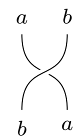
5. Unitary modular tensor categories

5.1. **Braided fusion categories.** Let $\mathcal{C}$ be a tensor category.

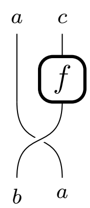
Definition 5.1.1. A *braiding* on $\mathcal{C}$ is a family β of isomorphisms



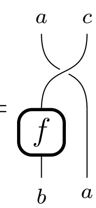
$$\left\{ \quad = \beta_{a,b} : b \otimes a \to a \otimes b \right\}_{a,b\in\mathcal{C}}$$

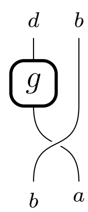
which satisfy the following axioms:

- (naturality) for all $f \in \mathcal{C}(b \to c)$ and $g \in \mathcal{C}(a \to d)$,

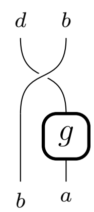


$$= (\mathrm{id}_a \otimes f) \circ \beta_{a,b} = \beta_{a,c} \circ (f \otimes \mathrm{id}_a) =$$

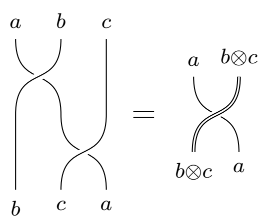




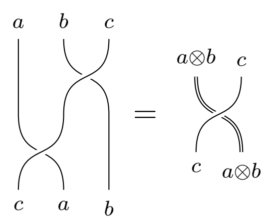
$$= (g \otimes \mathrm{id}_b) \circ \beta_{a,b} = \beta_{d,b} \circ (\mathrm{id}_b \otimes g) =$$



- (monoidality) For all $b, c \in \mathcal{C}$,



and



where we have suppressed the associators. More formally, the following diagrams should commute:

$$\begin{array}{ccccc} b \otimes (c \otimes a) & \xrightarrow{\mathrm{id}_b \otimes \beta_{a,c}} & b \otimes (a \otimes c) & \xrightarrow{\alpha} & (b \otimes a) \otimes c \\ \downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle \beta_{a,b} \otimes \mathrm{id}_c} \\ (b \otimes c) \otimes a & \xrightarrow{\beta_{a,b\otimes c}} & a \otimes (b \otimes c) & \xrightarrow{\alpha} & (a \otimes b) \otimes c \end{array} \tag{5.1.2}$$

$$\begin{array}{ccccc} (c \otimes a) \otimes b & \xrightarrow{\beta_{a,c} \otimes \mathrm{id}_b} & (a \otimes c) \otimes b & \xrightarrow{\alpha^{-1}} & a \otimes (c \otimes b) \\ \downarrow{\scriptstyle \alpha^{-1}} & & & & \downarrow{\scriptstyle \mathrm{id}_a \otimes \beta_{b,c}} \\ c \otimes (a \otimes b) & \xrightarrow{\beta_{a\otimes b,c}} & (a \otimes b) \otimes c & \xrightarrow{\alpha^{-1}} & a \otimes (b \otimes c) \end{array} \tag{5.1.3}$$

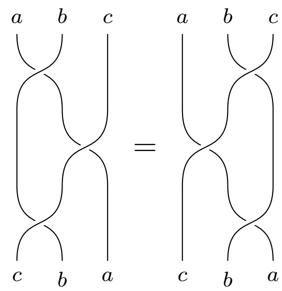
When $\mathcal{C}$ is unitary, we typically require a braiding to be unitary as well.

Remark 5.1.4. By [Gal14], every braiding on a unitary fusion category is *automatically* unitary.

Exercise 5.1.5. Prove that every braided multitensor category is a finite direct sum of braided tensor categories.
Hint: Show that $\mathcal{C}_{ij} = 0$ *for all* $i \neq j$.

Remark 5.1.6. Just as a monoidal category may be considered as a 2-category with one object, a braided monoidal category may be considered as a 3-category with one object and one 1-morphism. We refer the reader to [BS10, CG11, JPR20] for various discussions of this.

Remark 5.1.7. By naturality, the following *Yang-Baxter relation* holds in a braided tensor category $\mathcal{C}$:



(5.1.8)

Exercise 5.1.9. Find all the Yang-Baxter relations that hold by changing some over crossings to under crossings.

Remark 5.1.10. Braided tensor categories admit a 3D graphical calculus where diagrams are projected generically to the 2D plane. One may apply only Reidemeister moves (R2) and (R3); see Theorem 5.2.1 below for more details.

Example 5.1.11. The category sVec of *super-vector spaces* has underlying fusion category $\mathsf{Vec}(\mathbb{Z}/2)$ with braiding given by

$$\beta_{V,W} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} : \begin{pmatrix} W_0 \otimes V_0 & W_0 \otimes V_1 \\ W_1 \otimes V_0 & W_1 \otimes V_1 \end{pmatrix} \to \begin{pmatrix} V_0 \otimes W_0 & V_1 \otimes V_0 \\ V_0 \otimes W_1 & V_1 \otimes W_1 \end{pmatrix}$$

That is, $\beta_{V,W}(w \otimes v) = (-1)^{\operatorname{gr}(w)\operatorname{gr}(v)} v \otimes w$.

Exercise 5.1.12. Show that braidings on $\mathsf{Vec}_{\mathsf{fd}}(G)$ correspond to bicharacters on G.

Exercise 5.1.13. Find all braidings on $\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2, \omega)$ for both choices of cocycle given by $\omega(g, g, g) = \pm 1$ and all other values $+1$.

Remark 5.1.14. A braided fusion category with all objects invertible is classified by a finite abelian group A and the equivalence class of an Eilenberg-MacLane abelian 3-cocycle [EM54]. The abelian 3-cohomology group $H^3_{\text{ab}}(A, \mathbb{C}^\times)$ is in canonical bijective correspondence with *quadratic forms* $q : A \to \mathbb{C}^\times$, which satisfy $q(a) = q(-a)$ and

$$\chi(a, b) := \frac{q(a+b)}{q(a)q(b)}$$

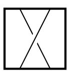
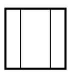
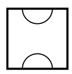
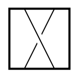
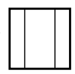
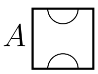
is a bicharacter.

A finite abelian group endowed with a quadratic form is called a *pre-metric group*; it is called a *metric group* if q is *non-degenerate*, i.e., the corresponding bicharacer χ is non-degenerate, i.e., the characters $\chi(a,-)$ and $\chi(-,a)$ are non-trivial when $a \neq 1$.

We remark that:

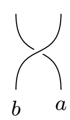
- Abelian 3-cocycles $(\omega, \beta) \in Z^3_{\text{ab}}(G, \mathbb{C}^\times)$ exactly give the associator and braiding data for such a braided fusion category, which we denote by $\mathcal{C}(A, q)$. Here, q corresponds to the class of (ω, β) under the canonical bijective correspondence.
- The associator ω is trivial on $\mathcal{C}(A, q)$ if and only if there is a bicharacter χ on A such that $q(a) = \chi(a, a)$.
- When $|A|$ is odd, every quadratic form q comes from a bicharacter.
- If $\mathcal{C}(A, q)$ is symmetric, then ω is trivial.

Exercise 5.1.15. Fix $A \in \mathbb{C}$ such that $d = -A^2 - A^{-2}$. Show that the tensor category $\mathsf{TLJ}(d)$ has braiding given by

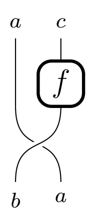
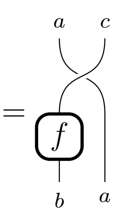
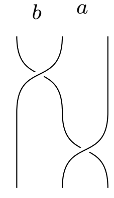
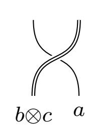
$\beta =$  $:= A$  $+ A^{-1}$  $\qquad \beta^{-1} =$  $:= A^{-1}$  $+ A$  . (5.1.16)

Example 5.1.17 (Drinfeld center)**.** The center of a tensor category should consist of objects $a \in \mathcal{C}$ such that $a \otimes b \cong b \otimes a$ for all $b \in \mathcal{C}$. These isomorphisms should be chosen coherently over the whole category. This gives rise to the notion of *half-braiding* that we saw when analyzing the excitations in the Levin-Wen string-net model.

The *Drinfeld center* $Z(\mathcal{C})$ of a fusion category $\mathcal{C}$ is the category whose objects are pairs (a, σ_a) where $a \in \mathcal{C}$ and σ_a is a half-braiding for a, i.e., σ_a is a family of isomorphisms

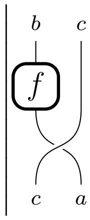
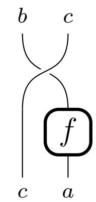
$\Bigg\{$  $= \sigma_{a,b} : b \otimes a \to a \otimes b \Bigg\}_{b \in \mathcal{C}}$

satisfying the following axioms:

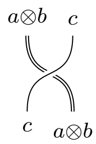
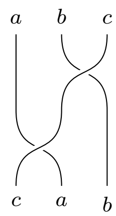
- (naturality) for all $f \in \mathcal{C}(b \to c)$,  $= (\text{id}_a \otimes f) \circ \sigma_{a,b} = \sigma_{a,c} \circ (f \otimes \text{id}_a) =$  .
- (monoidality) For all $b, c \in \mathcal{C}$,  $=$  , where we have suppressed the associators. More formally, the following diagram should commute:

$$\begin{array}{ccccc} b \otimes (c \otimes a) & \xrightarrow{\text{id}_b \otimes \sigma_{a,c}} & b \otimes (a \otimes c) & \xrightarrow{\alpha} & (b \otimes a) \otimes c \\ \big\downarrow{\scriptstyle \alpha} & & & & \big\downarrow{\scriptstyle \sigma_{a,b} \otimes \text{id}_c} \\ (b \otimes c) \otimes a & \xrightarrow{\sigma_{a,b\otimes c}} & a \otimes (b \otimes c) & \xrightarrow{\alpha} & (a \otimes b) \otimes c \end{array} \tag{5.1.18}$$

The hom spaces are defined by

$$Z(\mathcal{C})((a,\sigma_a)\to(b,\sigma_b)) := \left\{ f\in\mathcal{C}(a\to b) \;\middle|\; \right.$$

$$=$$

$$\left. \forall\, c\in\mathcal{C} \right\}.$$

The Drinfeld center is a tensor category with $(a,\sigma_a)\otimes(b,\sigma_b) := (a\otimes b, \sigma_{a\otimes b})$ where $\sigma_{a\otimes b}$ is given on $c\in\mathcal{C}$ by

$$\sigma_{a\otimes b,c} =$$

$$:=$$

$$= \alpha_{a,b,c}\circ(\mathrm{id}_a\otimes\sigma_{b,c})\circ\alpha^{-1}_{a,c,b}\circ(\sigma_{a,c}\otimes\mathrm{id}_b)\circ\alpha_{c,a,b} \qquad (5.1.19)$$

We have that $Z(\mathcal{C})$ is *braided* with braiding $\beta_{(a,\sigma_a),(b,\sigma_b)} := \sigma_{a,b}$. Indeed, (5.1.2) holds by (5.1.18), and (5.1.3) automatically holds by the definition (5.1.19). Finally, morphisms in $Z(\mathcal{C})$ were precisely defined so that both naturality axioms hold.

When $\mathcal{C}$ is unitary, we define $Z^\dagger(\mathcal{C})$ to be the full braided monoidal subcategory of $Z(\mathcal{C})$ on the objects (a,σ_a) whose half-braidings are unitary.

Exercise 5.1.20. Prove that $Z(\mathcal{C})$ is Karoubian and deduce that $Z(\mathcal{C})$ is a tensor category. When $\mathcal{C}$ is unitary, prove $Z^\dagger(\mathcal{C})$ is unitarily Karoubian and deduce that $Z^\dagger(\mathcal{C})$ is a unitary tensor category.

Remark 5.1.21. It turns out that $Z(\mathcal{C})$ is again fusion. This is typically proven by constructing an equivalence between $Z(\mathcal{C})$ and the representation category of the tube algebra [Müg03].

Exercise 5.1.22 ($\star\star$)**.** Suppose $\mathcal{C}$ is a UFC.

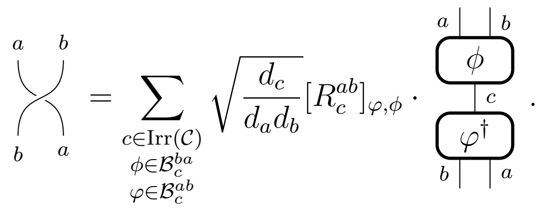
(1) Show that every object of $Z(\mathcal{C})$ is isomorphic in $Z(\mathcal{C})$ to an object of $Z^\dagger(\mathcal{C})$ [Müg03, Thm. 6.4]. Deduce that $Z^\dagger(\mathcal{C})^\natural$ (the braided fusion category obtained by forgetting the dagger) is braided equivalent to $Z(\mathcal{C})$.
(2) Prove that $Z^\dagger(\mathcal{C}) = Z(\mathcal{C})$, i.e., every half-braiding is automatically unitary [Gal14, Prop. 3.1].

Exercise 5.1.23. Compute the Drinfeld center of $\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2,\omega)$ for both choices of cocycle given by $\omega(g,g,g)=\pm1$ and all other values $+1$.

We now give the physicist's R-matrix definition of a unitary braiding for a UFC $\mathcal{C}$. Fix a set of simple representatives $\mathrm{Irr}(\mathcal{C})$. Suppose there is a unitary braiding on $\mathcal{C}$, so that for all $a,b\in\mathrm{Irr}(\mathcal{C})$, we have a coherent family of unitary isomorphisms $R^{ab} := \beta_{a,b} : b\otimes a\to a\otimes b$. We can decompose $a\otimes b$ and $b\otimes a$ into the same direct sum of simple objects

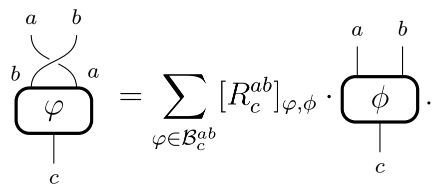
$$a\otimes b \cong \bigoplus_{c\in\mathrm{Irr}(\mathcal{C})} N_{ab}^c c \cong b\otimes a.$$

This decomposition gives us the *R-matrix* for the braiding determined by the formula



(5.1.24)

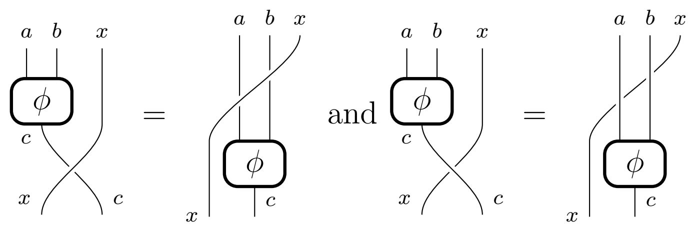
Equivalently, (5.1.24) can be expressed as



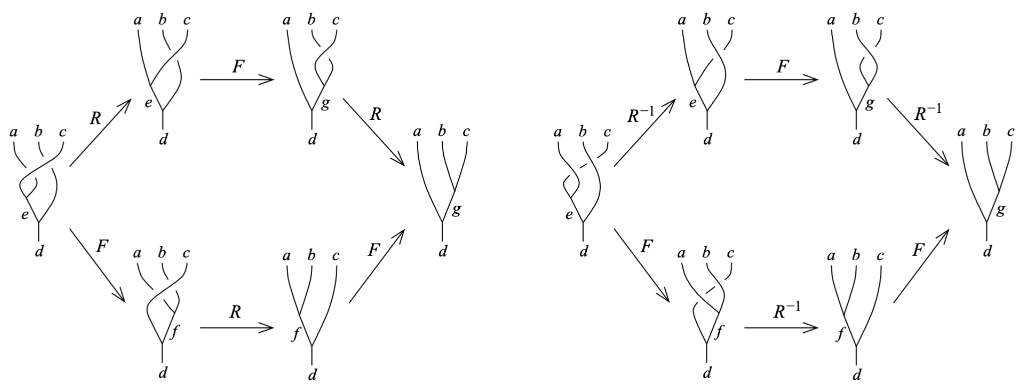
(5.1.25)

Thus R_c^{ab} is a unitary matrix for every $a, b, c \in \mathrm{Irr}(\mathcal{C})$ which are required to satisfy the following axioms:

- (naturality)



- (hexagon)



[BBCW19, Fig. 2]

As in the Fusion Categories module, one can interpret these diagrams in the symmetric monoidal category $\mathsf{Hilb}_{\mathsf{fd}}$, giving the following hexagon coherence axioms between the F- and R-matrices:

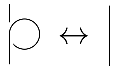
$$\sum_{\lambda,\gamma} [R_e^{ac}]_{\alpha\lambda} \left[F_d^{acb}\right]_{(e,\lambda,\beta)(g,\gamma,\nu)} \left[R_g^{bc}\right]_{\gamma\mu} = \sum_{f,\sigma,\delta,\psi} \left[F_d^{cab}\right]_{(e,\alpha,\beta)(f,\delta,\sigma)} \left[R_d^{fc}\right]_{\sigma\psi} \left[F_d^{abc}\right]_{(f,\delta,\psi)(g,\mu,\nu)},$$

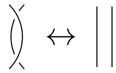
$$\sum_{\lambda,\gamma} \left[\left(R_e^{ca}\right)^{-1}\right]_{\alpha\lambda} \left[F_d^{acb}\right]_{(e,\lambda,\beta)(g,\gamma,\nu)} \left[\left(R_g^{cb}\right)^{-1}\right]_{\gamma\mu} = \sum_{f,\sigma,\delta,\psi} \left[F_d^{cab}\right]_{(e,\alpha,\beta)(f,\delta,\sigma)} \left[\left(R_d^{cf}\right)^{-1}\right]_{\sigma\psi} \left[F_d^{abc}\right]_{(f,\delta,\psi)(g,\mu,\nu)}$$

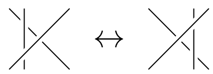
[BBCW19, (33,34)]

5.2. **Braid groups and representations.**

Theorem 5.2.1 (Reidemeister [Rei27]). *Two knot/link projections represent isotopic knots in* $\mathbb{R}^3$ *if and only if they are related by a finite number of the* Reidemeister moves*:*

(R1) 

(R2) 

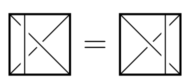
(R3) 

Definition 5.2.2. The algebraic braid group AB_n is the group generated by $\beta_1, \ldots, \beta_{n-1}$ subject to the relations

(B1) $\beta_i\beta_j = \beta_j\beta_i$ for $|i-j| > 1$ and
(B2) $\beta_i\beta_{i\pm1}\beta_i = \beta_{i\pm1}\beta_i\beta_{i\pm1}$.

Exercise 5.2.3. Show that AB_2 is isomorphic to $\mathbb{Z}$, but that AB_3 contains a group isomorphic to the free group $\mathbb{F}_2$.

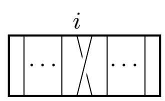
Definition 5.2.4. The diagramamtic braid group DB_n is the group whose elements consist of string diagrams with n boundary points on the lower and upper sides of a rectangle, and the lower points are paired to the upper points by smooth strings which only intersect at a finite number of points, where we indicate which string passes over the other as in a knot/link projection. Moreover, the strings are not allowed to have any critical points. All such diagrams are considered up to isotopy and Reidemeister moves (R2) and (R3). For example, the following elements of DB_3 are equal:



We multiply in DB_n by stacking boxes and smoothing out strings, similar to multiplication in TLJ_n, which is manifestly associative.

Exercise 5.2.5. Prove that DB_n is a group under the above multiplication. That is, find the identity element, and show every element has an inverse.

Exercise 5.2.6. Consider the distinguished elements of DB_n given by

$b_i :=$ 

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Prove that the elements $b_1, \ldots, b_{n-1} \in DB_n$ satisfy Relations (R2) and (R3). Deduce there is a well-defined group homomorphism $\Phi_n : AB_n \to DB_n$.

Exercise 5.2.7. Show that every element of B_n can be written as a product of $b_1, \ldots, b_{n-1}$ from Exercise 5.2.6. Deduce that Φ_n from Exercise 5.2.6 is surjective.

We will not prove the following theorem as it would take us too far afield.

Theorem 5.2.8 (Artin [Art25]). *The group homomorphism* $\Phi_n : AB_n \to DB_n$ *from Exercise 5.2.6 is an isomorphism.*

Notation 5.2.9. From this point forward, we simply write B_n to denote either AB_n or DB_n, which we identify under the group isomorphisms Φ_n.

Now fix objects a, b in a braided tensor category $\mathcal{C}$. Since (5.1.8) holds in $\mathcal{C}$, (R3) holds in $\mathrm{End}(b^{\otimes n})$ for the braiding $\beta_{b,b}$. Clearly (R2) holds as $\beta_{b,b}$ is invertible. Hence for each $n \in \mathbb{N}$, we get a representation $\Phi_n^{a,b} : B_n \to \mathrm{End}(\mathrm{Hom}(a \to b^{\otimes n}))$ given by postcomposition with the corresponding element of the diagrammatic braid group in $\mathrm{End}(b^{\otimes n})$.

5.3. **Aside: two definitions of the Jones polynomial.** Recall that the braiding on the Temperley-Lieb-Jones category was defined by

(5.1.16)

where $d = -A^2 - A^{-2}$.

Exercise 5.3.1. Show that the map $\Psi : B_n \to TL_n(d)$ given by

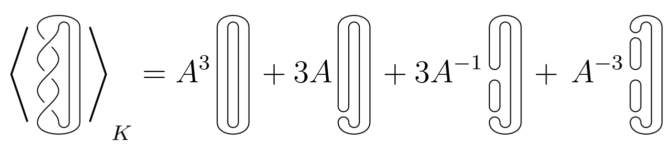
$$e \mapsto 1 \qquad \beta_i \mapsto A\,\mathrm{id}_n + A^{-1}E_i \qquad \beta_i^{-1} \mapsto A^{-1}\,\mathrm{id}_n + AE_i$$

where $d = -A^2 - A^{-2}$ preserves (B1) and (B2). Deduce that Ψ extends to a well-defined unital $*$-algebra homomorphism $\Psi : \mathbb{C}[B_n] \to TLJ_n(d)$, where the $*$ on the group algebra is the conjugate-linear extension of inversion.

Exercise 5.3.2. Determine when $\Psi(\beta_i)$ is a unitary in $U(TLJ_n(d))$ for $i - 1, \ldots, n-1$.

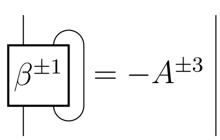
Definition 5.3.3 ([Kau87])**.** Given a link ℓ, we define an element $\langle \ell \rangle_K \in TL_0(d)$ called the *Kauffman bracket* of ℓ by replacing the crossings by $\beta^{\pm 1}$ as in (5.1.16).[1] Here, we identify $TL_0(d) = \mathbb{C}[A, A^{-1}]$, polynomials in A and A^{-1}. By (5.1.8) and Exercise 5.1.15, we see that $\langle \ell \rangle_K$ is *invariant* under applying (R2) and (R3) to ℓ anywhere locally. Thus the Kauffman bracket is almost an invariant of knots and links, modulo (R1).

Example 5.3.4. We calculate the Kauffman bracket of a trefoil knot as follows:



$$= -A^{-9} + A^{-1} + A^3 + A^7.$$

This proves a trefoil is not isotopic to its mirror image.

Exercise 5.3.5. Show that  $= -A^{\pm 3}$ (strand). Deduce that $\langle \ell \rangle_K$ is not invariant under (R1).

Definition 5.3.6. Let $\vec{\ell}$ be an *oriented* link. For each crossing in a projection of $\vec{\ell}$, we define the *sign* of the crossing as follows:

We define the *writhe factor* $\mathrm{wr}(\vec{\ell})$ to be the number of crossings, counted with their signs.

[1]This differs from Kauffman's original definition of the bracket polynomial [Kau87] by a normalization. Kauffman normalized so that the uknot has bracket equal to 1, whereas we normalize so that the unknot has bracket equal to δ.

Exercise 5.3.7. Let $\vec{\ell}$ be an oriented link and let ℓ be the link obtained from forgetting the orientation. Show that

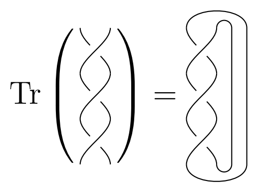
$$V_{\vec{\ell}}(A) := d^{-1}(-A)^{-3\,\mathrm{wr}(\vec{\ell})} \cdot \langle \ell \rangle_K \tag{5.3.8}$$

is invariant under (R1), (R2), and (R3).

Definition 5.3.9. The *Jones polynomial* of $\vec{\ell}$ is $V_{\vec{\ell}}(A)$ as in (5.3.8)

We now give Jones' original construction of his polynomial [Jon85], in modern diagrammatic language.

Definition 5.3.10. Given a braid b, we obtain a link ℓ by *closing/capping/tracing* the braid to the right. For example, we can represent a trefoil knot as follows:



Theorem 5.3.11 (Markov [Mar35])**.** *Every link is the closure of a braid. Moreover, two braids give the same link under closure if and only if they are related by a finite number of the following two moves:*

(M1) *If $b \in B_n$, we can swap $b \leftrightarrow aba^{-1}$ for some braid $a \in B_n$.*
(M2) *If $b \in B_n$, we can swap $b \leftrightarrow b\beta_n^{\pm 1}$, the n-th generator of B_n.*

Exercise 5.3.12. Prove that we get the same link under taking the closure of a braid under either (M1) or (M2).

Definition 5.3.13. Suppose $\vec{\ell}$ is an oriented link. Write $\vec{\ell} = \mathrm{Tr}(\vec{b})$ for some braid $b \in B_n$ where $\vec{b}$ is obtained from b by orienting all strands from bottom to top. Define

$$V_{\vec{\ell}}(A) := d^{-1}(-A^3)^{-\exp(b)} \cdot \mathrm{Tr}_{TLJ_n(d)}(\Psi(b)) \tag{5.3.14}$$

where $d = -A^2 - A^{-2}$, $\exp(b)$ is the *exponent sum* of b as a reduced word in $\beta_1, \ldots, \beta_{n-1}$, and $\Psi : \mathbb{C}[B_n] \to TLJ_n(d)$ is the unital $*$-algebra homomorphism from Exercise 5.3.1.

Exercise 5.3.15. Show that $\exp(b)$ is exactly the writhe factor of $\mathrm{Tr}(\vec{b})$.

Proposition 5.3.16. *The formula (5.3.14) for $V_{\vec{\ell}}$ is well-defined, i.e., it does not depend on the choice of b. Moreover, it agrees with (5.3.8).*

Proof. It is sufficient to show (5.3.14) agrees with (5.3.8), which is straightforward. However, for the sake of pedagogy, we will show that (5.3.14) is well-defined by showing it is invariant under the Markov moves (M1) and (M2).

(M1): This is immediate from $\exp(a) = -\exp(a^{-1})$ for all $a \in B_n$, together with the facts that Ψ is a homomorphism and Tr is a trace:

$$\mathrm{Tr}(\Psi(aba^{-1})) = \mathrm{Tr}(\Psi(a)\Psi(b)\Psi(a)^{-1}) = \mathrm{Tr}(\Psi(a)^{-1}\Psi(a)\Psi(b)) = \mathrm{Tr}(\Psi(b)).$$

(M2): We prove that $B_n \ni b \leftrightarrow b\beta_n \in B_{n+1}$ does not change (5.3.14), and the proof for $b \leftrightarrow b\beta_n^{-1}$ is similar. Note that $\exp(b\beta_n) = 1 + \exp(b)$. Expanding $\Psi(\beta_n) = A\,\mathrm{id}_n + A^{-1}E_n$, we have

$$\begin{aligned}
(-A^3)^{-\exp(b\beta_n)} \cdot \mathrm{Tr}_{TL_{n+1}(d)}(\Psi(b\beta_n)) \\
&= (-A^3)^{-1-\exp(b)} \cdot (A\,\mathrm{Tr}_{TL_{n+1}(d)}(\Psi(b)) + A^{-1}\,\mathrm{Tr}_{TL_n(d)}(\Psi(b)E_n)) \\
&= (-A^3)^{-1-\exp(b)} \cdot (Ad + A^{-1}) \cdot \mathrm{Tr}_{TL_n(d)}(\Psi(b)) \\
&= (-A^3)^{-1-\exp(b)} \cdot (A^3) \cdot \mathrm{Tr}_{TL_n(d)}(\Psi(b)) \\
&= (-A^3)^{-\exp(b)} \cdot \mathrm{Tr}_{TL_n(d)}(\Psi(b)).
\end{aligned}$$

This completes the proof. □

5.4. ***G*-crossed braided fusion categories.** For this section, $\mathcal{C}$ denotes a fusion category.

Definition 5.4.1. Let G be a finite group. A *G-grading* on $\mathcal{C}$ is a function $\mathrm{gr} : \mathrm{Irr}(\mathcal{C}) \to G$ such that whenever $N_{ab}^c \neq 0$, $\mathrm{gr}(c) = \mathrm{gr}(a)\,\mathrm{gr}(b)$. Another way to say this is that $\mathcal{C}$ is a direct sum $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ and $c_g \in \mathcal{C}_g$ and $c_h \in \mathcal{C}_h$ implies $c_g \otimes c_h \in \mathcal{C}_{gh}$. A G-grading is called *faithful* if $\mathrm{gr} : \mathrm{Irr}(\mathcal{C}) \to G$ is surjective, which is equivalent to $\mathcal{C}_g \neq 0$ for all $g \in G$.

Exercise 5.4.2. Define the *universal grading* group of $\mathcal{C}$ by

$$U = U_{\mathcal{C}} := \langle c \in \mathrm{Irr}(\mathcal{C}) | ab = c \text{ whenever } N_{ab}^c \neq 0\rangle.$$

(1) Show that the canonical map $\mathrm{Irr}(\mathcal{C}) \to U$ is a faithful grading.
(2) Show that any faithful G-grading gives a canonical surjective group homomorphism $U \to G$.

Exercise 5.4.3. Compute the universal grading groups of the following fusion categories: $\mathsf{Vec}_{\mathsf{fd}}(G, \omega)$, Fib, Ising, $\mathsf{Rep}(S_3)$.

Exercise 5.4.4. Find a canonical bijection $\mathrm{Aut}_\otimes(\mathrm{id}_{\mathcal{C}}) \cong \mathrm{Hom}(U \to \mathbb{C}^\times)$. Deduce that when $\mathcal{C}$ has a pivotal structure, the equivalence classes of pivotal structures on $\mathcal{C}$ are in bijective correspondence with $\mathrm{Hom}(U \to \mathbb{C}^\times)$.

Exercise 5.4.5 ([GJS15, Cor. 3.7])**.** Let $\mathrm{Inv}(Z(\mathcal{C}))$ and $\mathrm{Inv}(\mathcal{C})$ denote the groups of isomorphism clases of invertible objects in $Z(\mathcal{C})$ and $\mathcal{C}$ respectively.

(1) Prove that half-braidings on $1_{\mathcal{C}}$ are in bijective correspondence with $\mathrm{Hom}(U \to \mathbb{C}^\times)$.
(2) Prove that there is a short exact sequence

$$1 \to \mathrm{Hom}(U \to \mathbb{C}^\times) \to \mathrm{Inv}(Z(\mathcal{C})) \xrightarrow{f} \mathrm{Inv}(\mathcal{C})$$

where f is induced by the forgetful functor $F : (c, \sigma_c) \mapsto c$. Deduce that lifts of $g \in \mathrm{Inv}(\mathcal{C})$ to $Z(\mathcal{C})$ are in bijective correspondence with $\mathrm{Hom}(U \to \mathbb{C}^\times)$.
(3) Construct an *inner* tensor equivalence $\mathrm{Ad}(g)$ of $\mathcal{C}$ for $g \in \mathrm{Inv}(\mathcal{C})$ by $c \mapsto g \otimes c \otimes g^{-1}$.
(4) Prove that a monoidal isomorphism $\mathrm{Ad}(g) \cong \mathrm{id}_{\mathcal{C}}$ gives a half-braiding on g.

Exercise 5.4.6 ([Pen20, §3.3])**.** Suppose $\mathcal{C}$ is a multifusion category. Show that $\mathcal{C}$ has a universal grading *groupoid* U whose objects are the simple summands of $1_{\mathcal{C}}$. Then prove that any faithful G-grading by a finite groupoid G induces a surjective groupoid homomorphism (functor) $U \to G$.

Definition 5.4.7. Suppose $\mathcal{B}, \mathcal{C}$ are braided tensor categories. We call a tensor functor $F : \mathcal{B} \to \mathcal{C}$ *braided* if $F^2_{a,b} \circ \beta^{\mathcal{B}}_{F(a),F(b)} = F(\beta^{\mathcal{C}}_{a,b}) \circ F^2_{b,a}$. (Observe that there is no extra condition for a monoidal natural transformation between to braided tensor functors.)

Exercise 5.4.8. Suppose $\mathcal{B}, \mathcal{C}$ are braided tensor categories. Suppose that $F, G : \mathcal{B} \to \mathcal{C}$ are two monoidally equivalent monoidal functors. Show that F is braided if and only if G is braided.

Exercise 5.4.9. Suppose $\mathcal{C}$ is a fusion category.

(1) For each $c \in \mathcal{C}$, consider the representable functor $\mathcal{C}(c \to -) : \mathcal{C} \to \mathsf{Vec_{fd}}$. Prove that $\mathcal{C}(c \to -)$ has a two-sided adjoint $- \otimes c : \mathsf{Vec_{fd}} \to \mathcal{C}$.

(2) Suppose now $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ is a faithfully G-graded fusion category. For $(V, \pi) \in \mathsf{Rep}(G)$, show that $V \otimes 1_{\mathcal{C}}$ can be equipped with a half-braiding as follows:

$$\sigma_{\pi,c_g} : c_g \otimes V \otimes 1_{\mathcal{C}} \cong V \otimes c_g \xrightarrow{\pi_g \otimes \mathrm{id}_{c_g}} V \otimes 1_{\mathcal{C}} \otimes c_g \cong V \otimes c_g \qquad \forall\, c_g \in \mathcal{C}_g.$$

(3) With $\mathcal{C}$ as in (2), construct a fully faithful braided monoidal functor $\mathsf{Rep}(G) \to Z(\mathcal{C})$.

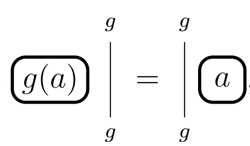
Repeat the above exercises when $\mathcal{C}$ is a UFC. In (1), replace $\mathsf{Vec_{fd}}$ with $\mathsf{Hilb_{fd}}$, and in (2) and (3), replace $\mathsf{Rep}(G)$ with $\mathsf{Rep}^\dagger(G)$, the category of unitary representations. (Observe that one must equip the hom spaces of $\mathcal{C}$ with Hilbert space structures in order to get a representable functor $\mathcal{C} \to \mathsf{Hilb_{fd}}$.)

Definition 5.4.10. Let $\mathcal{C}$ be a tensor category. The tensor autoequivalences $\mathrm{Aut}_\otimes(\mathcal{C})$ of $\mathcal{C}$ can be viewed as:

- an ordinary (1-)group, as tensor functors compose associatively on the nose, or
- a *2-group*, which is a monoidal category where every object and every morphism is invertible. Here, the morphisms $\eta : F \Rightarrow G$ are monoidal natural isomorphisms.

Unless otherwise stated, we will treat $\mathrm{Aut}_\otimes(\mathcal{C})$ as a 2-group.

Exercise 5.4.11. Suppose $\mathcal{G}$ is a 2-group. Show that $\mathcal{G}$ is classified by:

- the group G of isomorphism classes of objects of $\mathcal{G}$,
- the abelian group $A := \mathrm{End}_{\mathcal{G}}(e)$,
- the action of G on A determined by the fomula , and
- a 3-cocycle $[\omega] \in H^3(G, A)$.

Definition 5.4.12. Suppose $\mathcal{C}$ is a fusion category and G a finite group. An *action* of G on $\mathcal{C}$ is a monoidal functor $\mathrm{B}G \to \mathrm{Aut}_\otimes(\mathcal{C})$, where $\mathrm{Aut}_\otimes(\mathcal{C})$ is a 2-group and not an ordinary group. (In particular, an action is *not* an ordinary group homomorphism $G \to \mathrm{Aut}_\otimes(\mathcal{C})$!)

Definition 5.4.13 ([EGNO15, §8.24])**.** Suppose $\mathcal{C} = \bigoplus_{g \in G}$ is a faithfully G-graded fusion category. A *G-crossed braiding* on $\mathcal{C}$ consists of the following data:

- a G-action $\psi : \mathrm{B} \to \mathrm{Aut}_\otimes(\mathcal{C})$ such that for all $g, h \in G$, $\psi_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$, and
- a family $\beta^\times$ of isomorphisms

$$\beta^\times_{a,c_g} : c_g \otimes a \to \psi_g(a) \otimes c_g \qquad \forall c_g \in \mathcal{C}_g,$$

subject to the following axioms:

- (naturality) f or all $f : c_g \to d_g$ and $g : a \to b$, we have $\beta_{b,d_g} \circ (f \otimes g) = (\psi_g(g) \otimes f) \circ \beta_{a,c_g}$.
- (compatibility with G-graded action) For all $g \in G$ and $c_h \in \mathcal{C}_h$, the following diagram commutes:

$$
\begin{array}{ccc}
\psi_g(c_h) \otimes \psi_g(a) & \xrightarrow{\beta^\times} & \psi_{ghg^{-1}}\psi_g(a) \otimes \psi_g(c_h) \\
\downarrow{\scriptstyle \psi_g^2} & & \downarrow{\scriptstyle \psi^2 \otimes \mathrm{id}} \\
\psi_g(c_h \otimes a) & & \psi_{gh}(a) \otimes \psi_g(c_h) \\
\downarrow{\scriptstyle \psi_g(\beta^\times)} & & \uparrow{\scriptstyle \psi^2 \otimes \mathrm{id}} \\
\psi_g(\psi_h(a) \otimes c_h) & \xrightarrow{(\psi_g^2)^{-1}} & \psi_g\psi_h(a) \otimes \psi_g(c_h)
\end{array}
$$

- (heptagon coherences) For all $b_g \in \mathcal{C}_g$, $c_h \in \mathcal{C}_h$, and $a \in \mathcal{C}$,

$$
\begin{array}{ccccc}
 & \swarrow{\scriptstyle \alpha} & b_g \otimes (c_h \otimes a) & \searrow{\scriptstyle \mathrm{id} \otimes \beta^\times} & \\
(b_g \otimes c_h) \otimes a & & & & b_g \otimes (\psi_g(a) \otimes c_h) \\
\downarrow{\scriptstyle \beta^\times} & & & & \downarrow{\scriptstyle \alpha} \\
\psi_{gh}(a) \otimes (b_g \otimes c_h) & & & & (b_g \otimes \psi_h(a)) \otimes c_h \\
\downarrow{\scriptstyle (\psi^2)^{-1} \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \beta^\times \otimes \mathrm{id}} \\
\psi_g\psi_h(a) \otimes (b_g \otimes c_h) & & \xrightarrow{\alpha} & & (\psi_g\psi_h(a) \otimes b_g) \otimes c_h
\end{array}
$$

  For all $c_g \in \mathcal{C}_g$ and $a, b \in \mathcal{C}$,

$$
\begin{array}{ccccc}
 & \swarrow{\scriptstyle \alpha^{-1}} & (c_g \otimes a) \otimes b & \searrow{\scriptstyle \beta^\times \otimes \mathrm{id}} & \\
c_g \otimes (a \otimes b) & & & & (\psi_g(a) \otimes c_g) \otimes b \\
\downarrow{\scriptstyle \beta^\times} & & & & \downarrow{\scriptstyle \alpha^{-1}} \\
\psi_g(a \otimes b) \otimes c_g & & & & \psi_g(a) \otimes (c_g \otimes b) \\
\downarrow{\scriptstyle (\psi_g^2)^{-1} \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id} \otimes \beta^\times} \\
(\psi_g(a) \otimes \psi_g(b)) \otimes c_g & & \xrightarrow{\alpha^{-1}} & & \psi_g(a) \otimes (\psi_g(b) \otimes c_g)
\end{array}
$$

  These are essentially the two hexagon axioms 5.1.2 and 5.1.3, but with an additional tensorator added for the G-action.

A faithfully G-graded fusion category equipped with a G-action ψ satisfying $\psi_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$ and a G-crossed braiding is called a *G-crossed braided fusion category.*

Remark 5.4.14. The basic idea here is that if $b_g \in \mathcal{C}_g$ and $a_h \in \mathcal{C}_h$, then $b_g \otimes a_h \in \mathcal{C}_{gh}$, but $a_h \otimes b_g \in \mathcal{C}_{hg}$. We know gh and hg need not be equal in G, but $g \cdot h = ghg^{-1} \cdot g$. This trick is often employed in semi-direct products of groups and crossed products of algebras by groups. Thus the g-action should move $\mathcal{C}_h$ to $\mathcal{C}_{ghg^{-1}}$ so that $\psi_g(a_h) \otimes b_g \in \mathcal{C}_{gh}$ once again.

Remark 5.4.15. Building on Remark 5.1.6, a G-crossed braided monoidal category is equivalent to a monoidal 2-functor BG into a target 3-category [JPR20]. In the case of a G-crossed braided fusion category $\mathcal{C}$, this 3-category can be taken as the delooping of the Brauer-Picard 2-groupoid $\mathsf{BrPic}(\mathcal{C})$ [ENO10], which is the core of the endomorphism monoidal 2-category $\operatorname{End}(\mathcal{C})$ in the 3-category $\mathsf{MultFusCat}$ of multifusion categories [DSPS13].

Example 5.4.16 ([GNN09])**.** Suppose $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ is a faithfully G-graded fusion category. By Exercise 5.4.9, we get a canonical fully faithful braided monoidal functor $\mathsf{Rep}(G) \to Z(\mathcal{C})$. More is true; in fact, $Z(\mathcal{C})$ is again a G-graded category, and can canonically be equipped with the structure of a G-crossed braided category where the trivial graded component $Z(\mathcal{C})_e \cong Z(\mathcal{C}_e)$.

This result can be used together with *de-equivariantization* (which we will discuss later) in order to provide even more examples [Kir01, Müg04] (see also [GNN09, Thm. 2.12]). If $\mathcal{C}$ is a braided fusion category and $F : \mathsf{Rep}(G) \to \mathcal{C}$ is a fully faithful braided monoidal functor, then the de-equivariantization $\mathcal{C}_G = \mathsf{Mod}_{\mathcal{C}}(\mathcal{O}(G))$ is a G-crossed braided fusion category such that $Z(\mathcal{C}_G) \cong \mathcal{C} \boxtimes \overline{\mathcal{C}_G^{\mathrm{loc}}}$ [DMNO13]. [[more on this later]]

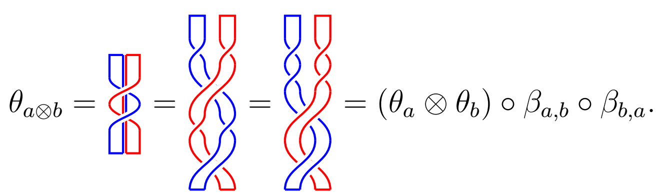
5.5. **Twists and ribbon categories.** Let $\mathcal{C}$ be a braided tensor category

Definition 5.5.1. A *twist* or on $\mathcal{C}$ is a natural transformation $\theta : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$ that satisfies the *balancing* axiom $\theta_{a\otimes b} = (\theta_a \otimes \theta_b) \otimes \beta_{a,b} \otimes \beta_{b,a}$. A braided tensor category equipped with a twist is called a *balanced* tensor category.

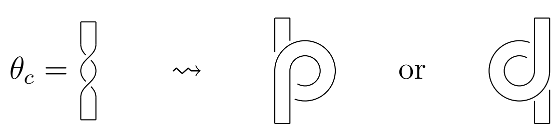
Remark 5.5.2. A twist on a braided monoidal category can be interpreted as thickening a strand slightly into the shape of a ribbon. We can then represent the twist as a 2π rotation of the ribbon along the y-axis.

(5.5.3)

One then sees why we enforce the balancing axiom by observing how such twists of ribbons behave in ambient space.



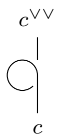
Remark 5.5.4. There are *two ways* a twisted ribbon can be flattened to the plane:

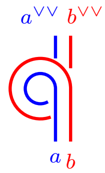


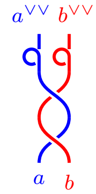
(5.5.5)

We will see in Exercises 5.5.7 and 5.5.8 below that in the context of a balanced tensor categories (which by assumption has duals), these two ways are *not* equivalent.

Definition 5.5.6. Suppose $\mathcal{C}$ is a braided tensor category so that in particular, $\mathcal{C}$ has duals. Given $c \in \mathcal{C}$, the *Drinfeld isomorphism* $\delta_c : c \to c^{\vee\vee}$ is given by

$\delta_c :=$ 

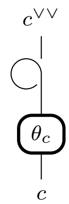
$\rightsquigarrow$ $\delta_{a\otimes b} =$ 

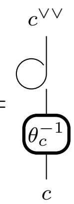
$=$ 

$= (\delta_a \otimes \delta_b) \circ \beta^{-1}_{a,b} \otimes \beta^{-1}_{b,a}.$

Exercise 5.5.7. Suppose $(\mathcal{C}, \theta)$ is a balanced tensor category.

(1) Show that we can define a pivotal structure on $\mathcal{C}$ in *two ways*:

$\varphi^{(1)}_c := \delta_c \circ \theta_c =$ 

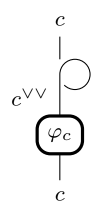
or $\varphi^{(2)}_c :=$ 

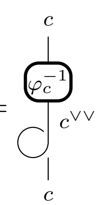
.

(2) Prove that $\varphi^{(1)} = \varphi^{(2)}$ if and only if $\theta^\vee_c = \theta_{c^\vee}$ for all $c \in \mathcal{C}$.

Exercise 5.5.8. Suppose $(\mathcal{C}, \vee, \varphi)$ is a braided pivotal category.

(1) Show that we can define a twist on $\mathcal{C}$ in *two ways*:

$\theta^{(1)}_c :=$ 

or $\theta^{(2)}_c :=$ 

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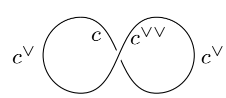
(2) Prove the constructions from (1) above are the inverses of the two constructions from Exercise 5.5.7(1).
(3) Prove that $(\theta^{(1)}_c)^\vee = \theta^{(2)}_{c^\vee}$ and $(\theta^{(2)}_c)^\vee = \theta^{(1)}_{c^\vee}$. Deduce that $\theta^{(1)}_c = \theta^{(2)}_c =: \theta_c$ if and only if $\theta^\vee_c = \theta_{c^\vee}$.
(4) Prove that if $\theta^{(1)} = \theta^{(2)}$, then $(\mathcal{C}, \vee, \varphi)$ is spherical.
(5) Suppose in addition $\mathcal{C}$ is semisimple. Prove that if $(\mathcal{C}, \vee, \varphi)$ is spherical, then $\theta^{(1)} = \theta^{(2)}$.

Definition 5.5.9. A balanced tensor category $(\mathcal{C}, \theta)$ is called a *ribbon* category if $\theta^\vee_c = \theta_{c^\vee}$.

Remark 5.5.10. In light of (5.5.5) and Exercises 5.5.7 and 5.5.8, we see that balanced rigid/braided pivotal tensor categories admit a graphical calculus where strings are thickened into ribbons, but the ribbons are fixed to the plane in the *blackboard framing*, in which the two ribbon loops from the right hand side of (5.5.5) are *not* equal. These two ribbon loops are equal if and only if $\theta^\vee_c = \theta_{c^\vee}$. Thus ribbon categories admit a graphical calculus where ambient 3D isotopies are allowed.

Exercise 5.5.11. Suppose $\mathcal{C}$ is ribbon. Show that the twist θ_c for $c \in \operatorname{Irr}(\mathcal{C})$ is given in diagrams by

$\theta_c = \frac{1}{d_c} \cdot$ 

.

Hint: Use Exercise 5.5.8.

5.5.1. *Unitary ribbon categories.* Suppose $\mathcal{C}$ is a unitary tensor category. By [Pen20, Thm. A], equivalence classes of unitary dual functors on $\mathcal{C}$ are in bijective correspondence with $\mathrm{Hom}(U_{\mathcal{C}} \to \mathbb{R}_{>0})$, where $U_{\mathcal{C}}$ is the universal grading group of $\mathcal{C}$. From a unitary dual functor $\vee$, one gets a group homomorphism $\pi : U_{\mathcal{C}} \to \mathbb{R}_{>0}$ by setting

$$\pi_{\vee}(c) := \frac{\dim_L^{\vee}(c)}{\dim_R^{\vee}(c)} \qquad\qquad \forall\, c \in \mathrm{Irr}(\mathcal{C}).$$

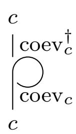
Given a group homomorphism $\pi : U_{\mathcal{C}} \to \mathbb{R}_{>0}$, there is a unique π-*balanced* unitary dual functor $\vee_\pi$ satisfying

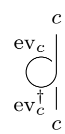
$$\dim_L^{\vee_\pi}(c) = \pi(c) \cdot \dim_R^{\vee_\pi}(c) \qquad\qquad \forall\, c \in \mathrm{Irr}(\mathcal{C}).$$

Recall that the canonical unitary pivotal structure associated to $\vee$ is given by

$$\varphi_c := (\mathrm{coev}_c^{\dagger} \otimes \mathrm{id}_{c^{\vee\vee}}) \circ (\mathrm{id}_c \otimes \mathrm{coev}_{c^{\vee}}).$$

Exercise 5.5.12. Suppose β is a unitary braiding on $\mathcal{C}$ and $\vee$ is a unitary dual functor. Recall the definitions of $\theta_c^{(1)}, \theta_c^{(2)}$ from Exercise 5.5.8.

(1) Prove that $\theta_c^{(1)} =$ 

and $\theta_c^{(2)} =$ 

for all $c \in \mathcal{C}$.

(2) Show that $(\theta_c^{(1)})^{\dagger} = (\theta_c^{(2)})^{-1}$ for all $c \in \mathcal{C}$. Deduce that θ is unitary if and only if $\theta_c^{(1)} = \theta_c^{(2)}$ for all $c \in \mathcal{C}$.

Definition 5.5.13. A *unitary ribbon category* is a unitary tensor category equipped with a unitary braiding and a unitary dual functor such that either of the canonical twists is unitary. Observe this automatically implies $(\mathcal{C}, \beta, \theta)$ is ribbon by Exercises 5.5.8 and 5.5.12.

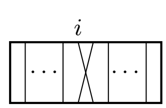
5.6. **Symmetric fusion categories.** For this section, $(\mathcal{C}, \beta)$ is a braided fusion category.

Definition 5.6.1. We call $\mathcal{C}$ *symmetric* if $\beta_{a,b} \circ \beta_{b,a} = \mathrm{id}_{a \otimes b}$ for all $a, b \in \mathcal{C}$.

Remark 5.6.2. Just as braided fusion categories give representations of the braid group, symmetric fusion categories give representations of the symmetric group. Indeed, recall that the symmetric group S_n has generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

(S1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$,
(S2) $\sigma_i \sigma_{i\pm1} \sigma_i = \sigma_{i\pm1} \sigma_i \sigma_{i\pm1}$, and
(S3) $\sigma_i^2 = 1$.

In S_n, there is no difference between a crossing and its inverse, as both are σ_i. We thus represent the generators σ_i graphically by

$\sigma_i :=$ 

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Example 5.6.3. Let G be a finite group. The fusion category $\mathsf{Rep}(G)$ of finite dimensional complex representations is symmetric with the swap braiding. Observe that $\mathsf{Rep}(G)$ has a canonical symmetric fiber functor to $\mathsf{Vec}_{\mathsf{fd}}$. A symmetric fusion category equivalent to $\mathsf{Rep}(G)$ is called *Tannakian*.

Example 5.6.4. A finite *super-group* is a pair (G, z) where G is a finite group and $z \in Z(G)$ is a distinguished order 2 element. The symmetric fusion category $\mathsf{sRep}(G, z)$ has objects representations of G on finite dimensional complex super-vector spaces such that z acts by the parity operator, i.e., $(V_0 \oplus V_1, \pi)$ such that $\pi_z|_{V_i} = (-1)^i$. The braiding on $\mathsf{sRep}(G, z)$ is the usual braiding on sVec from Exercise 5.1.11. Observe that $\mathsf{sRep}(G, z)$ has a canonical symmetric fiber functor to sVec. A symmetric fusion category equivalent to $\mathsf{sRep}(G, z)$ is called *super-Tannakian*.

Remark 5.6.5. Every Tannakian fusion category is super-Tannakian with $z = 1$.

Exercise 5.6.6. Suppose $\mathcal{C}$ is a symmetric fusion category.

(1) Show that setting $\theta_c := \mathrm{id}_c$ defines a ribbon twist on $\mathcal{C}$.
(2) Deduce that the induced pivotal structure from Exercise 5.5.7 is spherical.
(3) Show that for $\mathcal{C} = \mathsf{sRep}(G, z)$, the quantum dimension is the super-dimension of the representation, i.e., if $V = V_0 \oplus V_1$, then $\dim(V) = \dim(V_0) - \dim(V_1)$.

Theorem 5.6.7 ([Del02])**.** *Every symmetric fusion category is super-Tannakian.*

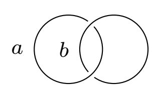
Proof. We omit the proof and refer the reader to [EGNO15, §9.9]. ☐

Corollary 5.6.8. *A symmetric fusion category $\mathcal{C}$ is Tannakian if and only if the canonical quantum dimension of every object from Exercise 5.6.6 is positive.*

5.7. **Modular tensor categories.** For this section, $\mathcal{C}$ denotes a ribbon fusion category unless stated otherwise. Our conventions for modular categories agree with [BBCW19] and disagree with [EGNO15, §8.13-14].

Definition 5.7.1. The *S-matrix* of $\mathcal{C}$ is the $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$ matrix whose a, b-th entry is given by

$$S_{a,b} := \frac{1}{D_{\mathcal{C}}} \cdot$$



where $D_{\mathcal{C}} := \sum_{c \in \mathrm{Irr}(\mathcal{C})} d_c^2$ is the *global dimension* of $\mathcal{C}$. Observe that $S_{a,1_{\mathcal{C}}} = S_{1_{\mathcal{C}},a} = d_a / D_{\mathcal{C}}$. We call $\mathcal{C}$ a *modular tensor category* (MTC) if S is invertible.

The *T-matrix* of $\mathcal{C}$ is the diagonal $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$ matrix whose (c, c)-th entry is θ_c, i.e.,

$$T := \mathrm{diag}(\theta_c)_{c \in \mathrm{Irr}(\mathcal{C})}.$$

Remark 5.7.2. By [BNRW16b], there are only finitely many MTCs with a given *rank* (number of simple objects). (By [JMNR19], there are only finitely many (G-crossed) braided fusion categories with a given rank.

A fusion category $\mathcal{C}$ is called *weakly integral* if

$$\mathrm{FPdim}(\mathcal{C}) := \sum_{c \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}(c)^2 \in \mathbb{Z}.$$

It can be shown using number theoretic techniques that the FP dimensions of simple objects in a weakly integral fusion category lie in a quadratic extension of $\mathbb{Q}$, so that $\mathcal{C}$ has a canonical $\mathbb{Z}/2$-grading (exercise!). Moreover, a weakly integral fusion category is pseudo-unitary, i.e., there exists a (unique) spherical structure in which all dimensions are positive [EGNO15, Prop. 9.6.5].

Classification by Rank.

Type	Rank	Citation
Fusion	2	[Ost03]
Pseudounitary fusion	3	[Ost13]
Ribbon fusion (premodular)	4	[Bru16]
MTC	4	[RSW09]
Weakly integral MTCs	7	[BGN+16]

Partial progress has been made on pseudounitary rank 4 fusion categories [Lar14] and rank 5 MTCs [BNRW16a]. There has even been some progress on rank 6 MTCs [Gre19].

Remark 5.7.3. A braided fusion category $\mathcal{C}$ has *Property (F)* if the images of the braid groups B_n in $\operatorname{End}(\mathcal{C}(a \to c^{\otimes n}))$ are finite. The *Property (F) Conjecture* of Naidu-Rowell [NR11] posits that Property (F) is equivalent to being weakly integral. At the time of writing, this conjecture has been verified for all known examples.

A simple example of a non-weakly integral braided fusion category is the Fib UMTC. It is well-known that not only are the braid group representations from Fib infinite, but they are also *universal* for topological quantum computation [FLW02].

Exercise 5.7.4. Calculate the S and T matrices of $Z(\mathsf{Vec}_{\mathsf{fd}}(\mathbb{Z}/n))$.

Example 5.7.5. By [Müg03, Thm. 1.2], the Drinfeld center $Z(\mathcal{C})$ for any spherical fusion category is automatically modular. See also Corollary 5.8.10 below.

Exercise 5.7.6. Show that $\mathsf{Vec}_{\mathsf{fd}}(G)$ has a unique spherical structure in which the dimension of every object is $+1$.

Exercise 5.7.7. Consider the braided fusion categories $\mathcal{C}(A, q)$ from Remark 5.1.14 equipped with the canonical spherical structure from Exercise 5.7.6.

(1) Show that if $\mathcal{C}(A, q)$ is symmetric, then ω is trivial.
(2) Show that $\mathcal{C}(A, q)$ is modular if and only if q is non-degenerate.

Exercise 5.7.8. Prove that $\mathcal{C}$ is symmetric if and only if $\operatorname{rank}(S) = 1$. In this sense, symmetric fusion categories are the 'opposite' of modular tensor categories.

Exercise 5.7.9. Show that S is a symmetric matrix, i.e., $S_{a,b} = S_{b,a}$ for all $a, b \in \operatorname{Irr}(\mathcal{C})$. *Hint: Ribbon categories are necessarily spherical. Realize $S_{a,b}$ as the trace of a product of braidings, and use the tracial property.*

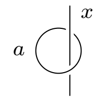
Exercise 5.7.10. Prove that for $a, b \in \operatorname{Irr}(\mathcal{C})$,

$$S_{a,b} = \frac{1}{D_{\mathcal{C}}} \sum_{c \in \operatorname{Irr}(\mathcal{C})} N_{\bar{a}b}^{c} \frac{\theta_c}{\theta_a \theta_b} d_c \tag{5.7.11}$$

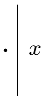
$$= \frac{1}{D_{\mathcal{C}}} \sum_{c \in \operatorname{Irr}(\mathcal{C})} N_{ab}^{c} \frac{\theta_a \theta_b}{\theta_c} d_c \tag{5.7.12}$$

Exercise 5.7.13. Suppose $\mathcal{C}$ is a ribbon fusion category.

(1) Prove that for all $a, x \in \mathrm{Irr}(\mathcal{C})$,



$$= \frac{S_{a,x}}{S_{1_\mathcal{C},x}} \cdot$$



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(2) Use the fusion relation to show that for all $a, b, x \in \mathrm{Irr}(\mathcal{C})$,
$$\frac{S_{a,x}}{S_{1_\mathcal{C},x}} \frac{S_{b,x}}{S_{1_\mathcal{C},x}} = \sum_{c \in \mathrm{Irr}(\mathcal{C})} N_{ab}^c \frac{S_{c,x}}{S_{1_\mathcal{C},x}}. \tag{5.7.14}$$
Deduce that $a \mapsto S_{a,x}/S_{1_\mathcal{C},x}$ defines an algebra map $K_0(\mathcal{C}) \to \mathbb{C}$.

(3) Deduce that when $\mathcal{C}$ is modular (so S is invertible), the *Verlinde formula* holds:
$$N_{ab}^c = \sum_{x \in \mathrm{Irr}(\mathcal{C})} \frac{S_{a,x} S_{b,x} (S^{-1})_{c,x}}{S_{1_\mathcal{C},x}}.$$

Exercise 5.7.15.

(1) Suppose K is a fusion ring with $\mathbb{Z}_+$-basis B and $\chi_1, \chi_2 : K \to \mathbb{C}$ are two distinct unital algebra homomorphisms. Prove that $\sum_{b \in B} \chi_1(b)\chi_2(b^*) = 0$.
Hint: Consider the χ_1-regular element $R_1 := \sum_{b \in B} \chi_1(b) b^*$, which is not zero. Show that $kR_1 = \chi_1(k) R_1$ for all $k \in K$. Then apply χ_2.
Note: This is [EGNO15, Lem. 8.14.1].

(2) Deduce from Exercise 5.7.13(2) that for all $a, b \in \mathrm{Irr}(\mathcal{C})$,
$$\sum_{a \in \mathrm{Irr}(\mathcal{C})} d_a d_b S_{a,b} = \delta_{b=1_\mathcal{C}}. \tag{5.7.16}$$

Exercise 5.7.17. Use (5.7.14) and Exercise 5.7.15 to prove that $(S^2)_{a,b} = \delta_{b=\overline{a}}$ and $(S^{-1})_{a,b} = S_{\overline{a},b}$. Deduce that $S^4 = I$.

Exercise 5.7.18. Prove that the S matrix *diagonalizes* the fusion rules.

Exercise 5.7.19. Suppose $\mathcal{C}$ is a unitary MTC. Show $S_{a,b} = \overline{S_{\overline{a},b}}$ for all $a, b \in \mathrm{Irr}(\mathcal{C})$. Deduce that S is unitary.

Definition 5.7.20. Suppose $\mathcal{C}$ is a modular category. The *Gauss sums* of $\mathcal{C}$ are given by
$$\tau^\pm(\mathcal{C}) := \sum_{c \in \mathrm{Irr}(\mathcal{C})} \theta_c^{\pm 1} d_c^2.$$
The *multiplicative central charge* of $\mathcal{C}$ is given by
$$\xi(\mathcal{C}) := \frac{\tau^+(\mathcal{C})}{\sqrt{D_\mathcal{C}}},$$

Exercise 5.7.21. Prove that $\tau^+(\mathcal{C})\tau^-(\mathcal{C}) = \dim(\mathcal{C})$. Deduce that $\xi(\mathcal{C})^2 = \tau^+(\mathcal{C})/\tau^-(\mathcal{C})$.

Recall that
$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$
The center of $SL_2(\mathbb{Z})$ is $\{-I\}$, and we define $PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z})/\{\pm I\}$. The group $PSL_2(\mathbb{Z})$ acts on the Riemann sphere via fractional linear transformations:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) := \frac{az+b}{cz+d} \qquad \forall z \in \widehat{\mathbb{C}}.$$

Remark 5.7.22. There is an ambiguity whether $SL_2(\mathbb{Z})$ or $PSL_2(\mathbb{Z})$ is called the *modular group*.

We saw in the Lattice Models notes that

$$SL_2(\mathbb{Z}) \cong \langle s, t | s^2 = (st)^3,\ s^4 = I \rangle,$$

and that the matrices $\mathfrak{s} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathfrak{t} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ satisfy these relations. Moreover, $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$.

Theorem 5.7.23. *Let $\mathcal{C}$ be a MTC and S, T its S and T matrices. Then $S^4 = I$ and $(ST)^3 = \xi(\mathcal{C})S^2$. Hence $\mathfrak{s} \mapsto S$ and $\mathfrak{t} \mapsto T$ define a projective representation of $SL_2(\mathbb{Z})$.*

Proof. That $S^4 = I$ follows immediately from Exercise 5.7.17. Clearly $(TST)_{a,b} = \theta_a \theta_b S_{a,b}$. We calculate

$$\begin{aligned}
(STS)_{a,b} &= \sum_{x \in \operatorname{Irr}(\mathcal{C})} S_{a,x} \theta_x S_{x,b} = \sum_{x \in \operatorname{Irr}(\mathcal{C})} \theta_x S_{a,x} S_{b,x} \\
&\underset{(5.7.14)}{=} \frac{1}{D_{\mathcal{C}}} \sum_{x \in \operatorname{Irr}(\mathcal{C})} \theta_x d_x \sum_{c \in \operatorname{Irr}(\mathcal{C})} N_{ab}^c S_{c,x} = \frac{1}{D_{\mathcal{C}}} \sum_{c \in \operatorname{Irr}(\mathcal{C})} N_{ab}^c \sum_{x \in \operatorname{Irr}(\mathcal{C})} \theta_x d_x S_{c,x} \\
&\underset{(5.7.11)}{=} \frac{1}{D_{\mathcal{C}}^2} \sum_{c \in \operatorname{Irr}(\mathcal{C})} N_{ab}^c \sum_{x \in \operatorname{Irr}(\mathcal{C})} \theta_x d_x \left(\sum_{y \in \operatorname{Irr}(\mathcal{C})} N_{\overline{x}c}^y \frac{\theta_y}{\theta_x \theta_c} d_y \right) \\
&= \frac{1}{D_{\mathcal{C}}^2} \sum_{c,y \in \operatorname{Irr}(\mathcal{C})} N_{ab}^c \frac{\theta_y}{\theta_c} d_y \sum_{x \in \operatorname{Irr}(\mathcal{C})} N_{c\overline{y}}^x d_x = \frac{1}{D_{\mathcal{C}}^2} \sum_{c,y \in \operatorname{Irr}(\mathcal{C})} N_{ab}^c \frac{\theta_y}{\theta_c} d_c d_y^2 \\
&= \frac{1}{D_{\mathcal{C}}^2} \left(\sum_{y \in \operatorname{Irr}(\mathcal{C})} \theta_y d_y^2 \right) \sum_{c \in \operatorname{Irr}(\mathcal{C})} \frac{N_{ab}^c d_c}{\theta_c} = \frac{1}{D_{\mathcal{C}}^2} \frac{\tau^+(\mathcal{C})}{\theta_a \theta_b} \sum_{c \in \operatorname{Irr}(\mathcal{C})} N_{ab}^c \frac{\theta_a \theta_b}{\theta_c} d_c \\
&\underset{(5.7.12)}{=} \frac{\xi(\mathcal{C})}{\theta_a \theta_b} S_{a,b}
\end{aligned}$$

The result follows. ∎

Exercise 5.7.24. In this exercise, we will prove Vafa's Theorem, which states that twists of simple objects in a ribbon fusion category are roots of unity. Our walkthrough follows [EGNO15, §8.18]. Let $w, x, y, z \in \operatorname{Irr}(\mathcal{C})$.

(1) Consider the spaces $\mathcal{C}(w \to x \otimes y \otimes z) \cong \mathcal{C}(w \to y \otimes x \otimes z)$. Consider the linear operators

$$\begin{aligned}
B_{xy,z} &:= \beta_{xy,z} \circ \beta_{z,xy}|_{\mathcal{C}(w \to x \otimes y \otimes z)} \\
B_{y,z} &:= \operatorname{id}_x \otimes (\beta_{y,z} \circ \beta_{z,y})|_{\mathcal{C}(w \to x \otimes y \otimes z)} \\
B_{x,z} &:= \operatorname{id}_y \otimes (\beta_{x,z} \circ \beta_{z,x})|_{\mathcal{C}(w \to y \otimes x \otimes z)}
\end{aligned}$$

Use the hexagon relation to prove that $\det(B_{xy,z}) = \det(B_{y,z}) \det(B_{x,z})$.

(2) Use the balance axiom to show that the identity in (1) is equivalent to

$$\prod_{c\in\operatorname{Irr}(\mathcal{C})}\left(\frac{\theta_c\theta_z}{\theta_w}\right)^{N_{xy}^cN_{cz}^w}=\prod_{a\in\operatorname{Irr}(\mathcal{C})}\left(\frac{\theta_x\theta_z}{\theta_a}\right)^{N_{xz}^aN_{ya}^w}\prod_{b\in\operatorname{Irr}(\mathcal{C})}\left(\frac{\theta_y\theta_z}{\theta_b}\right)^{N_{yz}^bN_{xb}^w}.$$

(3) Use the same variable $a=b=c$ above to deduce that

$$(\theta_w\theta_x\theta_y\theta_z)^{N_{xyz}^w}=\prod_{c\in\operatorname{Irr}(\mathcal{C})}\theta_c^{N_{xy}^cN_{czw}+N_{xz}^cN_{yc}^w+N_{yz}^cN_{xc}^w}.$$

(4) Write $t_c:=\log(\theta_c)2\pi i\in\mathbb{C}/\mathbb{Z}$ for some branch of the logarithm. Apply $2\pi i\log$ to the equation above to get

$$N_{xyz}^w\otimes(t_w+t_x+t_y+t_z)=\sum_{c\in\operatorname{Irr}(\mathcal{C})}(N_{xy}^cN_{cz^w}+N_{xz}^cN_{yc}^w+N_{yz}^cN_{xc}^w)\otimes t_c \qquad \text{in } \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{C}/\mathbb{Z}.$$

(5) It is a remarkable fact due to [dBG91, CGR00] that each d_c in a modular category is a cyclotomic integer, i.e., an algebraic integer in a cyclotomic field. Prove this result holds in a ribbon fusion category.
Hint: Find a fully faithful embedding of $\mathcal{C}$ into $Z(\mathcal{C})$ using the braiding. Then use the fact that $Z(\mathcal{C})$ is modular.

(6) Let K be the ring of integers in $\mathbb{Q}(d_c)_{c\in\operatorname{Irr}(\mathcal{C})}$. Multiply both sides by $d_wd_xd_y$, sum over $w,x,y\in\operatorname{Irr}(\mathcal{C})$, and simplify to see that $D_{\mathcal{C}}d_z\otimes t_z=0$ in $K\otimes_{\mathbb{Z}}\mathbb{C}/\mathbb{Z}$.

(7) Use the fact that d_z^2 divides $D_{\mathcal{C}}$ ($D_{\mathcal{C}}/d_z^2\in K$) to show that $t_z=0$ in $\mathbb{C}/\mathbb{Z}$. Deduce that θ_z is a root of unity.

Exercise 5.7.25. Find all MTCs with $\mathbb{Z}/2$ fusion rules.

Exercise 5.7.26 ($\star\star$)**.** Find all MTCs with Ising fusion rules:

$$\sigma\otimes\sigma\cong 1\oplus\psi \qquad\qquad \psi\otimes\psi\cong 1.$$

Exercise 5.7.27. Let $\mathcal{C}$ be an MTC.

(1) Prove that the multiplicative central charge $\xi(\mathcal{C})$ is a root of unity.

(2) The *chiral central charge (mod 8)* of $\mathcal{C}$ is the number $c_-\in\mathbb{R}/8\mathbb{R}$ such that

$$\xi(\mathcal{C})=\exp\left(\frac{2\pi i c_-}{8}\right).$$

Prove that the chiral central charge c_- is rational.

Remark 5.7.28. It was proven in [Müg03, Prop. 5.18 and Rem. 5.19] that given a spherical fusion category $\mathcal{C}$, $c_{Z(\mathcal{C})}\equiv 0 \mod 8$.

5.8. **Non-degenerate braided fusion categories.** In this section, we study non-degenerate braided fusion categories, which are basically similar to modular categories, but without any pivotal structure. For this section, $(\mathcal{C},\beta)$ denotes a braided fusion category.

Definition 5.8.1. Two objects $a,b\in\mathcal{C}$ *centralize* each other if $\beta_{a,b}\circ\beta_{b,a}=\mathrm{id}_{a\otimes b}$. Given a fusion subcategory $\mathcal{D}\subset\mathcal{C}$, the *centralizer* $\mathcal{D}'$ is the full subcategory of $\mathcal{C}$ of objects which centralize all objects in $\mathcal{D}$.

An object $a\in\mathcal{C}$ is called *transparent* if for all $b\in\mathcal{C}$, $\beta_{a,b}\circ\beta_{b,a}=\mathrm{id}_{a\otimes b}$. That is, $a\in\mathcal{C}$ is *transparent* if and only if $a\in\mathcal{C}'$. We call $\mathcal{C}'$ the *Müger center*, which is also denoted $Z_2(\mathcal{C})$.

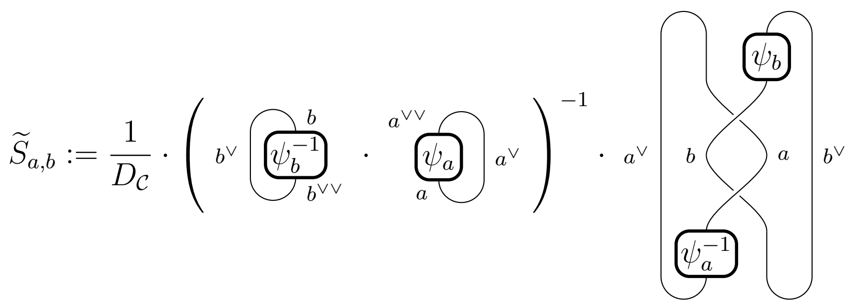
Remark 5.8.2. Observe that $Z_2(\mathcal{C})$ is a symmetric fusion category, so $\mathcal{C}$ is symmetric if and only if $\mathcal{C} = Z_2(\mathcal{C})$.

Definition 5.8.3. A ribbon category $\mathcal{C}$ is called *super modular* if $Z_2(\mathcal{C}) = \mathsf{sVec}$.

Remark 5.8.4. At the time of writing, an important open question is whether every super modular category $\mathcal{C}$ admits a *minimal modular extension*, i.e., a modular $\mathcal{D} \supset \mathcal{C}$ such that $\dim(\mathcal{D}) = 2\dim(\mathcal{C})$. Recently, a solution to this problem was announced in the affirmative by Johnson-Freyd and Reutter.

Even though a braided fusion category does not have a pivotal structure, we can still define an S-matrix.

Definition 5.8.5. For each $c \in \operatorname{Irr}(\mathcal{C})$, pick an *arbitrary* isomorphism $\psi_c : c \to c^{\vee\vee}$. Define the $\widetilde{S}$*-matrix* by



Observe that while ψ_a, ψ_b were arbitrary, the value of $\widetilde{S}_{a,b}$ does not depend on ψ_a nor ψ_b.

We call $\mathcal{C}$ *non-degenerate* if $\widetilde{S}$ is invertible.

Definition 5.8.6. Observe that $\mathcal{C}$ lifts to $Z(\mathcal{C})$ in two ways. That is, $c \mapsto (c, \beta_{c,-})$ extends to a braided tensor functor $\mathcal{C} \to Z(\mathcal{C})$, and $c \mapsto (c, \beta_{-,c}^{-1})$ extends to a braided tensor functor $\mathcal{C}^{\mathrm{rev}} \to Z(\mathcal{C})$. This gives a braided tensor functor

$$\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \to Z(\mathcal{C}).$$

We call $\mathcal{C}$ *factorizable* if this braided tensor functor is an equivalence.

Exercise 5.8.7 ($\star$)**.** Let $\mathcal{C}$ be a fusion category. Prove that $Z(\mathcal{C})$ is factorizable.

Theorem 5.8.8. *The following are equivalent for a braided fusion category $\mathcal{C}$.*

(1) $\mathcal{C}$ is non-degenerate.
(2) $\mathcal{C}' = Z_2(\mathcal{C}) = \mathsf{Vec}_{\mathsf{fd}}$.
(3) $\mathcal{C}$ is factorizable.

Proof. We omit the proof and refer the reader to [EGNO15, §8.20]. □

Corollary 5.8.9. *A ribbon category $\mathcal{C}$ is modular if and only if $Z_2(\mathcal{C}) = \mathsf{Vec}_{\mathsf{fd}}$.*

Corollary 5.8.10. *If $\mathcal{C}$ is a spherical fusion category, then $Z(\mathcal{C})$ is modular.*

Exercise 5.8.11. Suppose $\mathcal{C}$ is a braided fusion category. Let $\operatorname{Inv}(\mathcal{C})$ be the group of isomorphism classes of invertible objects of $\mathcal{C}$.

(1) Show that for every $a \in \operatorname{Inv}(\mathcal{C})$ and $c \in \mathcal{C}$, $\dim(\operatorname{End}(a \otimes c)) = 1$. Deduce there is a scalar $\gamma_a(c) \in \mathbb{C}^\times$ such that $\beta_{a,c} \circ \beta_{c,a} = \gamma_a \operatorname{id}_{a,c}$.

(2) Show that the assignment $\mathrm{Irr}(\mathcal{C}) \ni c \mapsto \gamma_a(c) \in \mathbb{C}^\times$ gives a group homomorphism $U_\mathcal{C} \to \mathbb{C}^\times$.
(3) Now suppose $\mathcal{C}$ is non-degenerate. Prove that every group homomorphism $U_\mathcal{C} \to \mathbb{C}^\times$ is of the form γ_a for some $a \in \mathrm{Inv}(\mathcal{C})$.
Hint: Combine Exercise 5.4.5 and Theorem 5.8.8.

Exercise 5.8.12. Suppose $\mathcal{C}, \mathcal{D}$ are braided fusion categories with $\mathcal{C}$ non-degenerate and $F : \mathcal{C} \to \mathcal{D}$ is a braided tensor functor.

(1) Show that F is fully faithful.
(2) We now identify $\mathcal{C}$ with a full braided fusion subcategory of $\mathcal{D}$ via F using (1). Show that $\mathcal{D}$ factorizes as $\mathcal{C} \boxtimes \mathcal{B}$ where $\mathcal{B}$ is another braided fusion category.

TODO: degree of difficulty?

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