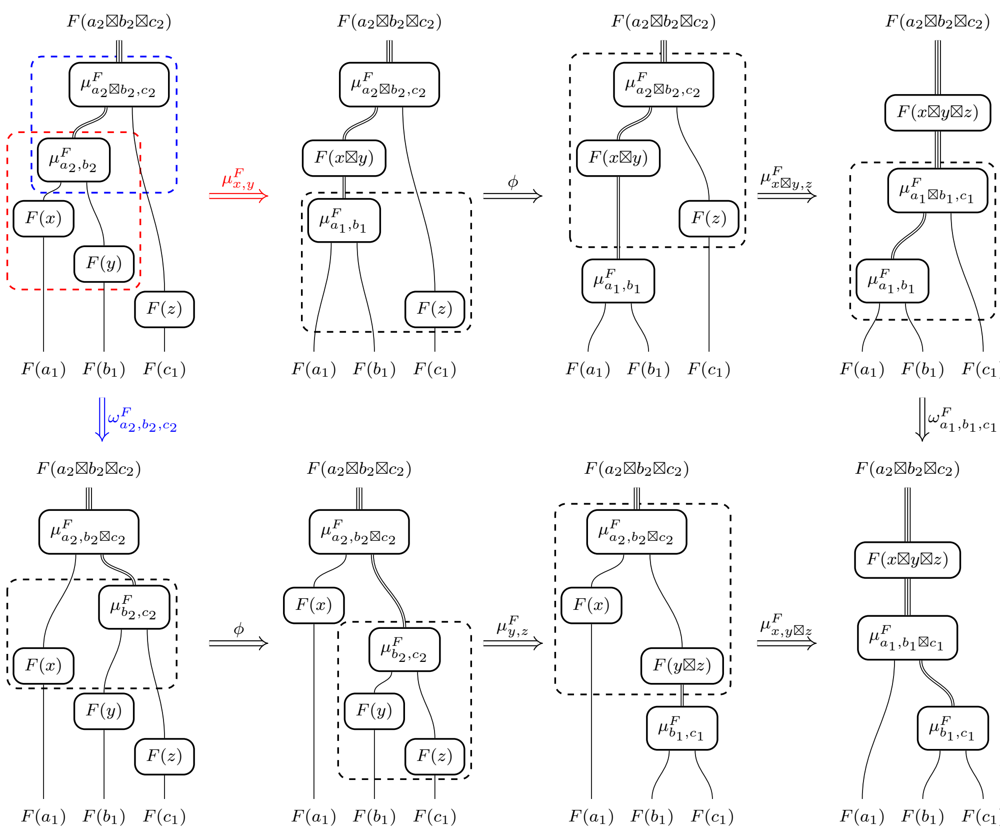


That ι^F is a 2-transformation implies that ι^F_1 equals the map on the right hand side above, which is a whiskering with F^1_e. This means ι^F_1 is automatically natural and compatible with F^2.

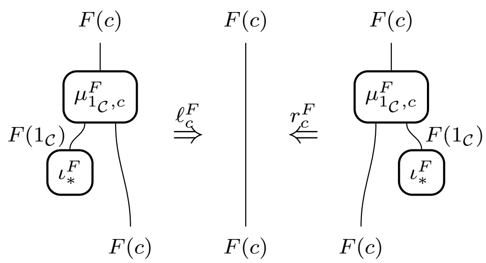
(Lax3) An invertible associator 2-modification ω^F. Explicitly, this is given by, for each $a, b, c \in \mathcal{C}$, an invertible 2-cell



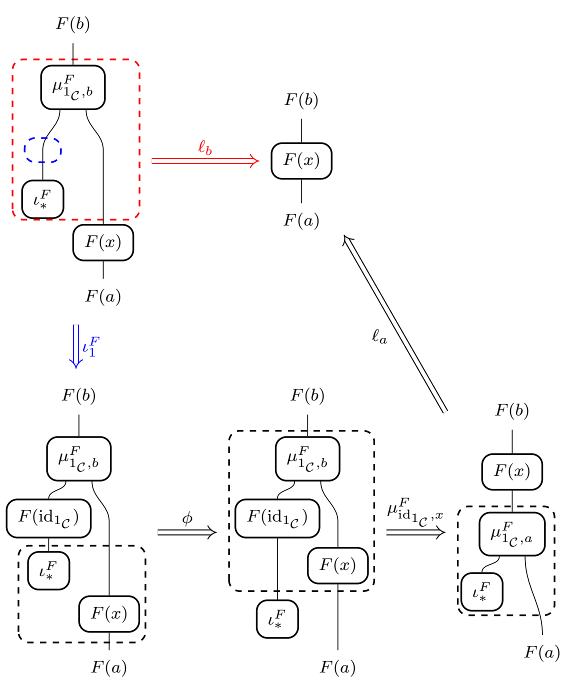
and the fact that ω is a 2-modification means that for all $x \in \mathcal{C}(a_1 \to a_2)$, $y \in \mathcal{C}(b_1 \to b_2)$, and $z \in \mathcal{C}(c_1 \to c_2)$,



(Lax4) invertible unitor 2-modifications ℓ^F and r^F, i.e., for each $c \in \mathcal{C}$, invertible 2-cells



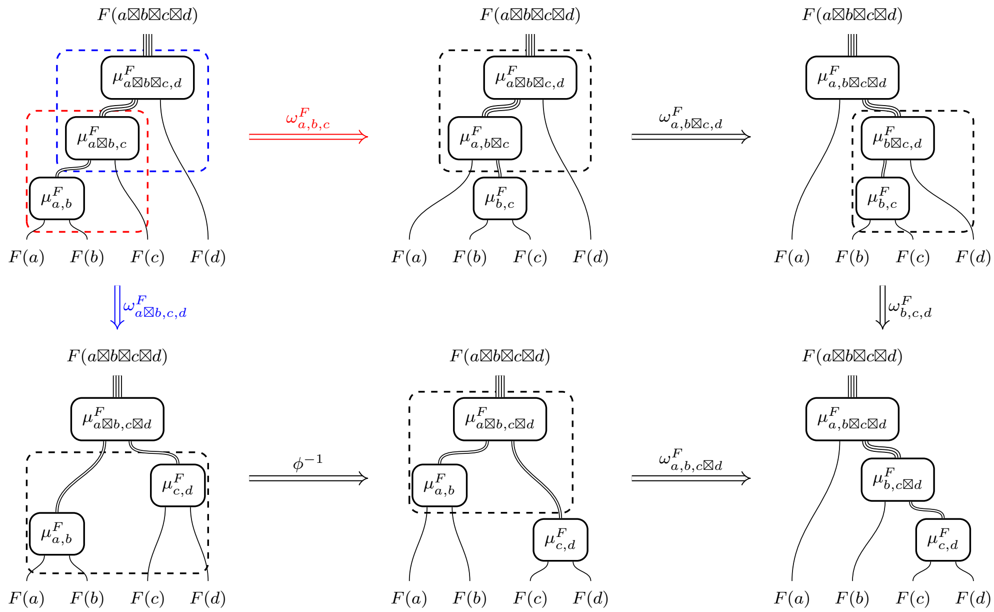
The fact that ℓ and r are 2-modifications means that for all $x \in \mathcal{C}(a \to b)$, the following diagram commutes:



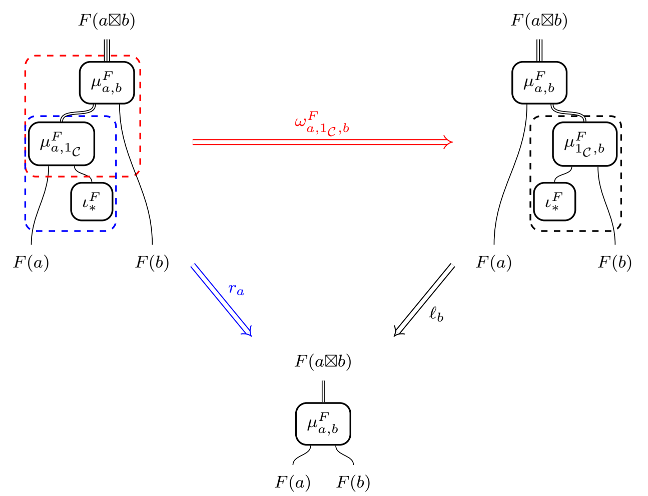
and a similar condition for r.

This data is subject to the additional two coherence conditions c.f. [Gur13, Def. 4.10]:

(F-1) For all $a,b,c,d \in \mathcal{C}$, the following diagram commutes:



(F-2) For all $a,b,c \in \mathcal{C}$, the following diagram commutes:



Definition 7.7.2. A lax 2-functor $(F, \mu^F, \iota^F, \omega^F, \ell^F, r^F) : \mathcal{C} \to \mathcal{D}$ between Gray-monoids is called *separable* if μ^F and ι^F admit right adjoints as 2-transformations in their respective 2-categories of 2-functors. Moreover, we require the right adjoint of μ^F to be a *bimodular* right adjoint. Observe that this is a property and not extra structure.

Exercise 7.7.3. Show that if F is separable, then any choice of right adjoints μ^R and ι^R satisfy various coherence axioms with ω^F, ℓ^F, r^F.

The point of introducing separable lax monoidal 2-functors is the following exercise.

Exercise 7.7.4. (Separable) lax monoidal 2-functors preserve (separable) 2-algebras.

7.8. **Separable 2-algebras in $\mathsf{Mod}(\mathcal{C})$.** Suppose $\mathcal{C}$ is a braided fusion category. The 2-category $\mathsf{Mod}(\mathcal{C})$ of (finitely semisimple) left $\mathcal{C}$-module categories can be endowed with a tensor product structure as follows.

Exercise 7.8.1. Show that the braiding β on $\mathcal{C}$ gives a canonical monoidal equivalence $\mathcal{C} \cong \mathcal{C}^{\mathrm{mp}}$, where the latter is the monoidal opposite with tensor product $a \otimes_{\mathrm{mp}} b := b \otimes a$.

Given a left $\mathcal{C}$-module category $(\mathcal{M}, \lambda_{\mathcal{M}})$, we get an organic $\mathcal{C} - \mathcal{C}$ bimodule structure by defining a left $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{mp}} \cong \mathcal{C} \boxtimes \mathcal{C}$-action on $\mathcal{M}$ by $(a \boxtimes b) \rhd m := (a \boxtimes b) \rhd m$. Observe that we have a canonical isomorphism $(a \boxtimes 1_{\mathcal{C}}) \rhd m \cong (1_{\mathcal{C}} \boxtimes a) \rhd m$, so the right action is canonically isomorphic to the left action. (One can also define the right action to be *equal* to the left action.)

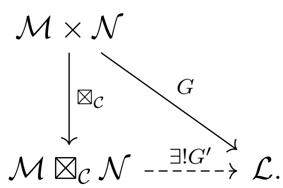
Exercise 7.8.2 ([DN13]). A *one-sided* $\mathcal{C} - \mathcal{C}$ bimodule is a $\mathcal{C} - \mathcal{C}$ bimodule category equipped with a natural isomorphism $\theta_{c,m} : c \rhd m \to m \lhd c$ for all $c \in \mathcal{C}$ and $m \in \mathcal{M}$ satisfying certain coherences.

(1) Work out what the coherences should be.
(2) Show that one-sided $\mathcal{C} - \mathcal{C}$ bimodules form a 2-category.
(3) Prove that this 2-category is equivalent to $\mathsf{Mod}(\mathcal{C})$.

Now given two left $\mathcal{C}$-modules $(\mathcal{M}, \lambda_{\mathcal{M}}), (\mathcal{N}, \lambda_{\mathcal{N}})$ in $\mathsf{Mod}(\mathcal{C})$, we equip them both with their canonical $\mathcal{C} - \mathcal{C}$ bimodule structures, and we form the *relative Deligne product* defined as *any* finitely semisimple category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ together with a $\mathcal{C}$-middle linear functor

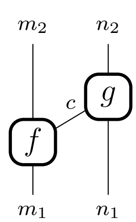
$$\boxtimes_{\mathcal{C}} : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$$

such that for any other finitely semsimple [[need abelian?]] category $\mathcal{L}$ equipped with a $\mathcal{C}$-middle linear functor $G : \mathcal{M} \times \mathcal{N} \to \mathcal{L}$, there exists a unique (*in a contractible sense*) linear functor $G' : \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathcal{L}$ such that the following diagram commutes:

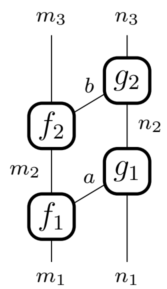


Exercise 7.8.3. Prove that $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ can be realized as any of the following finitely semisimple categories. In particular, the relative Deligne product exists.

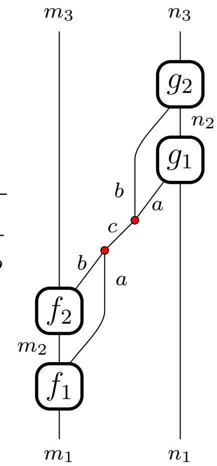
(1) $\mathsf{Fun}(\mathcal{M}^{\mathrm{op}} \to \mathcal{N})$
(2) $Z_{\mathcal{C}}(\mathcal{M} \boxtimes \mathcal{N}) := \mathrm{Hom}_{\mathcal{C}-\mathcal{C}}(\mathcal{C} \to \mathcal{M} \boxtimes \mathcal{N})$
(3) the *ladder category* $\mathsf{Lad}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ [BBJ19, Def. 7], whose objects are pairs of simples $(m \in \mathcal{M}, n \in \mathcal{N})$ and whose morphisms $(m_1, n_1) \to (m_2, n_2)$ are linear combinations of pairs $(f \in \mathcal{M}(m_1 \lhd c \to m_2), g \in \mathcal{N}(c \rhd n_1 \to n_2))$, which we view graphically as *basic ladder string diagrams*:



Composition is given by stacking ladders:



$$:= \sum_{c \in \mathrm{Irr}(\mathcal{C})} \sqrt{\frac{d_c}{d_a d_b}}$$



One should be careful to interpret this diagram as a pair of morphisms, one in $\mathcal{M}$ and one in $\mathcal{N}$.
Note: The scalars above are really only for the unitary setting, where we have chosen an orthogonal basis $\mathcal{B}_{ba}^c$ with a particular normalization with respect to the isometry inner product. Summing over any basis and dual basis with respect to a particular pairing works just the same.

Exercise 7.8.4. Endow $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ with the structure of a left $\mathcal{C}$-module.

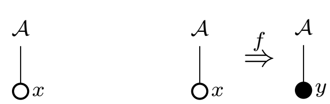
Remark 7.8.5. The 2-category $\mathsf{Mod}(\mathcal{C})$ becomes a monoidal 2-category under the monoidal operation of relative Deligne product $\boxtimes$. Observe that we have only defined this monoidal product on objects by universal property. For a complete discussion of the monoidal 2-category structure on $\mathsf{Mod}(\mathcal{C})$, we refer the reader to [Gre10].

Using a separable lax monoidal 2-functor $\mathsf{Mod}(\mathcal{C}) \to \mathsf{2Vec}$, we can classify separable 2-algebras in $\mathsf{Mod}(\mathcal{C})$.

Exercise 7.8.6. Prove that $\mathrm{Hom}(\mathcal{C} \to -) : \mathsf{Mod}(\mathcal{C}) \to \mathsf{2Vec}$ can be organically equipped with the structure of a separable lax monoidal 2-functor.

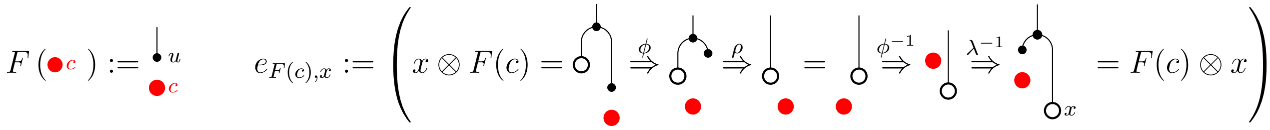
Theorem 7.8.7. *Separable 2-algebras in $\mathsf{Mod}(\mathcal{C})$ are equivalent to pairs $(\mathcal{A}, F^Z)$ where $\mathcal{A}$ is a multifusion category and $F^Z : \mathcal{C} \to Z(\mathcal{A})$ is a braided tensor functor.*

Proof. Let $\mathcal{M}$ be a separable 2-algebra in $\mathsf{Mod}(\mathcal{C})$. Define $\mathcal{A} := \mathrm{Hom}_{\mathsf{Mod}(\mathcal{C})}(\mathcal{C} \to \mathcal{M})$, which is a semisimple category. The objects and homs are denoted



Since $\mathrm{Hom}_{\mathsf{Mod}(\mathcal{C})}(\mathcal{C} \to -) : \mathsf{Mod}(\mathcal{C}) \to \mathsf{2Vec}$ is separable lax monoidal by Exercise 7.8.6, $\mathcal{A}$ is a separable 2-algebra in $\mathsf{2Vec}$ and thus multifusion.

There is a canonical braided monoidal functor $F^Z : \mathcal{C} \to Z(\mathcal{A})$ given by $c \mapsto (F(c), e_{F(c)})$ where



It is straightforward to check the coherences.

Conversely, given a multifusion $\mathcal{A}$ and a braided tensor functor $F^Z : \mathcal{C} \to Z(\mathcal{A})$, we can equip the underlying semisimple category of $\mathcal{A}$ with the structure of a one-sided $\mathcal{C} - \mathcal{C}$ bimodule category via $b \rhd a := F(b) \otimes a$. Since $\mathcal{C}$ maps into $Z(\mathcal{A})$, we see that the tensor product in $\mathcal{A}$ is $\mathcal{C}$-middle linear:

$$(a_1 \lhd c) \otimes a_2 \cong F(c) \otimes a_1 \otimes a_2 \cong a_1 \otimes F(c) \otimes a_2 \cong a_1 \otimes (c \rhd a_2).$$

Thus $\otimes : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ descends to map $\mathcal{A} \boxtimes_{\mathcal{C}} \mathcal{A} \to \mathcal{A}$. Again, as $\mathcal{C}$ includes into the center, we see that the image of the associator for $\mathcal{A}$ gives an associator in $\mathsf{Mod}(\mathcal{C})$. Hence $\mathcal{A}$ considered as an object of $\mathsf{Mod}(\mathcal{C})$ has the structure of a 2-algebra. One then checks separability.

Finally, one checks these two constructions are mutually inverse. □

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