Chapter 2

Operator algebras and their modules

In our study of finite dimensional operator algebras and in our future study of tensor categories, it will be necessary to discuss abstract finite dimensional complex algebras and their semisimplifications. The next two sections do this background work. Some of this presentation is inspired by [Lam01, §3-4].

2.1 Complex algebras and modules

Definition 2.1.1 — A *complex algebra* is a complex vector space A equipped with a compatible associative multiplication satisfying

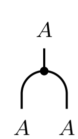
- (distributive) $(a+b)\cdot c = a\cdot c + b\cdot c$ and $a\cdot(b+c) = a\cdot b + a\cdot c$ for all $a,b,c\in A$, and
- (compatibility with scalars) $(\lambda a)\cdot(\mu b) = (\lambda\mu)(a\cdot b)$ for all $a,b\in A$ and $\lambda,\mu\in\mathbb{C}$.

These conditions just say that $\cdot : A^2 \to A$ is bilinear, equivalently, $\cdot : A\otimes A\to A$ is linear. We assume that complex algebras are finite dimensional unless stated otherwise.

An algebra A is called:

- *unital* if there is an element $1\in A$ called the *unit* such that $1\cdot a = a = a\cdot 1$ for all $a\in A$.
- *commutative* if $a\cdot b = b\cdot a$ for all $a,b\in A$.

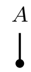
Remark 2.1.2. Using the diagrammatic calculus from §1.6, we may represent the multiplication map $\cdot : A\otimes A\to A$ as a trivalent vertex:



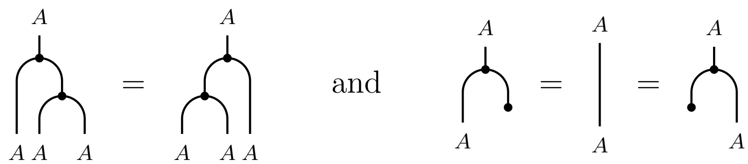
Note that we may identify $a \in A$ with the linear map $\mathbb{C} \to A$ determined by $1_{\mathbb{C}} \mapsto a$:

$a =$ 

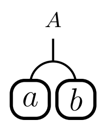
In particular, we view the unit $1 \in A$ as a map $i\colon \mathbb{C} \to A$ which we represent as a univalent vertex:

$i =$ 

Using this diagrammatic calculus for the algebra A, we may then describe the associativity and unitality axioms as follows.



In what follows, we usually omit the dot and simply write ab for multiplication, e.g.,

$ab =$  .

Exercise 2.1.3. Prove that unitality is a property of an algebra and not extra structure, i.e., the unit of an algebra is unique when it exists.

Example 2.1.4 (Matrix algebras) — Our main example of a complex algebra is $B(H)$ for H a finite dimensional Hilbert space, which is isomorphic to a matrix algebra $M_n(\mathbb{C})$ with $n = \dim(H)$.

Example 2.1.5 (Opposite algebra) — If A is an algebra, then the *opposite algebra* A^{op} has the same underlying vector space as A, but multiplication is given by $a \cdot_{\text{op}} b := ba$.

Example 2.1.6 — The *center* of an algebra A is

$$Z(A) := \{z \in A \,|\, za = az \text{ for all } a \in A\}\,.$$

It is always a commutative (unital) subalgebra. Recall that $Z(M_n(\mathbb{C})) = \mathbb{C} \cdot 1$, where 1 is the identity matrix.

Example 2.1.7 (Group algebras) — For a finite group G, the *group algebra* $\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C}g$ with multiplication $(\lambda g)(\mu h) := (\lambda\mu)(gh)$ is a unital complex algebra.

Exercise 2.1.8. Prove that $\mathbb{C}[G]$ is commutative if and only if G is abelian.

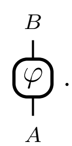
Definition 2.1.9 — We say that a linear operator $\varphi \colon A \to B$ between algebras A, B is an *algebra homomorphism* when

$$\varphi(a_1)\varphi(a_2) = \varphi(a_1 a_2) \qquad \forall\, a_1, a_2 \in A. \tag{2.1.10}$$

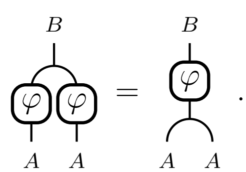
When A and B are unital, we further require $\varphi(1_A) = 1_B$.

- An invertible algebra map is called an *algebra isomorphism*.
- An algebra map $A \to A$ is called an *algebra endomorphism*.
- An invertible algebra endomorphism is called an *algebra automorphism*.

Remark 2.1.11. As in §1.6 and Remark 2.1.2, we graphically represent an algebra homomorphism $\varphi \colon A \to B$ by



Equation (2.1.10) is expressed graphically by



Proposition 2.1.12 — Every algebra automorphism of $M_n(\mathbb{C})$ is *inner*, that is, if $\theta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a complex algebra isomorphism, then there is an invertible $h \in M_n(\mathbb{C})$ such that $\theta(x) = h^{-1}xh$ for all $x \in M_n(\mathbb{C})$.

Proof. Let $\{e_j\}$ be an ONB for $\mathbb{C}^n$. Then $\{|e_i\rangle\langle e_j|\}$ is a system of matrix units (see Example 1.4.15) for $M_n(\mathbb{C})$. Since θ is an algebra map, $\{p_{ij} := \theta(|e_i\rangle\langle e_j|)\}$ is a collection of rank one operators satisfying (SMU1) and (SMU2), i.e., $p_{ij}p_{k\ell} = \delta_{j=k}p_{i\ell}$ and $\sum_j p_j = 1$. Pick $f_1 \in \operatorname{im}(p_{11})$ and set $f_j := p_{j1}f_1$ for all $j > 1$. Note that

$$p_{ij}f_k = p_{ij}p_{k1}f_1 = \delta_{j=k}p_{i1}f_1 = \delta_{j=k}f_i. \tag{2.1.13}$$

Now define $h \in M_n(\mathbb{C})$ by $hf_j := e_j$. Then since $f_k = p_{kk} f_k$ for all k,

$$h^{-1}|e_i\rangle\langle e_j|hf_k = h^{-1}|e_i\rangle\langle e_j|e_k = \delta_{j=k}h^{-1}e_i = \delta_{j=k}f_i \underset{(2.1.13)}{=} p_{ij}f_k = \theta(|e_i\rangle\langle e_j|)f_k \qquad \forall\, i,j,k.$$

Since $\{|e_i\rangle\langle e_j|\}$ is a basis for $M_n(\mathbb{C})$ and $\{f_k\}$ is a basis for $\mathbb{C}^n$, the result follows. □

Definition 2.1.14 — Suppose A is an algebra. A *right ideal* of A is a subspace $I \subset A$ such that $Ia \subset I$ for all $a \in A$. Similarly, we can define left and 2-sided ideals.

We call A *simple* if the only 2-sided ideals of A are 0 and A itself.

Exercise 2.1.15. Prove that any descending or ascending chain of ideals in a finite dimensional algebra A eventually stabilizes. (This means that A is both *Artinian* and *Noetherian*.) Deduce that every ideal both contains a minimal ideal and is contained in a maximal ideal. Here, ideals can be taken to be right, left, or 2-sided, as desired.

Example 2.1.16 — Suppose $\varphi : A \to B$ is a (unital) algebra homomorphism. Then $\ker(\varphi) \subset A$ is a 2-sided ideal.

Proposition 2.1.17 — The complex algebra $M_n(\mathbb{C})$ is simple. Hence any algebra map from $M_n(\mathbb{C})$ into another complex algebra is either injective or the zero map.

Proof. Suppose I is a 2-sided ideal, and let $x \in I$ be non-zero. Pick a unit vector $\xi \in \mathbb{C}^n$ such that $x\xi \neq 0$, and set $\eta := x\xi/\|x\xi\|$. Then $|\eta\rangle\langle\eta| \cdot x \cdot |\xi\rangle\langle\xi| \in I$ is non-zero, so I contains the rank one operator $|\eta\rangle\langle\xi|$ and the minimal projection $|\xi\rangle\langle\xi|$.

Extend ξ to an ONB $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ with $e_1 = \xi$. Observe that $|e_j\rangle\langle e_j| = |e_j\rangle\langle e_1| \cdot |e_1\rangle\langle e_1| \cdot |e_1\rangle\langle e_j| \in I$ for all j, so $1 = \sum_{j=1}^n |e_j\rangle\langle e_j| \in I$.

The last statement follows by analyzing the kernel of such a map as in Example 2.1.16. □

Just as groups act on sets, algebras act on modules.

Definition 2.1.18 — Suppose A is as unital algebra. A *(right) module* M_A for A is a vector space M equipped with a bilinear map $\lhd\colon M \times A \to M$ satisfying

- (associativity) $(m \lhd a) \lhd b = m \lhd (a \cdot b)$ for $m \in M$ and $a, b \in A$;
- (unitality) $m \lhd 1 = m$ for $m \in M$.

The data of a (right) module is equivalent to an algebra homomorphism $\rho\colon A^{\mathrm{op}} \to \mathrm{End}(M)$. We say M_A is *faithful* if the map ρ is injective.

We can similarly define the notion of a left module.

Since modules have underlying vector spaces, it makes sense to take the direct sum of modules. If M_A and N_A are two modules, then $(M \oplus N)_A$ has right action given by

$$(m, n) \lhd a := (m \lhd a, n \lhd a) \quad \text{for} \quad m \in M, n \in N, a \in A.$$

A module M_A is called:

- *simple* or *irreducible* if its only submodules are 0 and M itself, and
- *indecomposable* if it cannot be written as the direct sum of two non-zero submodules.

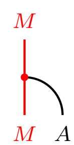
Clearly simple implies irreducible, but not conversely.

Definition 2.1.19 — We say that a linear operator $\varphi\colon M_A \to N_A$ between (right) A-modules is an *A-module map* when

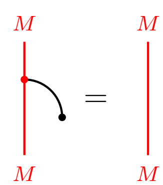
$$\varphi(m \triangleleft a) = \varphi(m) \triangleleft a \qquad \forall m \in M, a \in A. \tag{2.1.20}$$

We write $\mathrm{Hom}(M_A, N_A)$ for the set of A-module maps and $\mathrm{End}(M_A) := \mathrm{Hom}(M_A, M_A)$.

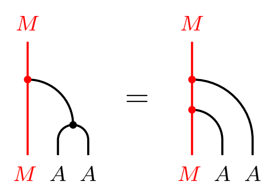
Remark 2.1.21. Using the diagrammatic calculus from §1.6, we represent a module M_A by a colored strand labeled M, and we use the black strand for A. The linear action map $\triangleleft : M \otimes A \to M$ is represented by a trivalent vertex.



The action map satisfies the following axioms with the unit and multiplication of A.

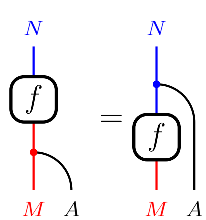


and

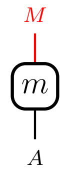


Topologically, these pictures look like those in Remark 2.1.2 above, but the strand corresponding to M has been straightened in order to emphasize the action of A on M.

We represent a module map $f : M_A \to N_A$ graphically by a coupon between an M-strand and an N-strand. Compatibility with the A-module structures is denoted graphically by



As before, we may identify $m \in M$ with the A-module map $A_A \to M_A$ given by $a \mapsto ma$.



Remark 2.1.22. In mathematics, functions are usually applied on the *left*, whereas in computer science, functions/methods are usually applied on the *right*. Thus the reader may be more comfortable with left modules rather than right modules. For example, taking $A = M_n(\mathbb{C})$, we are usually more comfortable with the left action on the space of kets $\mathbb{C}^n$:

$$x \rhd |\xi\rangle \coloneqq x|\xi\rangle = |x\xi\rangle.$$

The corresponding right module for $M_n(\mathbb{C})$ is the space of bras $\overline{\mathbb{C}^n}$:

$$\langle\xi| \lhd x \coloneqq \langle\xi|x = \langle x^*\xi|.$$

However, observe that a right A-module M_A is always a left module for the algebra $\operatorname{End}(M_A)$. This convention is especially helpful when $A = \mathbb{C}$ and $M = \mathbb{C}^n$, so that $\operatorname{End}(M_A) = M_n(\mathbb{C})$. We will see analogous settings later on in Part[[II]] §[[]].

Example 2.1.23 — Suppose V is a right $M_n(\mathbb{C})$-module. Then $V \cong Ve_{11} \otimes \overline{\mathbb{C}^n}$ as $M_n(\mathbb{C})$-modules where $M_n(\mathbb{C})$ acts naturally on $\overline{\mathbb{C}^n}$, (e_{ij}) is the standard system of matrix units, and Ve_{11} is a *multiplicity space* which carries the trivial $M_n(\mathbb{C})$-action. Indeed, if $\{\xi_i\}$ is a basis for Ve_{11} and $\{\langle\epsilon_j|\}$ is the standard basis for $\overline{\mathbb{C}^n}$, then the map

$$\begin{aligned} f\colon Ve_{11} \otimes \overline{\mathbb{C}^n} &\to V \\ \xi_i \otimes \langle\epsilon_j| &\mapsto \xi_i \lhd e_{ij} \end{aligned}$$

is a linear isomorphism which intertwines the $M_n(\mathbb{C})$-actions:

$$f(\xi_i \otimes \langle\epsilon_j|) \lhd e_{k\ell} = \xi_i \lhd e_{ij}e_{k\ell} = \delta_{j=k}\xi_i \lhd e_{i\ell} = \delta_{j=k} f(\xi_i \otimes \langle\epsilon_\ell|) = f(\xi_i \otimes \langle\epsilon_j| \lhd e_{k\ell}).$$

Thus there is only one simple $M_n(\mathbb{C})$-module up to isomorphism, and all modules are direct sums of simple modules.

Exercise 2.1.24. For a unital algebra A, verify that $M = A$ determines a module with $\lhd = \cdot : A \times A \to A$, the algebra multiplication. Then verify that the submodules of A_A are exactly the right ideals of A.

Exercise 2.1.25. Show that if $f : A \to A$ is right A-linear, then f is left multiplication by an element of A. That is, $\operatorname{End}(A_A) = A$.

Exercise 2.1.26. Prove that for algebras A, B, every module for $A \oplus B$ decomposes canonically as a direct sum of an A-module and a B-module.

Fact 2.1.27. A *division algebra* is a unital algebra A such that every non-zero element is invertible. It is well-known that the only finite dimensional complex division algebra is $\mathbb{C}$ itself. Indeed, suppose A is such an algebra and consider the embedding $L \colon A \hookrightarrow \mathrm{End}(A) \cong M_{\dim(A)}(\mathbb{C})$ by left multiplication operators: $L_a(b) := ab$. For $a \in A$, we saw in §1.7 how $\mathrm{spec}(L_a) \neq \emptyset$, so $\lambda\, \mathrm{id}_A - L_a = L_{\lambda 1 - a}$ is not invertible for some $\lambda \in \mathbb{C}$. This only occurs when $\lambda 1 - a = 0$, i.e., $a = \lambda 1$, so $A \cong \mathbb{C}$.

Lemma 2.1.28 (Schur) — Suppose M_A and N_A are two simple A-modules. Then any A-linear map $T : M \to N$ is either an isomorphism or zero. In particular, $\mathrm{End}(M_A) \cong \mathbb{C}$.

Proof. If $T \colon M \to N$ is non-zero, then $\ker(T) \subset M$ and $\mathrm{im}(T) \subset N$ are submodules. Since T is non-zero and M, N are simple, $\ker(T) = M$ and $\mathrm{im}(T) = N$, and thus T is an isomorphism. The last claim follows by setting $M = N$. Indeed, $\mathrm{End}(M_A)$ is a finite dimensional complex division algebra, and is thus $\mathbb{C}$. □

Definition 2.1.29 — We call an element e of an algebra A an *idempotent* if $e^2 = e$. (Idempotents are 'non-unitary' projections.)

Exercise 2.1.30. Suppose that $r : M_A \to N_A$ and $s : N_A \to M_A$ are maps such that $rs = \mathrm{id}_{N_A}$. Prove that $sr \in \mathrm{End}(M_A)$ is an idempotent whose image is isomorphic to N_A.

Exercise 2.1.31. Suppose e, f are idempotents such that $e + f$ is also an idempotent. Prove that e, f are *orthogonal*, i.e., $ef = 0 = fe$.

Similar to Proposition 1.4.24 above, we have the following correspondence between idempotents and direct summands of A_A.

Exercise 2.1.32. Prove that the map $e \mapsto eA$ gives a bijective correspondence between the idempotents in A and the direct summands of A_A.

Definition 2.1.33 — For subspaces U, V of an algebra A, we write $U \cdot V$ or UV for the span of elements of the form uv where $u \in U$ and $v \in V$. Similarly, V^n is the span of elements of the form $v_1 v_2 \cdots v_n$ where each $v_i \in V$.

Lemma 2.1.34 (Rieffel, care of [Lam01, Thm. 3.11]) — Suppose A is simple and $I \subset A$ is a non-zero right ideal. Define $B := \mathrm{End}(I_A)$, and consider I as a left B-module. The right action map $\rho : A^{\mathrm{op}} \to \mathrm{End}(I)$ is an isomorphism onto the subspace $E := \mathrm{End}({}_B I)$.

Proof. Clearly $\rho_a : I \to I$ commutes with the left B-action on I. Since I is non-zero, the map $A \to E$ is non-zero and thus injective as A is simple. To see this map is surjective, we show that $\rho A = \operatorname{im}(\rho)$ is a left ideal of E, which contains the identity ρ_1.

Step 1: Since left multiplication λ_y by an element $y \in I$ is contained in B, if $T \in E$ and $x \in I$, then $T\rho_x = \rho_{Tx}$. Indeed, for $y \in I$, we have

$$(T\rho_x)(y) = T(yx) = (T\lambda_y)(x) = (\lambda_y T)(x) = y(Tx) = \rho_{Tx}(y).$$

Thus $E \cdot \rho I = \rho I := \operatorname{im}(\rho|_I)$.

Step 2: Since $A = AI$ by simplicity, $\rho A = \rho I \cdot \rho A$. Indeed, if $a \in A$ and $x \in I$, then $\rho_{xa} = \rho_a \rho_x$.

Step 3: $E \cdot \rho A = E \cdot \rho I \cdot \rho A = \rho I \cdot \rho A = \rho A$. □

Theorem 2.1.35 — A finite dimensional unital complex algebra A is simple if and only if it is of the form $M_n(\mathbb{C})$.

Proof. Suppose A is simple. Let $I \subset A$ be a minimal right ideal, which exists as A is finite dimensional. Then I_A is a simple right module, so $B := \operatorname{End}(I_A) \cong \mathbb{C}$ by Schur's Lemma 2.1.28. By Rieffel's Lemma 2.1.34 above, the map $R : A \to \operatorname{End}({}_B I) = \operatorname{End}(I) = M_{\dim(I)}(\mathbb{C})$ is an isomorphism.

The other direction is Proposition 2.1.17. □

Exercise 2.1.36. Prove that the result of Lemma 2.1.34 holds if $I \subset A$ is replaced with any faithful A-module M_A.

2.2 Semisimple algebras

In this section, all algebras are assumed to be finite dimensional and unital over $\mathbb{C}$. As in the previous section, our treatment is inspired by [Lam01, §3-4].

Definition 2.2.1 — We say a complex algebra A is *semisimple* if A_A is a direct sum of simple modules.

Definition 2.2.2 — A direct sum of full matrix algebras $A = \bigoplus_{i=1}^k M_{m_i}(\mathbb{C})$ is called a *multimatrix algebra*. We call the row vector $m_A := (m_1, \ldots, m_k)$ the *dimension row vector* for A.

Exercise 2.2.3. Suppose A is a multimatrix algebra. Find a bijective correspondence between 2-sided ideals of A and central idempotents, i.e., idempotents in $Z(A)$.

Exercise 2.2.4. Consider the multimatrix algebra $A = \bigoplus_{i=1}^k M_{m_i}(\mathbb{C})$.

(1) Use Exercises 2.1.23 and 2.1.26 to prove that A has exactly k simple right modules up to isomorphism.

(2) Prove that every finite dimensional right A-module M_A is a direct sum of simple modules.

(3) Deduce that A is semisimple.

Theorem 2.2.5 (Artin-Wedderburn) — An algebra is semisimple if and only if it is a multimatrix algebra.

Proof. Write $A_A = \bigoplus_{i=1}^n m_i(M_i)_A$ as a direct sum of simple modules, where M_i, M_j are not isomorphic if $i \neq j$, and $m_i \in \mathbb{N}$ is the multiplicity with which M_i occurs in A. Then

$$A = \operatorname{End}(A_A) = \operatorname{End}\left(\bigoplus_{i=1}^n m_i(M_i)_A\right) \cong \bigoplus_{i=1}^n M_{m_i}(\operatorname{End}((M_i)_A)) \cong \bigoplus_{i=1}^n M_{m_i}(\mathbb{C}).$$

The converse direction follows by Exercise 2.2.4. □

Exercise 2.2.4 and Theorem 2.2.5 give the following immediate corollary.

Corollary 2.2.6 — Suppose A is a finite dimensional complex semisimple algebra with n simple summands. Then A has exactly n simple modules up to isomorphism, and all modules are direct sums of simple modules.

Exercise 2.2.7. Prove that when A is semisimple and M_A, N_A are two right modules, then $\dim \operatorname{Hom}(M_A \to N_A) = \dim \operatorname{Hom}(N_A \to M_A)$.

Corollary 2.2.8 — Every corner of a semisimple algebra (eAe where $e \in A$ is an idempotent) is semisimple.

Proof. By taking direct sums, it suffices to consider the case of an idempotent $e \in M_n(\mathbb{C})$. Since $eM_n(\mathbb{C})e = \operatorname{End}(e\mathbb{C}^n) \cong M_{\operatorname{rank}(e)}(\mathbb{C})$, the result follows. □

It will be helpful in §2.4 below to have a description of semisimplicity in terms of the Jacobson radical.

Definition 2.2.9 — The *Jacobson radical* $J(A)$ of A is the intersection of all maximal right ideals of A.

The following characterization of elements in the Jacobson radical is well-known; see [Lam01, Lem. 4.1].

Lemma 2.2.10 — For $a \in A$, the following are equivalent.

(1) $a \in J(A)$.

(2) $1 + ab$ is right invertible for all $b \in A$.

(3) $Ma = 0$ for all simple right A-modules M_A.

Proof.

$\neg(2) \Rightarrow \neg(1)$: Suppose $1 + ab$ does not admit a right inverse for some $b \in A$. Then $(1 + ab)A$ is a right ideal not equal to A, and thus $1 + ab$ is contained in some maximal right ideal $M \subset A$. Since $1 \notin M$, it follows that $a \notin M$, and we conclude $a \notin J(A)$.

$\neg(3) \Rightarrow \neg(2)$: Suppose M_A is a simple right A-module and $m \in M$ and $a \in A$ with $ma \neq 0$. Since M_A is simple, $maA = M$, and thus there is a $b \in A$ such that $mab = -m$. But then $m + mab = m(1 + ab) = 0$, so $1 + ab$ does not admit a right inverse.

$(3) \Rightarrow (1)$: Suppose $Ma = 0$ for all simple right A-modules. Let $N \subset A$ be a maximal right ideal. Then A/N with right A-action given by $(b+N)c := bc+N$ is a simple right A-module, and thus $(b+N)a = N$ for all $b \in A$. In particular, $N = (1+N)a = a+N$, and thus $a \in N$. Since N was arbitrary, $a \in J(A)$. □

Corollary 2.2.11 — The Jacobson radical $J(A)$ is a 2-sided ideal.

Proof. Clearly $J(A)$ is a right ideal. Suppose $b \in J(A)$ and $a \in A$. Then for every right A-module M, $Mab \subset Mb = 0$, so $ab \in J(A)$ by Lemma 2.2.10. □

Exercise 2.2.12. Show that $J(M_n(\mathbb{C})) = 0$. Deduce that $J(A) = 0$ when A is semisimple.

Exercise 2.2.13. Prove that $a \in J(A)$ if and only if $1 + abc$ is invertible for every $a, c \in A$. Deduce that $aM = 0$ for every left A-module, and $J(A)$ is the intersection of all maximal left ideals.

Lemma 2.2.14 — Suppose A is a finite dimensional unital complex algebra. Every element of $J(A)$ is nilpotent.

Proof. Suppose $a \in J(A)$. Since A is finite dimensional, eventually a^n is a linear combination of the a^k for $k < n$. Thus there is a polynomial of the form

$$p(x) = x^n + \lambda_{n-1}x^{n-1} + \cdots + \lambda_1 x + \lambda_0$$

such that $p(a) = 0$. Let j be minimal such that $\lambda_j \neq 0$; by Lemma 2.2.10, $j > 0$. Then

$$\begin{aligned} 0 &= \frac{1}{\lambda_j} p(a) \\ &= \frac{1}{\lambda_j} a^n + \frac{\lambda_{n-1}}{\lambda_j} a^{n-1} + \cdots + \frac{\lambda_{j+1}}{\lambda_j} a^{j+1} + a^j \\ &= a^j \underbrace{\left(1 + \frac{\lambda_{j+1}}{\lambda_j} a + \cdots \frac{\lambda_{n-1}}{\lambda_j} a^{n-1-j} + \frac{1}{\lambda_j} a^{n-j}\right)}_{\text{right invertible as } a \in J(A)}. \end{aligned}$$

Since $a \in J(A)$, the the term on the right hand side is invertible by Lemma 2.2.10, and thus $a^j = 0$, so a is nilpotent. □

Exercise 2.2.15. Prove that the ideal $J(A)$ is nilpotent, i.e., there exists an $n \in \mathbb{N}$ such that $J(A)^n = 0$.
Hint following [Lam01, Thm. 4.12]: Since A is finite dimensional and $J(A)^n$ is an ideal for all n, eventually $J(A)^n$ stabilizes, i.e., $I := J(A)^n = J(A)^{n+1}$ at some n, and so $I = I^2$. Show that $I = 0$. One can do this by contradiction; if $I \neq 0$, choose a right ideal $K \subset A$ which is minimal with respect to the property that $KI \neq 0$. (Why does one exist?) Now proceed to deduce a contradiction.

Exercise 2.2.16 (Quotient modules and algebras)**.** Recall the construction of the quotient space V/W for a subspace $W \subset V$ from Exercise 1.5.5.

(1) If M_A is a right A-module and $N_A \subset M_A$ is a submodule, prove that M/N is a right module with right A-action given by $(m + N) \lhd a := ma + N$. In this case, observe that the canonical surjection $M \to M/N$ is an A-module map.

(2) If A is a complex algebra and $I \subset A$ is a 2-sided ideal, prove that A/I is an algebra with multiplication given by $(a + I)(b + I) := ab + I$. In this case, observe that the canonical surjection $A \to A/I$ is a (unital) algebra map.

Definition 2.2.17 — Suppose A is an algebra. A *short exact sequence* of A-modules

$$0 \longrightarrow M_A \xrightarrow{\iota} N_A \xrightarrow{\pi} P_A \longrightarrow 0 \tag{2.2.18}$$

consists of an injective A-module map $\iota : M_A \to N_A$ and a surjective A-module map $\pi : N_A \to P_A$ such that $\operatorname{im}(\iota) = \ker(\pi)$.

Exercise 2.2.19. Prove that the following are equivalent for a short exact sequence of A-modules as in (2.2.18).

(SES1) The sequence *right splits*, i.e., there is a *section* $\sigma : P_A \to N_A$ such that $\pi \circ \sigma = \mathrm{id}_P$.

(SES2) The sequence *left splits*, i.e., there is a *retract* $\rho : N_A \to M_A$ such that $\rho \circ \iota = \mathrm{id}_M$.

(SES3) The sequence *splits*, i.e., ι, π can be extended to an isomorphism $N_A \cong M_A \oplus P_A$.

Theorem 2.2.20 (Fundamental theorem of semisimple algebras) — For a finite dimensional unital complex algebra, the following are equivalent.

(SS1) A is semisimple.

(SS2) A is a multimatrix algebra.

(SS3) Every finite dimensional right A-module is a direct sum of simple modules.

(SS4) $J(A) = 0$.

(SS5) Every minimal right ideal of A is a direct summand.

(SS6) Every right A-module is a summand of a free module, i.e., given a right A-module M, there is a $k \in \mathbb{N}$ and an idempotent $e \in M_k(A)$ such that $M \cong eA^k$ as right A-modules.

(SS7) Every finite dimensional right A-module P_A is *projective*, i.e., whenever $T : M_A \twoheadrightarrow N_A$ is a surjective map and $S : P_A \to N_A$ is an arbitrary map, there is a lift $R : P_A \to M_A$ such that $TR = S$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists R"'] \arrow[d, "S"] & \\ M \arrow[r, two heads, "T"'] & N \arrow[r] & 0 \end{tikzcd}$$

(SS8) Every short exact sequence of A-modules splits.

Proof. First, we note that the cycle of implications (SS1)$\Rightarrow$(SS2)$\Rightarrow$(SS3)$\Rightarrow$(SS1) is exactly how we proved the Artin-Wedderburn Theorem 2.2.5 using Exercise 2.2.4, which relied on Example 2.1.23. For the rest of the proof, we proceed as follows:

$$\begin{tikzcd} & \text{(SS1)} \arrow[r, Leftrightarrow] \arrow[dl, Rightarrow] & \text{(SS2)} \arrow[r, Rightarrow] & \text{(SS6)} \arrow[d, Rightarrow] \\ \text{(SS4)} \arrow[r, Rightarrow] & \text{(SS5)} \arrow[u, Rightarrow] & \text{(SS8)} \arrow[l, Rightarrow] & \text{(SS7)} \arrow[l, Rightarrow] \end{tikzcd}$$

(SS1)$\Rightarrow$(SS4): This is Exercise 2.2.12.

(SS4)$\Rightarrow$(SS5): Let $I \subset A$ be a minimal right ideal. Since $J(A) = 0 \neq I$, there is a maximal right ideal M which does not contain I. Since I is minimal, $I \cap M = 0$. Since M is maximal, $I + M = A$. Hence $A \cong I \oplus M$.

(SS5)$\Rightarrow$(SS1): Since every right ideal of A contains a minimal right ideal, we can co-inductively split off every minimal right ideal of A as a summand. This process realizes A as a direct sum of simple right modules.

(SS2)⇒(SS6): Similar to Exercise 2.2.4 using Exercise 2.1.26.

(SS6)⇒(SS7): Summands of free modules are always projective.

(SS7)⇒(SS8): Given a short exact sequence of A-modules as in (2.2.18), since P_A is projective, the identity map id_P admits a section, and thus (2.2.18) right splits. Now apply Exercise 2.2.19.

(SS8)⇒(SS5): Let I be a minimal right ideal of A. Since the sequence of right A-modules

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

is exact, it splits by (SS8), and thus I is a direct summand of A. □

Exercise 2.2.21. For an idempotent $e \in M_n(A)$, prove that $\mathrm{End}(eA_A^n) = eM_n(A)e$.

Corollary 2.2.22 — If A is semisimple and M_A is a right A-module, then there is a finite set $\{m_i\}_{i=1}^n \subset M$ and right A-linear maps $f_i : M_A \to A_A$ for $i = 1, \ldots, n$ such that $m = \sum_{i=1}^n m_i f_i(m)$ for all $m \in M$. (We sometimes call (m_i, f_i) a *projective basis* for M_A.)

Proof. By the Fundamental Theorem of Semisimple Algebras 2.2.20, M is a summand of a free module A^k, i.e., there is an idempotent $e \in M_k(A)$ such that $M = eA^k$. By a slight abuse of notation, we write $e : A_A^k \to M_A$ for the canonical surjection. Since M_A is projective, we have the following commutative diagram.

$$\begin{tikzcd} & M \arrow[dl, dashed, "\exists f"'] \arrow[d, "\mathrm{id}_M"] & \\ A^k \arrow[r, "e"] & M \arrow[r] & 0 \end{tikzcd}$$

A surjection $e : A_A^k \to M_A$ is a choice of $m_i \in M$ for each $i = 1, \ldots, k$ such that each $m \in M$ can be written as $\sum m_i a_i$. Indeed, we just need to specify where each $\delta_i \in A^k$ goes, where δ_i is the tuple with 1_A in the i-th spot and zeroes everywhere else. A map $f : M_A \to A_A^k$ is a k-tuple of maps $(f_1, \ldots, f_k)$ where each $f_i : M_A \to A_A$. That $ef = \mathrm{id}_M$ is exactly the condition that $\sum_i m_i f_i(m) = m$ for all $m \in M$. □

We have the following strengthening of Lemma 2.1.34.

Corollary 2.2.23 — Suppose A is semisimple and M_A is a faithful right module. Define $B := \mathrm{End}(M_A)$, and consider M as a left B-module. The right action map $\rho : A^{\mathrm{op}} \to \mathrm{End}(M)$ is an isomorphism onto the subspace $\mathrm{End}({}_BM)$.

Proof. If A is simple, this is Exercise 2.1.36. When A is semisimple, decompose $A = \bigoplus A_i$ into simple summands and decompose $M = \bigoplus M_i$ into faithful modules for these summands. Since $\mathrm{End}(M_A) = \bigoplus \mathrm{End}((M_i)_{A_i})$, we may apply Exercise 2.1.36 to each $(M_i)_{A_i}$ and then take direct sum to get the desired result. □

Corollary 2.2.24 — For any 2-sided ideal $I \subset A$ with $I \subset J(A)$, $J(A/I) = J(A)/I$. In particular, $J(A/J(A)) = 0$, so $A/J(A)$ is semisimple.

Proof. It is well-known (and sometimes called the *Third Isomorphism Theorem for non-commutative rings*) that the map $J \mapsto J/I$ is a bijective correspondence between right ideals of A containing I and right ideals of A/I. Since $I \subset J(A)$, I is contained in every maximal right ideal. Thus the maximal right ideals of A/I correspond bijectively to the maximal right ideals of A containing I. It immediately follows that $a \in J(A/I)$ if and only if $a \in J(A)/I$.

The last claim is immediate from Theorem 2.2.20. □

Corollary 2.2.25 — Suppose A is an algebra and $I \subset A$ is a 2-sided ideal. If A/I is semisimple, then $J(A) \subset I$.

Proof. We prove the contrapositive. If $a \in J(A) \setminus I$, then $a + I \neq 0$. For all $b \in A$, $1 + ab$ admits a right inverse in A, and thus $(1+ab)+I$ admits a right inverse in A/I. We conclude that $a + I \in J(A/I) \neq 0$. □

With more work, one can combine Corollaries 2.2.24 and 2.2.25 above into one result.

Exercise 2.2.26. Prove that for any 2-sided ideal $I \subset A$, $J(A/I) = (J(A) + I)/I$. Then deduce Corollaries 2.2.24 and 2.2.25 from this result.

Exercise 2.2.27. Suppose A is a finite dimensional complex algebra whose only idempotents are zero and 1. Prove that $J(A)$ is the unique maximal right ideal of A. Deduce that every element of A is either nilpotent or invertible.

We end this section with a brief discussion of non-degenerate traces on algebras.

Definition 2.2.28 — A *trace* on an algebra A is a non-zero linear map $\mathrm{Tr} : A \to \mathbb{C}$ such that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ for all $a, b \in A$. We call Tr *nondegenerate* if the associated bilinear form $(a, b) := \mathrm{Tr}(ab)$ is nondegenerate, i.e., the map $a \mapsto (b \mapsto (a, b) = \mathrm{Tr}(ab))$ is an isomorphism $A \to A^\vee := \mathrm{Hom}(A \to \mathbb{C})$.

Given an algebra A equipped with a trace Tr, we say $a \in A$ is *negligible* if $\mathrm{Tr}(ab) = 0$ for all $b \in A$. The set of all negligible elements forms a 2-sided ideal called the *negligible ideal*, denoted $N = N(\mathrm{Tr})$.

Warning 2.2.29 — A non-degenerate trace may not vanish on nilpotent elements. Indeed, consider the non-semisimple algebra

$$A := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a, b \in \mathbb{C} \right\} \subset M_2(\mathbb{C}) \qquad A \cong \mathbb{C}1 \oplus \mathbb{C}\varepsilon \qquad \varepsilon := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since A is commutative, any non-zero linear map $A \to \mathbb{C}$ is a trace. Defining $\mathrm{Tr} : A \to \mathbb{C}$ by $\mathrm{Tr}(1) = 1$ and $\mathrm{Tr}(\varepsilon) = 1$ gives a non-degenerate trace. Observe that ε is nilpotent.

Lemma 2.2.30 — If A is equipped with a trace Tr_A which is zero on nilpotent elements, then the negligible ideal N contains the Jacobson radical $J(A)$. In particular, A/N is semisimple.

Proof. By Lemma 2.2.14, every element of the Jacobson radical is nilpotent and thus has zero trace. The result now follows as $J(A)$ is a 2-sided ideal. □

Corollary 2.2.31 — If A admits a non-degenerate trace Tr_A which is zero on nilpotent elements, then A is semisimple.

Example 2.2.32 — Since A is unital, we may embed $\lambda : A \hookrightarrow \mathrm{End}(A)$ by left multilpication operators: $\lambda_a b := ab$. If the trace Tr on $\mathrm{End}(A) \cong M_{\dim(A)}(\mathbb{C})$ is non-degenerate on $\lambda A \subset \mathrm{End}(A)$, then A is semisimple.

2.3 Complex $*$-algebras and states

Definition 2.3.1 — A *complex $*$-algebra* is a (unital) complex algebra A equipped with an anti-linear involution $* : A \to A$ satisfying $(ab)^* = b^* a^*$ and $a^{**} = a$ for all $a, b \in A$.

Exercise 2.3.2. Show that if A is a unital complex $*$-algebra, then $1^* = 1$.

Theorem 2.3.3 (Classification of involutions on $M_n(\mathbb{C})$) —

(1) Any involution $*$ on $M_n(\mathbb{C})$ is of the form $x^* = hx^\dagger h^{-1}$ for some invertible $h \in M_n(\mathbb{C})$ which is self-adjoint, i.e., $h = h^\dagger$.

(2) For the involution $x^* = hx^\dagger h^{-1}$ associated to a self-adjoint invertible $h \in M_n(\mathbb{C})$, the following are equivalent.

   (a) $(M_n(\mathbb{C}), *) \cong (M_n(\mathbb{C}), \dagger)$ as $*$-algebras.

   (b) ($*$-definite) $x^* x = 0$ implies $x = 0$.

   (c) h is positive or negative definite.

Proof. To prove (1), observe that $x \mapsto x^{*\dagger}$ is an automorphism of $M_n(\mathbb{C})$, and is thus inner by Proposition 2.1.12. Thus there is a $k \in M_n(\mathbb{C})$ such that $x^{*\dagger} = k^{-1} x k$. Taking adjoints and setting $h = k^\dagger$, we have $x^* = hx^\dagger h^{-1}$. The condition that $x^{**} = x$ for all $x \in M_n(\mathbb{C})$ is then

$$x = x^{**} = (hx^\dagger h^{-1})^* = h(h^\dagger)^{-1}xh^\dagger h^{-1} \qquad \Longleftrightarrow \qquad xh^\dagger h^{-1} = h^\dagger h^{-1}x \qquad \forall\, x \in M_n(\mathbb{C}).$$

Thus $h^\dagger h^{-1} \in Z(M_n(\mathbb{C})) = \mathbb{C}1$, so $h^\dagger = \lambda h$ for some $\lambda \in \mathbb{C}$. Taking adjoints,

$$h = \overline{\lambda}h^\dagger = |\lambda|^2 h,$$

so $\lambda \in U(1)$, the unimodular complex scalars. Replacing h by $\lambda^{1/2}h$ for some choice of square root $\lambda^{1/2}$, we may assume $h = h^\dagger$. (Note that the choice of square root $-\lambda^{1/2}$ does not affect the operation of conjugating by h.)

We now prove (2).

(a)⇒(b): It suffices to prove that $x^\dagger x = 0$ implies $x = 0$. Indeed, if $\xi \in \mathbb{C}^n$,

$$\|x\xi\|^2 = \langle x\xi|x\xi\rangle = \langle x^\dagger x\xi|\xi\rangle = 0 \qquad \Longrightarrow \qquad x\xi = 0.$$

Since $\xi \in \mathbb{C}^n$ was arbitrary, $x = 0$.

¬(c)⇒ ¬(b): If h is not positive or negative definite, choose $-\infty < r < 0 < s < \infty$ such that $r, s \in \operatorname{spec}(h)$, and pick unit length eigenvectors $\eta, \xi \in \mathbb{C}^n$ for h corresponding to r, s respectively. Observe that η, ξ are also eigenvectors of h^{-1} corresponding to eigenvalues $\frac{1}{r}, \frac{1}{s}$ respectively. Since η, ξ are eigenvectors corresponding to distinct eigenvalues, $\eta \perp \xi$, i.e., $\langle \eta|\xi\rangle = 0$. Setting

$$x := \begin{bmatrix} \sqrt{-r}\eta + \sqrt{s}\xi & 0 & \cdots & 0 \end{bmatrix} \in M_n(\mathbb{C}),$$

we have

$$\begin{aligned} hx^\dagger h^{-1}x &= h \begin{bmatrix} \sqrt{-r}\eta^\dagger + \sqrt{s}\xi^\dagger \\ 0 \\ \vdots \\ 0 \end{bmatrix} h^{-1} \begin{bmatrix} \sqrt{-r}\eta + \sqrt{s}\xi & 0 & \cdots & 0 \end{bmatrix} \\ &= h \begin{bmatrix} \sqrt{-r}\eta^\dagger + \sqrt{s}\xi^\dagger \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{-r}}{r}\eta + \frac{\sqrt{s}}{s}\xi & 0 & \cdots & 0 \end{bmatrix} \\ &= h \begin{bmatrix} \frac{-r}{r} + \frac{s}{s} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = 0. \end{aligned}$$

Thus $x^*x = hx^\dagger h^{-1}x = 0$, but $x \neq 0$.

(c)⇒(a): Suppose h is positive or negative definite. We may assume h is positive definite by replacing h with $-h$ if necessary. The map $x \mapsto h^{-1/2}xh^{1/2}$ is the desired $*$-algebra isomorphism $(M_n(\mathbb{C}), *) \to (M_n(\mathbb{C}), \dagger)$. $\square$

Definition 2.3.4 — Let A be a unital complex $*$-algebra. We call a linear functional $\varphi : A \to \mathbb{C}$:

- a *trace* or *tracial* if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$.
- *positive* if $\varphi(a^*a) \geq 0$ for all $a \in A$.
- a *state* if φ is positive and $\varphi(1) = 1$.
- *faithful* if φ is positive and $\varphi(a^*a) = 0$ implies $a = 0$.

A positive linear functional is also called a *weight*.

Example 2.3.5 — For $x \in M_n(\mathbb{C})$, we write $\operatorname{Tr}(x) = \sum_{j=1}^n x_{jj}$ and $\operatorname{tr}(x) = \frac{1}{n}\operatorname{Tr}(x)$. Then Tr is tracial, positive, and faithful, and tr is furthermore a state.

Exercise 2.3.6. For $\eta, \xi \in \mathbb{C}^n$, show that $\operatorname{Tr}(|\eta\rangle\langle\xi|) = \langle\xi|\eta\rangle$.

Lemma 2.3.7 — The complex $*$-algebra $(M_n(\mathbb{C}), \dagger)$ has a unique normalized trace.

Proof. Suppose $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ is another trace with $\varphi(1) = 1$. Then

$$\varphi(|e_i\rangle\langle e_i|) = \varphi(|e_i\rangle\langle e_j| \cdot |e_j\rangle\langle e_i|) = \varphi(|e_j\rangle\langle e_i| \cdot |e_i\rangle\langle e_j|) = \varphi(|e_j\rangle\langle e_j|) \qquad \forall\, i, j.$$

Moreover,

$$\varphi(|e_i\rangle\langle e_j|) = \varphi(|e_i\rangle\langle e_j|\cdot|e_j\rangle\langle e_j|) = \varphi(|e_j\rangle\langle e_j|\cdot|e_i\rangle\langle e_j|) = \langle e_j|e_i\rangle\varphi(|e_j\rangle\langle e_j|) = 0 \qquad \forall\, i \neq j.$$

The result follows. □

Exercise 2.3.8. Show that the set of faithful tracial weights on a multimatrix algebra $A = \bigoplus_{i=1}^k M_{m_i}(\mathbb{C})$ is a torsor for the group of positive invertible operators $Z(A)_+^\times$.

Exercise 2.3.9. Suppose $\varphi : A \to \mathbb{C}$ is a linear functional on a unital complex $*$-algebra. Use Exercise 1.4.7 to prove that φ is a trace if and only if $\varphi(a^*a) = \varphi(aa^*)$ for all $a \in A$.

Example 2.3.10 — Show that if $d \geq 0$ in $M_n(\mathbb{C})$ has $\operatorname{Tr}(d) = 1$, then $\varphi(x) := \operatorname{Tr}(dx)$ is a state. Such a d is called a *density matrix*. (In Proposition 2.3.16 below, we will show every state on $M_n(\mathbb{C})$ is of this form.)
Exercise: What happens if we use tr *instead of* Tr*?*

Exercise 2.3.11. Let $A = \mathbb{C}^2$ with coordinate-wise multiplication and $(a, b)^* := (\overline{b}, \overline{a})$. Prove that A has no states.

Exercise 2.3.12. The group algebra $\mathbb{C}[G]$ of a finite group has an involution given by the anti-linear extension of $g \mapsto g^{-1}$. Prove that the map $\mathrm{tr} : \sum_g \lambda_g g \mapsto \lambda_e$ is a faithful tracial state on $\mathbb{C}[G]$.

Exercise 2.3.13. Consider the space $C(G) := \{\xi \colon G \to \mathbb{C}\}$ of complex-valued functions on a finite group G, where multiplication is given by convolution:

$$(\xi_1 * \xi_2)(g) := \sum_{g=hk} \xi_1(h)\xi_2(k),$$

the involution is given by $\xi^*(g) := \overline{\xi(g^{-1})}$, and a faithful tracial state is given by $\mathrm{tr}(\xi) := \xi(e)$. Verify the map $C(G) \to \mathbb{C}[G]$ given by

$$\xi \longmapsto \sum_{g\in G} \xi(g)g$$

is a trace-preserving $*$-isomorphism.

Lemma 2.3.14 — Suppose A is a complex $*$-algebra and $\varphi : A \to \mathbb{C}$ is a state.

(1) $\langle a|b\rangle_\varphi := \varphi(a^*b)$ is a positive sesquilinear form on A which is faithful (or definite) if and only if φ is faithful.

(2) $N_\varphi := \{a \in A \,|\, \varphi(a^*a) = 0\}$ is a left ideal of A.

(3) For all $a \in A$, $\varphi(a^*) = \overline{\varphi(a)}$.

Proof.

(1) Obvious.

(2) By the Cauchy-Schwarz Inequality (C-S),

$$N_\varphi = \{a \in A \,|\, \varphi(b^*a) = 0 \text{ for all } b \in A\},$$

which is clearly a left ideal.

(3) Since $\langle \cdot | \cdot \rangle_\varphi$ is positive, it is also self-adjoint. Hence $\varphi(a^*) = \langle a|1\rangle_\varphi = \overline{\langle 1|a\rangle_\varphi} = \overline{\varphi(a)}$. □

Construction 2.3.15 (GNS) — Suppose φ is a weight on a complex $*$-algebra A. On A/N_φ, we get an inner product given by:

$$\langle a + N_\varphi | b + N_\varphi\rangle_\varphi := \varphi(a^*b),$$

which is thus a Hilbert space. We denote the corresponding Hilbert space by $L^2(A, \varphi)$; this is called the *GNS-Hilbert space* (where GNS stands for Gelfand-Naimark-Segal).

We write $\Omega := 1 + N_\varphi \in L^2(A, \varphi)$, i.e., Ω is the image of 1 under the canonical surjection $A \to A/N_\varphi = L^2(A,\varphi)$. Thus $a\Omega$ is the image of $a \in A$ under the canonical surjection.

Proposition 2.3.16 — For any weight φ on $M_n(\mathbb{C})$, there exists a unique $d \in M_n(\mathbb{C})$ with $d \geq 0$ called the *density matrix of* φ such that $\varphi(a) = \operatorname{Tr}(da)$ for all $a \in M_n(\mathbb{C})$. Moreover, φ is a state if and only if $\operatorname{Tr}(d) = 1$, and φ is faithful if and only if d is invertible.

Proof. Since tr is a state by Lemma 2.3.7, $L^2(M_n(\mathbb{C}), \operatorname{tr})$ is a Hilbert space. Since tr is faithful, by the Riesz-Representation Theorem 1.3.18, every linear map $M_n(\mathbb{C}) \to \mathbb{C}$ can be uniquely expressed as $\langle d|$ for some $d \in M_n(\mathbb{C})$ for the trace inner product. Thus there is a unique $d \in M_n(\mathbb{C})$ such that $\varphi(x) = \langle d|x\rangle = \operatorname{Tr}(d^\dagger x)$ for all $x \in M_n(\mathbb{C})$. Taking $x = |\xi\rangle\langle\xi|$ for a unit vector $\xi \in H$, we have

$$0 \leq \varphi(|\xi\rangle\langle\xi|) = \operatorname{Tr}(d^\dagger \cdot |\xi\rangle\langle\xi|) = \operatorname{Tr}(|\xi\rangle\langle\xi| \cdot d^\dagger \cdot |\xi\rangle\langle\xi|) = \langle\xi|d^\dagger\xi\rangle \operatorname{Tr}(|\xi\rangle\langle\xi|) \underset{\text{(Exer. 2.3.6)}}{=} \langle\xi|d^\dagger\xi\rangle,$$

so $d = d^\dagger \geq 0$. Note that $\varphi(1) = \operatorname{Tr}(d)$, which is equal to one if and only if φ is a state.

If d is not invertible, then $\lambda = 0$ is an eigenvalue of d with a corresponding eigenvector $\eta \in \mathbb{C}^n$, and thus $\varphi(|\eta\rangle\langle\eta|) = \operatorname{Tr}(d|\eta\rangle\langle\eta|) = 0$. Conversely, if d is invertible, then $\ker(d^{1/2}) = \ker(d) = 0$, so for every non-zero $\eta \in H$,

$$\varphi(|\eta\rangle\langle\eta|) = \operatorname{Tr}(d|\eta\rangle\langle\eta|) = \operatorname{tr}(d^{1/2}|\eta\rangle\langle\eta|d^{1/2}) > 0.$$

Since every positive operator is a positive linear combination of rank one projections by the spectral theorem, φ is faithful. $\square$

Remark 2.3.17. Operator algebras are often viewed as algebras of functions on *non-commutative spaces* (see Ethos 2.4.30 below). Under this analogy, a state on an operator algebra is often viewed as a *noncommutative measure/integral*. One can then think of a density matrix as a *non-commutative Radon-Nikodym derivative*, and thus it would make sense to denote the density d so that $\varphi = \operatorname{Tr}(d\,\cdot\,)$ as $\frac{d\varphi}{d\operatorname{Tr}}$, i.e.,

$$\varphi(x) \;=: \int x\, d\varphi \text{ “=” } \int x \frac{d\varphi}{d\operatorname{Tr}} d\operatorname{Tr} \;:=\; \operatorname{Tr}\left(x \frac{d\varphi}{d\operatorname{Tr}}\right).$$

Proposition 2.3.18 — Suppose φ is a weight on A. For $a \in A$, the map given by $b\Omega \mapsto ab\Omega$ defines a left multiplication operator $\lambda_a \in B(L^2(A,\varphi))$. The adjoint of this operator is λ_{a^*} given by $b\Omega \mapsto a^*b\Omega$.

Proof. First, λ_a is well-defined as $N_\varphi \subset A$ is a left ideal. We compute that

$$\langle b\Omega|\lambda_a c\Omega\rangle_\varphi = \langle b\Omega|ac\Omega\rangle_\varphi = \varphi(b^*ac) = \varphi((a^*b)^*c) = \langle a^*b\Omega|c\Omega\rangle_\varphi = \langle\lambda_{a^*}b\Omega|c\Omega\rangle_\varphi.$$

It follows that $\lambda_a^\dagger = \lambda_{a^*}$. $\square$

Exercise 2.3.19. Let G be a finite group, and consider the Hilbert space

$$\ell^2G := \{\xi : G \to \mathbb{C}\} \qquad \text{with} \qquad \langle \eta | \xi \rangle := \sum_g \overline{\eta(g)}\xi(g).$$

Observe that G acts on ℓ^2G by $(\lambda_g\xi)(h) := \xi(g^{-1}h)$.

(1) Verify that $\lambda_g^\dagger = \lambda_{g^{-1}}$, and thus $\lambda : G \to B(\ell^2(G))$ is a unitary representation which extends linearly to an algebra map $\lambda : \mathbb{C}[G] \to B(\ell^2(G))$.

(2) Construct a unitary isomorphism $\ell^2G \cong L^2(\mathbb{C}[G], \mathrm{tr})$ which intertwines the left $G/\mathbb{C}[G]$-actions.

Exercise 2.3.20. Prove that if $a \in A$, the map given by $b\Omega \mapsto ba\Omega$ defines a right multiplication operator $\rho_a \in B(L^2(A, \varphi))$. Calculate the adjoint of ρ_a. Determine when $\rho_a^\dagger = \rho_{a^*}$. Deduce that $\rho_a^\dagger = \rho_{a^*}$ for all $a \in A$ if φ is a trace.

Remark 2.3.21. Since $\mathrm{End}(A_A) = A$, $\lambda A = \{\lambda_a \,|\, a \in A\} \subset B(L^2(A, \varphi))$ is the set of all operators which commute with $\rho A = \{\rho_a \,|\, a \in A\}$. Indeed, observe that on $b\Omega \in L^2(A, \varphi)$ we have

$$\lambda_c\rho_a b\Omega = cba\Omega = \rho_a\lambda_c b\Omega,$$

and thus λ_c commutes with ρ_a for all $a, c \in A$. Now suppose $x \in B(L^2(A, \varphi))$ commutes with ρA. Notice there exists some (not necessarily unique) $c \in A$ such that $c\Omega = x\Omega$. Observe $x = \lambda_c$ as for every $b\Omega \in L^2(A, \varphi)$ we have

$$xb\Omega = x\rho_b\Omega = \rho_b x\Omega = \rho_b c\Omega = \lambda_c b\Omega.$$

2.4 Operator algebras

We now have all the background material necessary to study finite dimensional operator algebras. For this section, A is a unital complex $*$-algebra (always assumed to be finite dimensional).

Definition 2.4.1 — We call A a C*-*algebra* if there exists a norm $\|\cdot\|$ on A which is submultiplicative ($\|ab\| \leq \|a\| \cdot \|b\|$) such that

$$\|a^*a\| = \|a\|^2 \qquad\qquad \forall\, a \in A. \tag{C*}$$

(We have omitted the completeness condition, as A was assumed to be finite dimensional.)

Example 2.4.2 — For $x \in B(H)$, define

$$\|x\| := \sup_{\substack{\eta \in H \\ \|\eta\|=1}} \|x\eta\|,$$

and observe that $\|x\xi\| \leq \|x\| \cdot \|\xi\|$ for all $\xi \in H$ (divide both sides by $\|\xi\|$ assuming $\xi \neq 0$). One verifies this defines a norm. Submultiplicativity follows from the fact that

$$\|xy\eta\| \leq \|x\| \cdot \|y\eta\| \leq \|x\| \cdot \|y\| \cdot \|\eta\| \qquad\qquad \forall \eta \in H.$$

To prove the C*-axiom (C*), First note that

$$\|x\eta\|^2 = \langle x\eta|x\eta\rangle = \langle \eta|x^\dagger x\eta\rangle \underset{\text{(C-S)}}{\leq} \|\eta\| \cdot \|x^\dagger x\eta\| \leq \|x^\dagger x\| \cdot \|\eta\|^2 \qquad \forall\, \eta \in H.$$

Thus $\|x\|^2 \leq \|x^\dagger x\| \leq \|x\| \cdot \|x^\dagger\|$. Similarly, $\|x^\dagger\|^2 \leq \|xx^\dagger\| \leq \|x\| \cdot \|x^\dagger\|$. These two sets of inequalities together imply $\|x\| = \|x^\dagger\|$, and thus these inequalities are all equalities.

Proposition 2.4.3 — The only C* norm on $\mathbb{C}^n = C(\{1, \dots, n\})$ is $\|f\|_\infty := \max_{j=1}^n |f_j|$.

Proof. We leave it to the reader to verify $\|\cdot\|_\infty$ is a C* norm.

Suppose $\|\cdot\|$ is another C* norm. By (C*), $\|\cdot\|$ is completely determined by its values on elements of the form $\overline{f}f$, which only take positive values.

First, observe that for an orthogonal projection $p \in \mathbb{C}^n$, $\|p\| = \|p^*p\| = \|p\|^2$, so $\|p\| \in \{0, 1\}$. Consider a positive function $f = (f_1, \dots, f_n)$. By replacing f with $f_j^{-1}f = (f_1/f_j, \dots, f_n/f_j)$ where $f_j = \max(f)$, we may assume that $f_i \in [0, 1]$ for all i, and at least one f_j is equal to 1. The C* axiom (C*) tells us that $\|f^2\| = \|f\|^2$, and iterating, $\|f^{2^n}\| = \|f\|^{2^n}$ for all n. For $f_i \in [0, 1)$, $f_i^n \to 0$ as $n \to \infty$, so f^{2^n} converges point-wise (and thus in some norm!) to some non-zero orthogonal projection p. Since all norms are equivalent on $\mathbb{C}^n$ by Proposition 1.2.11, $\|f\|^{2^n} = \|f^{2^n}\| \to \|p\| = 1$. This is only possible if $\|f\| = 1$. We conclude that $\|f\| = \max_{j=1}^n f_j$. □

We now completely characterize all (f.d.) operator algebras. Condition (C*6) below comes from [Müg00, Prop. 2.1].

Theorem 2.4.4 (Fundamental Theorem of finite dimensional operator algebras) — The following conditions are equivalent for a finite dimensional unital complex $*$-algebra A.

(C*1) A is a C*-algebra.

(C*2) (multimatrix) There exists a $*$-isomorphism $A \cong \bigoplus_{i=1}^k M_{m_i}(\mathbb{C})$ where each summand has the usual conjugate transpose $\dagger$ operation.

(C*3) (matrix $\dagger$-subalgebra) There exists an injective unital $*$-homomorphism $\iota\colon A \hookrightarrow$

$M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, where $M_n(\mathbb{C})$ has the usual conjugate transpose $\dagger$ operation.

(C*4) ($\exists$ faithful trace) There exists a faithful tracial state $\mathrm{tr} : A \to \mathbb{C}$.

(C*5) ($\exists$ faithful state) There exists a faithful state $\varphi : A \to \mathbb{C}$.

(C*6) ($*$-definite) For every $a \in A$, $a^*a = 0$ implies $a = 0$.

(C*7) (no positive nilpotents) If $a^*a \in A$ is *nilpotent* ($(a^*a)^n = 0$ for some n), then $a = 0$.

Proof. We prove the following implications:

$$\begin{tikzcd}
(\mathrm{C}^*2) \arrow[dd, Rightarrow] & (\mathrm{C}^*7) \arrow[l, Rightarrow] & (\mathrm{C}^*6) \arrow[l, Rightarrow] \\
& (\mathrm{C}^*1) \arrow[ur, Rightarrow] & \\
(\mathrm{C}^*3) \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & (\mathrm{C}^*4) \arrow[r, Rightarrow] & (\mathrm{C}^*5) \arrow[uu, Rightarrow]
\end{tikzcd}$$

The interesting part is proving (C*6) $\Rightarrow$ (C*2).

(C*2) $\Rightarrow$ (C*3): Set $n := \sum_{i=1}^{k} m_i$ and embed A as block-diagonal matrices.

(C*3) $\Rightarrow$ (C*1): By Example 2.4.2, $M_n(\mathbb{C})$ is a C*-algebra, so restrict its norm to ιA.

(C*3) $\Rightarrow$ (C*4): Restrict tr from Example 2.3.5 to ιA.

(C*4) $\Rightarrow$ (C*5): Trivial.

(C*1) $\Rightarrow$ (C*6): If $a^*a = 0$, then by (C*), $\|a\|^2 = \|a^*a\| = 0$, so $a = 0$.

(C*5) $\Rightarrow$ (C*6): If $a^*a = 0$, then $\varphi(a^*a) = 0$, so $a = 0$.

(C*6) $\Rightarrow$ (C*7): We prove the contrapositive, i.e., if $a \neq 0$, then a^*a is not nilpotent. Indeed, if $a \neq 0$, then by (C*6), $a^*a \neq 0$. But then by (C*6), $(a^*a)^2 \neq 0$. By a simple induction argument, we have $(a^*a)^{2^k} \neq 0$ for all $k \in \mathbb{N}$. We conclude that a^*a is not nilpotent.

(C*7) $\Rightarrow$ (C*2): Before we begin, observe that (C*7) trivially implies (C*6), as $a^*a = 0$ is obviously nilpotent. We are thus free to use (C*6) in this proof. We now proceed in 3 steps.

Step 1: $J(A) = 0$, so A is semisimple. Thus by the Artin-Wedderburn Theorem 2.2.5, A is a multimatrix algebra.

Proof. Since $J(A)$ is an ideal, $b \in J(A)$ implies $b^*b \in J(A)$. But b^*b is nilpotent by Lemma 2.2.14, so $b = 0$ by (C*7). $\square$

Step 2: Each full matrix algebra summand $M_n(\mathbb{C})$ of A is preserved under $*$.

Proof. We now know that $A \cong \bigoplus_{i=1}^{k} M_{m_i}(\mathbb{C})$ as algebras by the fundamental theorem of semisimple algebras 2.2.20. Consider the k mutually orthogonal central idempotents $p_1, \ldots, p_k$ where p_i corresponds to the unit of $M_{m_i}(\mathbb{C})$. Then $p_1^*, \ldots, p_k^*$ are also mutually orthogonal central idempotents, so $p_i^* = p_j$ for some $j = 1, \ldots, n$. Since each $p_j \neq 0$, we also have $p_j^* p_j \neq 0$ by (C*6), so $p_j^* = p_j$ for all j. □

Step 3: Since $x^*x = 0$ implies $x = 0$ on each full matrix summand $M_n(\mathbb{C})$ of A, by Theorem 2.3.3, $(M_n(\mathbb{C}), *) \cong (M_n(\mathbb{C}), \dagger)$, and thus $(A, *) \cong \left(\bigoplus_{i=1}^{k} M_{m_i}(\mathbb{C}), \dagger\right)$ as complex $*$-algebras. □

Definition 2.4.5 — A *unitary algebra* is a finite dimensional unital complex $*$-algebra that satisfies the equivalent conditions of Theorem 2.4.4. Note that unitary algebras are more commonly called *finite dimensional* C*-*algebras.*

Example 2.4.6 — By Exercise 2.3.12, the group algebra $\mathbb{C}[G]$ of a finite group G admits a faithful tracial state. Hence $\mathbb{C}[G]$ is unitary and thus semisimple.

Example 2.4.7 — If A, B are two unitary algebras, then so is $A \otimes B$ with $(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1 a_2 \otimes b_1 b_2$ and $(a \otimes b)^* := a^* \otimes b^*$. Indeed, given faithful unital $*$-algebra maps $A \hookrightarrow M_k(\mathbb{C})$ and $B \hookrightarrow M_n(\mathbb{C})$, we get a faithful unital $*$-algebra map

$$A \otimes B \hookrightarrow M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \cong M_{kn}(\mathbb{C}).$$

Exercise 2.4.8. Prove that if A is a unitary algebra, then we have a $*$-algebra isomorphism

$$M_n(\mathbb{C}) \otimes A \cong M_n(A).$$

Hint: It can be helpful to represent 'matrix elements' of either side as $|i\rangle \otimes a \otimes \langle j|$.

Corollary 2.4.9 (Spectral permanence) — Suppose A is a unitary algebra and $\iota : A \hookrightarrow M_n(\mathbb{C})$ is an injective unital $*$-homomorphism as in (C*3). For all $a \in A$,

$$\operatorname{spec}_A(a) = \operatorname{spec}_{M_n(\mathbb{C})}(\iota(a)).$$

Proof. If $a \in A$ is invertible, then $\iota(a)$ is as well. Hence $\operatorname{spec}_{M_n(\mathbb{C})}(\iota(a)) \subseteq \operatorname{spec}_A(a)$. Conversely, if $\iota(a)$ is invertible, its inverse clearly lies in ιA. Hence a is invertible, and $\operatorname{spec}_A(a) \subseteq \operatorname{spec}_{M_n(\mathbb{C})}(\iota(a))$. □

Corollary 2.4.10 — Every unitary algebra A has a unique C* norm.

Proof. By (C*), every C* norm is completely determined by its values on positive operators. Suppose $a \in A$ is positive, and consider A unitally $*$-embedded in $M_n(\mathbb{C})$ from (C*3). The Gelfand Theorem 1.7.13 says that the unital $*$-algebra generated by a is isomorphic to $C(\operatorname{spec}(a))$, so it suffices to prove the result for C*-algebras of the form $C(X)$ for $X \subset \mathbb{C}$ a finite set. Since $C(X) \cong \mathbb{C}^n$ as a unital complex $*$-algebra, the result now follows from Proposition 2.4.3. □

Proposition 2.4.11 — Every unitary algebra A is closed under the functional calculus and polar decomposition.

Proof. Identify A with a $*$-closed subalgebra of $M_n(\mathbb{C})$ by (C*3). If $a \in A$ is normal and $f\colon \operatorname{spec}(a) \to \mathbb{C}$, then $f(a)$ is in the unital $*$-algebra generated by a and $a^\dagger$ as in Gelfand's Theorem 1.7.13, which again lies in A.

Next, by identifying $A \cong \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$ with (C*2), each $a \in A$ corresponds to a tuple $(x_i) \in \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$. Then $a_i = u_i|a_i|$ is the polar decomposition in $M_{n_i}(\mathbb{C})$ seen in Definition 1.7.24, and $a = u|a|$ where $u = (u_i)$ and $|a| = (|a_i|)$. □

Exercise 2.4.12. Suppose A is a unitary algebra and $a \in A$. Prove that $\operatorname{Re}(a), \operatorname{Im}(a), a_+, a_-$ from Exercise 1.7.19 all lie in A. Deduce that every element in a is a linear combination of 4 unitaries in A.

Exercise 2.4.13. Suppose A is a unitary algebra equipped with a faithful trace Tr. Show that any other weight φ on A is of the form $a \mapsto \operatorname{Tr}(da)$ for a unique positive $d \in A$. Then prove:

- φ is a trace if and only if $d \in Z(A)$, and
- φ is faithful if and only if d is invertible.

Corollary 2.4.14 — Suppose A is a $*$-algebra with a state $\varphi : A \to \mathbb{C}$, which is not necessarily faithful. If $a \in J(A)$, then $\varphi(a^*a) = 0$.

Proof. Recall that $L^2(A, \varphi)$ is a Hilbert space equipped with a unital $*$-homomorphism $\lambda\colon A \to B(L^2(A, \varphi))$ where λ_a is left multiplication by a and $\lambda_a^\dagger = \lambda_{a^*}$ by Proposition 2.3.18. Observe that $A/\ker(\lambda) \cong \operatorname{im}(\lambda) \subset B(L^2(A, \varphi))$ is a unital $*$-subalgebra and is thus semisimple. Hence $J(A) \subset \ker(\lambda)$ by Corollary 2.2.25. If $a \in \ker(\lambda)$, then necessarily

$$0 = \|\lambda_a\Omega\|^2 = \|a\Omega\|^2 = \varphi(a^*a).$$

We conclude that $\varphi(a^*a) = 0$ for all $a \in J(A)$. □

Our next task is to show that unitary algebras are also von Neumann algebras.

Definition 2.4.15 — For a subset $S \subset B(H)$, the *commutant* of S is

$$S' := \{x \in B(H) \mid xs = sx \text{ for all } s \in S\}.$$

Exercise 2.4.16. Show that if $S \subset T \subset B(H)$, then $T' \subset S'$, $S \subset S''$, and $S' = S'''$.

Definition 2.4.17 — Let $A \subset B(H)$ be a $*$-closed subalgebra. For $k \in \mathbb{N}$, the *k-amplification* of H is the Hilbert space $\bigoplus_{j=1}^k H$. The algebra A acts on the amplified Hilbert space $\bigoplus_{j=1}^k H$ by diagonal operators. That is, as in Remark 1.4.28, we may think of $B\left(\bigoplus_{j=1}^k H\right)$ as $k \times k$ matrices over $B(H)$. The A-action is given by

$$a \cdot \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_k \end{bmatrix} := \begin{bmatrix} a\eta_1 \\ \vdots \\ a\eta_k \end{bmatrix} = \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_k \end{bmatrix}.$$

Alternatively, one may view the k-amplification of H as $\mathbb{C}^k \otimes H$, which admits an action of $B(\mathbb{C}^k \otimes H) \cong M_k(\mathbb{C}) \otimes B(H)$. In particular, there is an action of A given by the inclusion $A \hookrightarrow M_k(\mathbb{C}) \otimes A \subset M_k(\mathbb{C}) \otimes B(H)$ given by $a \mapsto 1 \otimes a$ (insert $a \in A$ into $1 \in M_k(\mathbb{C})$ as in the Kronecker product) for any $*$-closed subalgebra $A \subset B(H)$.

Exercise 2.4.18. Suppose $S \subseteq B(H)$ is a subset, and let $\alpha : B(H) \to M_n(B(H))$ be the amplification

$$x \longmapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix}.$$

Prove that:

(1) $\alpha(S)' = M_n(S')$, and

(2) If $0, 1 \in S$, then $M_n(S)' = \alpha(S')$.

(3) Deduce that when $0, 1 \in S$, $\alpha(S)'' = \alpha(S'')$.

Exercise 2.4.19. Use Exercise 2.1.23 to prove that any unital $*$-algebra map $M_k(\mathbb{C}) \to M_n(\mathbb{C})$ is unitarily conjugate to an amplification.
Hint: For a proof, see Example 4.6.1.

Theorem 2.4.20 (von Neumann Bicommutant) — If $A \subset B(H)$ is a unital $*$-subalgebra, then $A = A''$.

We give two proofs: one using purely algebraic techniques, and one using operator algebraic techniques.

Algebraic proof. Since $A \subset B(H)$ is a $*$-algebra, $A \xrightarrow{*} A^{\mathrm{op}}$ is an (antilinear) algebra isomorphism. Now A^{op} is semisimple and $H_{A^{\mathrm{op}}}$ is a faithful right module. By Corollary 2.2.23, $A \cong A'' = \mathrm{End}({}_{A'}H)$ where $A' = \mathrm{End}(H_{A^{\mathrm{op}}})$. □

Operator algebraic proof. This proof follows [Jon15, Thm. 3.2.1]. Consider the n-amplification $\bigoplus_{j=1}^n H$ where $n = \dim(H)$ which carries the diagonal A-action $\alpha : A \to M_n(B(H))$.[1] Let $\{e_i\}$ be an ONB of H, and consider the vector

$$e := \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \in \bigoplus_{j=1}^n H,$$

i.e., e is the j-th standard basis vector e_j in the j-th summand of the amplified Hilbert space. Consider the subspace $Ae := \alpha(A)e \subset \bigoplus_{j=1}^n H$, and let $p_{Ae} \in B\left(\bigoplus_{j=1}^n H\right)$ be the projection onto Ae. Since A is $*$-closed, $p_{Ae} \in \alpha(A)' = M_n(A')$ by Corollary 1.4.27(2) and Exercise 2.4.18(1).

If $x \in A''$, then $\alpha(x) \in M_n(A')'$ by Exercise 2.4.18(2) and thus commutes with p_{Ae}. Thus $\alpha(x)Ae \subseteq Ae$. Since A is unital, there is an $a \in A$ such that $\alpha(x)\alpha(1)e = \alpha(a)e$. In particular, $xe_j = ae_j$ for all j, so $x = a \in A$. Hence $A'' \subseteq A$, so $A = A''$. □

Definition 2.4.21 — Unital $*$-subalgebras $A \subset B(H)$ such that $A = A''$ are called *von Neumann algebras* or W**-algebras*.

By Exercise 2.4.16, A' is also a von Neumann algebra, so von Neumann algebras always come in pairs: A and A'. Combining Theorems 2.4.4 and 2.4.20, we immediately have the following corollary.

Corollary 2.4.22 — Unitary algebras are the same thing as finite dimesional von Neumann algebras.

Although the following corollary was already proven in Proposition 2.4.11 above, we provide a second von Neumann algebraic proof.

Corollary 2.4.23 — Suppose $A \subset B(H)$ is a unitary algebra. For $a \in A$, let $a = u|a|$ be the polar decomposition from Definition 1.7.24. Then $|a|$ and u are again in A.

Proof. We know $|a| = \sqrt{a^*a} \in A$ by Proposition 2.4.11. Recall from Remark 1.7.25 that the u constructed in Definition 1.7.24 commutes will all unitaries v which commute with a.

[1] To compare with the previous proof, notice this amplification is precisely

$$H \otimes \mathbb{C}^{\dim(H)} = H \otimes \mathbb{C}^{\dim(H^\vee)} \cong H \otimes H^\vee \cong B(H),$$

where the vector e corresponds to the identity 1_H and the subspace Ae corresponds to the subalgebra A.

This means that u commutes with all unitaries $v \in A'$. Since A' is a unitary algebra, it is spanned by its unitaries by Exercise 1.7.19. This means u commutes with all of A', so $u \in A'' = A$. □

Exercise 2.4.24. Suppose A is a unitary algebra acting on a Hilbert space H with commutant A'. A vector $\eta \in H$ is called

- *cyclic* for A if $H = A\eta$, and
- *separating* for A if $a\eta = b\eta$ for $a, b \in A$ implies $a = b$.

Prove that η is cyclic for A if and only if η is separating for A'.

Exercise 2.4.25. Suppose A is a unitary algebra.

(1) Show that if φ is a faithful weight on A, then $\Omega_\varphi \in L^2(A, \varphi)$ is cyclic and separating for A.

(2) Show that if A acts on the Hilbert space H and $\Omega \in H$ is cyclic and separating, then $\varphi_\Omega(x) := \langle \Omega | x\Omega \rangle$ is a faithful weight such that $H \cong L^2(A, \varphi_\Omega)$ as left A-modules.

(3) Can you turn the above two statements into a bijective correspondence?

Exercise 2.4.26. Suppose φ is a weight on a unitary algebra A. We say another weight $\psi \leq \varphi$ if $\psi(x) \leq \varphi(x)$ for all $x \geq 0$. Use Exercise 1.4.22/Corollary 1.6.22 to construct a bijective correspondence between weights $\psi \leq \varphi$ and operators $0 \leq x \leq 1$ in $A' \subset B(L^2(A, \varphi))$. Deduce that $A' = \mathbb{C}$ if and only if $\psi \leq \varphi$ implies $\psi = \varphi$.

Definition 2.4.27 — Suppose A is a unitary algebra and $p \in A$ is a projection. The *central support* of p is the smallest central projection $z(p) \in Z(A)$ such that $p \leq z(p)$.

Trick 2.4.28 — Suppose A is a unitary algebra and $p \in A$ is a projection. Then there are partial isometries $u_1, \ldots, u_n \in A$ such that the central support is given by

$$z(p) = \sum u_i p u_i^*.$$

To see this, it suffices to consider the case of a projection $p \in M_n(\mathbb{C})$. Assuming $p \neq 0$, there is a rank one projection $p_1 \leq p$, corresponding to some unit vector $|\xi_1\rangle \in \mathbb{C}^n$. Complete to an ONB, which gives a system of matrix units e_{ij} for $M_n(\mathbb{C})$ such that $p_1 = e_{11}$, $\sum p_{ii} = 1$, and $p_{ii} = e_{i1} p_{11} e_{i1}^* = e_{i1} p e_{i1}^*$ for each i.

Facts 2.4.29. We have the following facts about central supports of projections.

(Z1) For projections $p, q \in A$, $pAq = 0$ if and only if $z(p)z(q) = 0$.

Proof. Suppose $pAq = 0$. Using Trick 2.4.28 for p, we have that

$$z(p)q = \sum_i u_i p u_i^* q \in \sum_i u_i p A q = 0.$$

We now use Trick 2.4.28 for q to see that

$$z(p)z(q) = \sum_j z(p) v_j q v_j^* = \sum_j v_j z(p) q v_j^* = 0.$$

Conversely, if $z(p)z(q) = 0$, then $pAq = pz(p)Az(q)q = pAz(p)z(q)q = 0$. □

(Z2) For projections $p \in A$ and $q \in A'$, $pq = 0$ if and only if $z(p)z(q) = 0$.

Proof. Omitted, as it is highly similar to, but easier than, the proof of (Z1). □

Note: for infinite dimensional von Neumann algebras, one replaces the sum of projections $\sum_{i=1}^n u_i p u_i^*$ with the sup of projections $\bigvee_{u \in U(A)} upu^*$ for the above arguments.

Ethos 2.4.30 — A commutative unitary algebra is always of the form

$$\mathbb{C}^n = \mathrm{Func}(\{1, \ldots, n\}),$$

the space of functions $\{1, \ldots, n\} \to \mathbb{C}$, with pointwise addition and scalar multiplication and complex conjugation as the involution. Here, one may identify $\{1, \ldots, n\}$ with the 'spectrum' $\mathrm{spec}(\mathbb{C}^n)$ of $\mathbb{C}^n$ of unital algebra homomorphisms to $\mathbb{C}$. However, for infinite dimensional commutative operator algebras, we get different function spaces for different versions of operator algebra.

For infinite dimensional commutative C*-algebras, the Gelfand isomorphism is a canonical unital $*$-algebra isomorphism

$$A \cong C(\mathrm{spec}(A)),$$

where $\mathrm{spec}(A)$ is the compact Hausdorff space (in the weak* topology) of unital algebra homomorphisms $A \to \mathbb{C}$. In this sense, one often views a non-commutative C*-algebra as the algebra of functions on a *non-commutative topological space*. This intuition leads to many useful analogies and results about general C*-algebras.

Similarly, there is an analogous result for W*-algebras; any commutative W*-algebras is (non-canonically!) isomorphic to $L^\infty(X, \mu)$ for some measure space (X, μ). There are ways to make this canonical, but they are beyond the scope of this book and this Ethos environment. Nonetheless, one often views a non-commutative W*-algebra as the algebra of functions on a *non-commutative measure space*. This intuition leads to many useful analogies and results about general W*-algebras.

"Hot Take" 2.4.31 — In quantum physics, we measure observables, which are modeled by self-adjoint operators acting on some Hilbert space. We will discuss this in more detail in §5 below. Our knowledge of the universe comes from measurements; it is only through observation that we gain information from the world around us. One might be tempted to conclude that it is not the underlying space that is real, but rather the algebra of observables, as the latter describes our reality. More sharply and solipsistically put: there is no underlying space, only the observations we make.

2.5 Standard forms for unitary algebras

In Construction 2.3.15 above, we took a weight φ on a unitary algebra A and obtained the GNS Hilbert space $H_\varphi := L^2(A, \varphi)$ which admits the left A-action seen in Proposition 2.3.18. In this section, we give a weight-free description of the *standard form* of a unitary algebra due to Haagerup [Haa75], when we view A as a von Neumann algebra. Importantly, we will see that standard forms of a von Neumann algebra form a *contractible space*.

Warning 2.5.1 — Contractibility for standard forms is independent from the fact that the space of faithful weights on a unitary algebra is contractible. (See Exercise 1.7.16 together with the correspondence between weights and densities from Proposition 2.3.16.)

Definition 2.5.2 — A *standard form* (H, J, P) for a unitary algebra A consists of:

- a Hilbert space H with a faithful action of A;
- a conjugate-linear unitary involution $J : H \to H$ such that $J^2 = 1$ and $A' = JAJ$; and
- a *positive cone* $P \subset H$ closed under addition and scaling by $\mathbb{R}_{\geq 0}$ satisfying:
  - (P1) $J\xi = \xi$ for all $\xi \in P$,
  - (P2) $aJaJP \subseteq P$ for all $a \in A$, and
  - (P3) P is *self-dual*, i.e., $P = P^\circ := \{\eta \in H \mid \langle \eta | \xi \rangle \geq 0 \text{ for all } \xi \in P\}$.

Example 2.5.3 — For $\mathrm{tr} : A \to \mathbb{C}$ a faithful tracial state, one readily verifies:

- $H_{\mathrm{tr}} := L^2(A, \mathrm{tr})$,
- $J_{\mathrm{tr}} : H_{\mathrm{tr}} \to H_{\mathrm{tr}}$ by $x\Omega_{\mathrm{tr}} \mapsto x^*\Omega_{\mathrm{tr}}$,

- $P_{\text{tr}} := \{x\Omega_{\text{tr}} \mid x \in A_+\}$

assembles into a standard form $(H_{\text{tr}}, J_{\text{tr}}, P_{\text{tr}})$ for A.

Exercise 2.5.4. Verify $(H_{\text{tr}}, J_{\text{tr}}, P_{\text{tr}})$ as in the previous Example 2.5.3 does indeed assemble into a standard form for A.
Hint: you could proceed as follows.

(1) *Show that for $a \in A$, $J_{\text{tr}}a^*J_{\text{tr}}$ is right multiplication by a on H_{tr}. Deduce $J_{\text{tr}}AJ_{\text{tr}} \subseteq A'$.*

(2) *Show that for all $x \in B(H_{\text{tr}})$, $(J_{\text{tr}}xJ_{\text{tr}})^* = J_{\text{tr}}x^*J_{\text{tr}}$. Deduce that $J_{\text{tr}}x\Omega_{\text{tr}} = x^*\Omega_{\text{tr}}$ for all $x \in B(H_{\text{tr}})$.*

(3) *Show that $JA'J \subseteq A''$, i.e., $[JxJ, y] = 0$ for all $x, y \in A'$. Deduce that $A' = JAJ$.*

(4) *To prove (P3), show that $\operatorname{tr}(ab) \geq 0$ for all $a \geq 0$ if and only if $b \geq 0$.*

Where did you use traciality above?

Proposition 2.5.5 — Consider the standard form $(H_{\text{tr}}, J_{\text{tr}}, P_{\text{tr}})$ from Example 2.5.3. If $u \in A' \subseteq B(H_{\text{tr}})$ is unitary such that $uJ_{\text{tr}} = J_{\text{tr}}u$ and $uP_{\text{tr}} = P_{\text{tr}}$, then $u = 1$.

Proof. Since $u \in A' = J_{\text{tr}}AJ_{\text{tr}}$, $J_{\text{tr}}uJ_{\text{tr}} \in A$. Since u commutes with J_{tr}, $u = J_{\text{tr}}uJ_{\text{tr}} \in A' \cap A = Z(A)$. Since $u\Omega_{\text{tr}} \in P_{\text{tr}}$ and Ω_{tr} is separting, $u \geq 0$. The only positive unitary is 1. $\square$

In fact, one gets a standard form from *any* choice of faithful weight, not just a trace. In order to see this, we rapidly recall the construction of Tomita and Takesaki [Tak03].

Exercise 2.5.6. Suppose $T : H \to K$ is an *anti-linear* map, i.e., $T(\lambda\eta + \xi) = \overline{\lambda}T\eta + T\xi$ for all $\eta, \xi \in H$ and $\lambda \in \mathbb{C}$.

(1) Show that anti-linear maps admit anti-linear adjoints uniquely determined by the equation
$$\langle T\eta|\xi\rangle_K = \langle T^*\xi|\eta\rangle_H \qquad \forall\, \eta \in H \text{ and } \xi \in K.$$
Then show that $T^*T \in B(H)$ is linear and positive.

(2) Prove that anti-linear maps admit polar decompositions $T = U|T|$ where $|T| = \sqrt{T^*T} \in B(H)$ and $U : H \to K$ is an anti-linear partial isometry (U^*U is a projection). Then deduce a uniqueness statement for the polar decomposition similar to Exercise 1.7.26.

Hint: An anti-linear map $H \to K$ is the same thing as a linear map $H \to \overline{K}$.

Exercise 2.5.7. Let $H = L^2(M_n(\mathbb{C}), \operatorname{tr})$, and suppose $d \in M_n(\mathbb{C})$ is positive and invertible. Consider the operator $\operatorname{Ad}(d) \in B(H)$ given by $h \mapsto dhd^{-1}$.

(1) Show that under the isomorphism $B(H) \cong M_n(\mathbb{C}) \otimes M_n(\mathbb{C})^{\mathrm{op}}$, $\mathrm{Ad}(d)$ corresponds to $d \otimes d^{-1}$, which is positive and invertible.

(2) Deduce that if $d = \sum \lambda_j p_j$ is the spectral decomposition of d in $M_n(\mathbb{C})$, then the spectral decomposition of $d \otimes d^{-1}$ is given by $\sum_{j,k} \lambda_j \lambda_k^{-1} p_j \otimes p_k$.

(3) Deduce that for all $z \in \mathbb{C}$, $\mathrm{Ad}(d)^z = \mathrm{Ad}(d^z)$.

Construction 2.5.8 (Tomita-Takesaki for unitary algebras) — Suppose A is a unitary algebra equipped with a faithful weight φ. Fix a faithful tracial state $\mathrm{tr} : A \to \mathbb{C}$, and observe that $\varphi(a) = \mathrm{tr}(ad)$ for some positive invertible $d \in A$ by Proposition 2.3.16. Set $H_\varphi := L^2(A, \varphi)$ with cyclic separating vector Ω_φ, and consider the anti-linear map $S_\varphi : x\Omega_\varphi \mapsto x^*\Omega_\varphi$. We compute the polar decomposition $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$, where $\Delta_\varphi := S_\varphi^* S_\varphi$. First, we compute that $S_\varphi^*(y\Omega_\varphi) := dy^*d^{-1}\Omega_\varphi$ as

$$\begin{aligned}\langle S_\varphi x\Omega_\varphi | y\Omega_\varphi\rangle &= \langle x^*\Omega_\varphi | y\Omega_\varphi\rangle = \varphi(xy) = \mathrm{tr}(xyd)\\ &= \mathrm{tr}(d^{-1}ydxd) = \varphi(d^{-1}ydx) = \langle dy^*d^{-1}\Omega_\varphi | x\Omega_\varphi\rangle \qquad \forall\, x, y \in A.\end{aligned}$$

We thus see that $\Delta_\varphi = S_\varphi^* S_\varphi$ is given by $\Delta_\varphi(x\Omega_\varphi) = dxd^{-1}\Omega_\varphi$, and for every $t \in \mathbb{R}$, $\Delta_\varphi^{it}(x\Omega_\varphi) = d^{it}xd^{-it}\Omega_\varphi$ by the functional calculus (cf. Exercise 2.5.7). The corresponding *modular automorphism* of A is given by

$$\sigma_t^\varphi(a) := \Delta_\varphi^{it} a \Delta_\varphi^{-it} = d^{it} a d^{-it},$$

which can be analytically continued to any $t \in \mathbb{C}$. The polar decomposition of S_φ is then given by $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$ where

$$J_\varphi x\Omega_\varphi := \sigma_{i/2}^\varphi(x)^*\Omega_\varphi = \sigma_{-i/2}^\varphi(x^*)\Omega_\varphi = d^{1/2}x^*d^{-1/2}\Omega_\varphi.$$

The anti-linear unitary J_φ is often called the *modular conjugation*. We define a right action of A on H_φ by

$$(x\Omega_\varphi) \lhd a := J_\varphi a^* J_\varphi x\Omega_\varphi = x\sigma_{-i/2}^\varphi(a)\Omega_\varphi = xd^{1/2}ad^{-1/2}\Omega_\varphi. \tag{2.5.9}$$

Clearly $A' \cap B(H_\varphi) = J_\varphi A J_\varphi$. (This is just a fancy way of saying $\mathrm{End}(A_A) = A$.)

Exercise 2.5.10. Show that the formulas for S_φ and $\sigma_{\pm i/2}^\varphi$ above are independent of the choice of faithful tracial state tr on A.
Hint: Any other faithful tracial state tr' *is of the form* $\mathrm{tr}'(a) = \mathrm{tr}(az)$ *for some positive invertible* $z \in Z(A)$ *with* $\mathrm{tr}(z) = 1$ *by Exercise 2.4.13.*

Exercise 2.5.11. Show that for $a \in A$, $a \rhd \Omega_\varphi = \Omega_\varphi \lhd \sigma_{i/2}^\varphi(a)$.

Example 2.5.12 — Continuing the notation of Construction 2.5.8, set

$$P_\varphi := \operatorname{span}\{aJ_\varphi aJ_\varphi\Omega_\varphi \,|\, a\in A\} = \operatorname{span}\left\{ad^{1/2}a^*d^{-1/2}\Omega_\varphi \,\middle|\, a\in A\right\}.$$

Then $(H_\varphi, J_\varphi, P_\varphi)$ is a standard form for A. Indeed, the map

$$H_{\mathrm{tr}} \to H_\varphi \qquad \text{given by} \qquad a\Omega_{\mathrm{tr}} \mapsto ad^{-1/2}\Omega_\varphi$$

is a unitary isomorphism which

- intertwines the left and right A-actions, i.e.,

  $$u(a \rhd x\Omega_{\mathrm{tr}} \lhd b) = u(axb\Omega_{\mathrm{tr}}) = axbd^{-1/2}\Omega_\varphi = aJ_\varphi b^* J_\varphi xd^{-1/2}\Omega_\varphi = a \rhd u(x\Omega_{\mathrm{tr}}) \lhd b,$$

- intertwines J_{tr} and J_φ, i.e., $uJ_{\mathrm{tr}}x\Omega_{\mathrm{tr}} = J_\varphi ux\Omega_{\mathrm{tr}}$, and
- maps P_{tr} onto P_φ, i.e., $uP_{\mathrm{tr}} = P_\varphi$.

In fact, there is exactly one such map. Indeed, given any two unitary isomorphisms $u, v : H_{\mathrm{tr}} \to H_\varphi$, $u^*v : H_{\mathrm{tr}} \to H_{\mathrm{tr}}$ commutes with J_{tr} and maps P_{tr} onto itself. Thus $u^*v = 1$ by Proposition 2.5.5.

Exercise 2.5.13. Suppose (H, J, P) is a standard form for A.

(1) Prove that if $\eta \in H$ such that $\langle \eta|\xi\rangle = 0$ for all $\xi \in P$, then $\eta = 0$.
Hint: Observe $\eta \in P^\circ = P$.

(2) Deduce that $xP = 0$ implies $x = 0$ for $x \in B(H)$.
Hint: First show $x^* = 0$.

(3) Deduce that if $p \in B(H)$ is a projection such that $P \subseteq pH$, then $p = 1$.

The next lemma first appeared as [AH14, Lem. 3.19] and showed that one of Haagerup's original axioms for a standard form is superfluous. We give a simple proof worked out with André Henriques which can be adapted for any von Neumann algebra.

Lemma 2.5.14 — Suppose (H, J, P) is a standard form for the unitary algebra A. Then $JxJ = x^*$ for all $x \in Z(A)$.

Proof. Observe that $p \mapsto JpJ$ is an involution on the minimal central projections of A, i.e., the minimal projections in the commutative unitary algebra $Z(A)$. Since the sum of all the minimal central projections is 1, we can write $1 = p + q + JqJ$ where $Jp_0J = p_0$ for every $p_0 \le p$ and $JqJ \perp q$. (Just look at the $\mathbb{Z}/2$-orbits of the minimal central projections.)

Since $J(1-p)J = 1-JpJ = 1-p$, for any $\xi \in P$, we have $(1-p)\xi = (1-p)J(1-p)J\xi \in P$. Now consider $a := q - JqJ$, for which $JaJ = -a$, and observe that

$$aJaJ = (q - JqJ)(JqJ - q) = -q - JqJ = -(1-p).$$

Then for all $\xi \in P$, $aJaJ\xi = -(1-p)\xi \in P$, and thus $\pm(1-p)\xi \in P$. We conclude that $(1-p)\xi = 0$ for all $\xi \in P$, so $p = 1$ by Exercise 2.5.13(3). $\square$

Corollary 2.5.15 — Suppose (H, J, P) is a standard form for the unitary algebra A. For any projection $p \in A$, the central support $z(p)$ is equal to $z(JpJ)$. In particular, $pJpJ = 0$ if and only if $p = 0$.

Proof. Use Trick 2.4.28 for p to write $z(p) = \sum u_i p u_i^*$ for partial isometries $u_1, \ldots, u_n \in A$. Then since J is an involution, the minimal central projection larger than JpJ must be given by

$$z(JpJ) = \sum Ju_iJ(JpJ)Ju_i^*J = \sum Ju_ipu_i^*J = Jz(p)J \underset{\text{(Lem. 2.5.14)}}{=} z(p).$$

The second statement now follows from (Z2), since $pJpJ = 0$ if and only if $z(p)z(JpJ) = z(p) = 0$ if and only if $p = 0$. $\square$

Proposition 2.5.16 (cf. [Haa75, Lem 2.8]) — Suppose (H, J, P) is a standard form for A. There is a cyclic separating vector Ω for A that lies in P.

Proof. Let $\eta_1, \ldots, \eta_n \in P$ be a maximal family of unit vectors such that the orthogonal projections $p_i \in A$ onto the subspaces $A'\eta_i$ are mutually orthogonal. By maximality, $P \subseteq \bigoplus A'\eta_i$ so that $\bigoplus A'\eta_i = H$ and $\sum p_i = 1$ by Exercise 2.5.13. Indeed, setting $p := 1 - \sum p_i$, for any $\xi \in P$, we have $pJpJ\xi \in P$ must be orthogonal to $p_iH = A'\eta_i$ for each i. So by maximality, $pJpJ\xi = 0$ for all $\xi \in P$. But then $pJpJ = 0$, so $p = 0$ by Corollary 2.5.15.

We claim that $\Omega := \sum \eta_i \in P$ is cyclic and separating. Since $J\eta_i = \eta_i$ as each $\eta_i \in P$, we have $A'\eta_i = JAJ\eta_i = JA\eta_i$, so the subspaces $A\eta_i$ are also mutually orthogonal. Thus if $a\Omega = 0$ for $a \in A$, we have $a\eta_i = 0$ for each i by orthogonality. But then $ax\eta_i = 0$ for each $x \in A'$, and thus $a = 0$ on $A'\eta_i$. We conclude $a = 0$ on H, so Ω is separating for A.

By the same argument swapping A and A', we see Ω is also separating for A' and thus cyclic for A by Exercise 2.4.24. Indeed, if $x \in A'$, as the $A'\eta_i$ are orthogonal $x\Omega = 0$ if and only if $x\eta_i = 0$ for all i. Then $xa\eta_i = 0$ for all $a \in A$, so $x = 0$ on $A\eta_i$. But we have

$$H = \bigoplus A'\eta_i = \bigoplus JAJ\eta_i = \bigoplus JA\eta_i \qquad \Longleftrightarrow \qquad H = JH = \bigoplus A\eta_i,$$

so $x = 0$ on H. $\square$

Lemma 2.5.17 — Suppose (H, J, P) is a standard form for A, and $\Omega \in P$ is a chosen cyclic separating vector. Define $S_\Omega : H \to H$ by $S_\Omega x\Omega = x^*\Omega$, and let $\Delta_\Omega = S_\Omega^* S_\Omega$. Then $S_\Omega = J\Delta_\Omega^{1/2}$ is the polar decomposition.

Proof. Since Ω is cyclic and separating, S_Ω is well-defined and invertible. By uniqueness of the polar decomposition, it suffices to prove that $JS_\Omega \geq 0$; it then follows that $S_\varphi = J(JS_\varphi)$ and thus $JS_\varphi = \Delta_\varphi^{1/2}$. Indeed, for all $x \in A$,

$$\langle x\Omega | JS_\Omega x\Omega\rangle = \langle S_\Omega x\Omega | Jx\Omega\rangle = \langle x^*\Omega | JxJ\Omega\rangle = \langle \Omega | xJxJ\Omega\rangle \geq 0$$

as both $\Omega, xJxJ\Omega \in P$. $\square$

Corollary 2.5.18 — Given a standard form (H, J, P) for A together with a chosen cyclic separating vector $\Omega \in P$, set $\varphi(a) := \langle \Omega | a\Omega \rangle$. Then φ is a faithful weight, and the map $u : \Omega \mapsto \Omega_\varphi$ uniquely extends to the unique left A-linear unitary isomorphism $H \to H_\varphi$ intertwining J with J_φ such that $uP = P_\varphi$.

Proof. The unitary u clearly intertwines the left A-actions. By Lemma 2.5.17, the unitary u intertwines S_Ω with S_φ and $\Delta_\Omega^{1/2}$ with $\Delta_\varphi^{1/2}$, so u intertwines J with J_φ by uniqueness of the polar decomposition. Then $uxJxJ\Omega = xJ_\varphi xJ_\varphi \Omega_\varphi$ for all $x \in A$, so $uP = P_\varphi$.

Uniqueness follows immediately from Proposition 2.5.5 as in Example 2.5.12. $\square$

Theorem 2.5.19 ([Haa75]) — Given standard forms (H_i, J_i, P_i) for A, there is a unique unitary $u : H_1 \to H_2$ such that $uJ_1 = J_2 u$, and $P_2 = uP_1$. Thus standard forms for a unitary algebra form a contractible space.

Proof. Without loss of generality, $H_1 = L^2(A, \mathrm{tr})$ for some faithful tracial state tr on A. By Proposition 2.5.16, there is a cyclic separating vector in P_2 for A. By Corollary 2.5.18, there is a unique unitary isomorphism $(H_2, J_2, P_2) \cong (H_\varphi, J_\varphi, P_\varphi)$ of standard forms, and by Example 2.5.12, there is a unique unitary isomorphism $(H_\varphi, J_\varphi, P_\varphi) \cong (H_{\mathrm{tr}}, J_{\mathrm{tr}}, P_{\mathrm{tr}})$ of standard forms. $\square$

Corollary 2.5.20 — Let (H, J, P) be a standard form for A. Then $a \in A$ is positive if and only if $\langle \xi | a\xi \rangle \geq 0$ for all $\xi \in P$.

Proof. By Example 2.5.3 and Theorem 2.5.19, we may identify $H = L^2(A, \mathrm{Tr})$ and $P = \{x\Omega_{\mathrm{Tr}} \,|\, x \in A_+\}$ for some faithful tracial weight Tr. Then $\langle \xi | a\xi \rangle \geq 0$ for all $\xi \in P$ if and only if $\mathrm{Tr}(ax) \geq 0$ for all $x \in A_+$, which is clearly equivalent to $a \geq 0$. $\square$

Notation 2.5.21 — Given a unitary algebra A, we denote its standard form by $L^2 A$.

Warning 2.5.22 — Even though the space of standard forms for A is contractible, and $A \cong L^2 A$ as vector spaces, one cannot identify $L^2 A$ with A without making a choice. Indeed, the space of invertible $A - A$ bimodular maps ${}_A A_A \to {}_A L^2 A_A$ which map $A_+ \to P = L^2 A_+$ is a torsor for $Z(A)_+^\times$ (pick a faithful trace and look at the image of 1_A), and is thus in bijection with faithful tracial weights on A.

We provide the following *weight independent* version of the standard form which is an adaptation of [Yam92].

Construction 2.5.23 — Let A be a unitary algebra. First, consider the set of formal symbols

$$P := \{\sqrt{\varphi} \,|\, \varphi \text{ is a weight on } A\}.$$

We define a function $P \times P \to [0, \infty)$ by

$$\langle\sqrt{\varphi}|\sqrt{\psi}\rangle := \mathrm{Tr}(d_\varphi^{1/2} d_\psi^{1/2})$$

where Tr is a choice of faithful tracial weight on A, d_φ is the density of φ with respect to Tr so that $\varphi(x) = \mathrm{Tr}(x d_\varphi)$ for all $x \in A$, and similarly for ψ.

We claim that this function is *independent* of the choice of trace. Indeed, if Tr' is another faithful tracial weight, then $\mathrm{Tr}'(x) = \mathrm{Tr}(xz)$ for some positive invertible $z \in Z(A)_+^\times$. This means

$$\mathrm{Tr}(x d_\varphi) = \varphi(x) = \mathrm{Tr}'(x d'_\varphi) = \mathrm{Tr}(x d'_\varphi z)$$

so $d_\varphi = d'_\varphi z$, and $d_\varphi^{1/2} = d_\varphi'^{1/2} z^{1/2}$ as z is central. Thus

$$\mathrm{Tr}'(d_\varphi'^{1/2} d_\psi'^{1/2}) = \mathrm{Tr}(d_\varphi'^{1/2} d_\psi'^{1/2} z) = \mathrm{Tr}(d_\varphi'^{1/2} z^{1/2} d_\psi'^{1/2} z^{1/2}) = \mathrm{Tr}(d_\varphi^{1/2} d_\psi^{1/2}).$$

We now consider the formal $\mathbb{C}$-linear combinations of elements of P equipped with the formal sesquilinear extension of $\langle\,\cdot\,|\,\cdot\,\rangle$, i.e.,

$$\left\langle \sum a_i \sqrt{\varphi_i} \middle| \sum b_j \sqrt{\psi_j} \right\rangle := \sum_{i,j} \overline{a_i} b_j \, \mathrm{Tr}(d_{\varphi_i}^{1/2} d_{\psi_j}^{1/2}) = \mathrm{Tr}\left(\left(\sum a_i d_{\varphi_i}^{1/2} \right)^* \sum b_j d_{\psi_j}^{1/2} \right).$$

Observe that the following are equivalent:

- $\sum a_i d_{\varphi_i}^{1/2} = \sum b_j d_{\psi_j}^{1/2}$, and
- the sesquilinear extension of $\langle\,\cdot\,|\,\cdot\,\rangle$ satisfies

$$\left\langle \sum a_i \sqrt{\varphi_i} - \sum b_j \sqrt{\psi_j} \middle| \sum a_i \sqrt{\varphi_i} - \sum b_j \sqrt{\psi_j} \right\rangle = 0.$$

We thus define $L^2 A$ as the quotient of these formal $\mathbb{C}$-linear combinations under the equivalence relation

$$\sum a_i \sqrt{\varphi_i} = \sum b_j \sqrt{\psi_j} \qquad \Longleftrightarrow \qquad \sum a_i d_{\varphi_i}^{1/2} = \sum b_j d_{\psi_j}^{1/2}.$$

Clearly $\langle\,\cdot\,|\,\cdot\,\rangle$ is an inner product, and P is a positive cone. We define J as complex conjugation on formal linear combinations of elements of P.

We define the left A-action on L^2A by the following functional equation, which is still independent of the choice of Tr:

$$\left\langle \sqrt{\varphi} \middle| x \cdot \sqrt{\psi} \right\rangle = \mathrm{Tr}(d_\varphi^{1/2} x d_\psi^{1/2}).$$

This action is clearly unital and linear, and $*$-preserving follows by the formula

$$\overline{\left\langle \sqrt{\varphi} \middle| x \cdot \sqrt{\psi} \right\rangle} = \overline{\mathrm{Tr}(d_\varphi^{1/2} x d_\psi^{1/2})} = \mathrm{Tr}(d_\psi^{1/2} x^* d_\varphi^{1/2}) = \left\langle \sqrt{\psi} \middle| x^* \cdot \sqrt{\varphi} \right\rangle.$$

What is not obvious at this point is why $x \cdot y \cdot \sqrt{\varphi} = xy \cdot \sqrt{\varphi}$. However, this follows from the following theorem.

Theorem 2.5.24 — The weight independent definition of (L^2A, J, P) above is a standard form for A.

Proof. Let Tr be a faithful tracial weight on A. The map $\sqrt{\varphi} \mapsto d_\varphi^{1/2}\Omega_{\mathrm{Tr}}$ uniquely extends to a unitary A-module isomorphism which is compatible with J, P. We check A-linearity, and we leave the rest of the details to the reader. Indeed, for all weights φ, ψ,

$$\langle \sqrt{\varphi} | x \cdot \sqrt{\psi} \rangle_{L^2A} = \mathrm{Tr}(d_\varphi^{1/2} x d_\psi^{1/2}) = \langle d_\varphi^{1/2}\Omega_{\mathrm{Tr}} | x d_\psi^{1/2}\Omega_{\mathrm{Tr}} \rangle_{L^2(A,\mathrm{Tr})}.$$ □

2.6 Complete positivity

In this section, we define the notion of a *completely positive* operator, and we give several characterizations, including theorems by Stinespring and Choi.

Definition 2.6.1 — Let A be a unitary algebra. A linear map $\Phi\colon A \to B(H)$ is called *completely positive* if for all $n \in \mathbb{N}$ and all positive $x \in M_n(A)$, $(\Phi(x_{ij})) \in M_n(B(H))$ is positive. Under the isomorphism $M_n(A) \cong M_n(\mathbb{C}) \otimes A$ from Exercise 2.4.8, this is equivalent to saying that for all n,

$$\mathrm{id} \otimes \Phi : M_n(\mathbb{C}) \otimes A \to M_n(\mathbb{C}) \otimes B(H) \qquad (a_{ij}) \mapsto (\Phi(a_{ij}))$$

is positive (maps positive operators to positive operators).

Remark 2.6.2. Under the representation

$$M_n(\mathbb{C}) \otimes A = \{|i\rangle \otimes a \otimes \langle j| \mid a \in A,\ i, j = 1, \ldots, n\}$$

from Exercise 2.4.8, the map $\mathrm{id} \otimes \Phi$ is given by $|i\rangle \otimes a \otimes \langle j| \mapsto |i\rangle \otimes \Phi(a) \otimes \langle j|$.

Example 2.6.3 (Homomorphisms) — If $\Phi : A \to B(H)$ is a $*$-homomorphism, then it is completely positive as $\mathrm{id}_n \otimes \Phi$ is again a $*$-homomorphism for every n.

Example 2.6.4 (Conjugation) — If $v : H \to K$, then the map $\Phi : B(K) \to B(H)$ given by $\Phi(x) = v^\dagger x v$ is completely positive. Indeed, for all $x \in M_n(B(K))$,

$$(v^\dagger x_{ij} v) = \begin{pmatrix} v & & \\ & \ddots & \\ & & v \end{pmatrix}^\dagger (x_{ij}) \begin{pmatrix} v & & \\ & \ddots & \\ & & v \end{pmatrix}.$$

Exercise 2.6.5. Show that the transpose map $A \mapsto A^T$ on $M_2(\mathbb{C})$ is positive, but not completely positive.

The next theorem says that every completely positive map is the composite of a unital homomorphism and a conjugation.

Theorem 2.6.6 (Stinespring) — For every completely positive $\Phi\colon A \to B(H)$, there exists a Hilbert space K, a unital $*$-homomorphism $\pi : A \to B(K)$ and a map $v\colon H \to K$ such that $\Phi(a) = v^\dagger \pi(a) v$. If Φ is unital, then v is an isometry.

Proof. First, we endow the vector space $A \otimes H$ with a sesquilinear form determined on simple tensors by

$$\langle a \otimes \eta | b \otimes \xi \rangle := \langle \eta | \Phi(a^* b) \xi \rangle_H.$$

Since Φ is completely positive, it is readily checked that this sesquilinear form is positive. Let $K = (A \otimes H)/N$ where N is the subspace of length zero vectors for this form. Observe that $A \otimes H$ has an organic A-action determined by $a(b \otimes \eta) = (ab) \otimes \eta$. Moreover N is a sub-module since, for $a \in A$ and $\sum b_i \otimes \xi_i$, we have

$$0 \leq \Big\langle \sum_i ab_i \otimes \xi_i \Big| \sum_j ab_j \otimes \xi_j \Big\rangle = \sum_{ij} \langle \xi_i | \Phi(b_i^* a^* a b_j) \xi_j \rangle \leq \|a^* a\| \sum_{ij} \langle \xi_i | \Phi(b_i^* b_j) \xi_j \rangle = 0.$$

Therefore, this A-action descends to K by

$$\pi(a)(b \otimes \eta + N) := ab \otimes \eta + N.$$

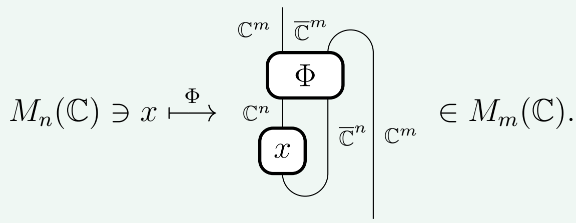
We then define the map $v\colon H \to K$ by $\xi \mapsto 1_A \otimes \xi + N$. For all $\xi, \eta \in H$,

$$\langle \xi | v^\dagger \pi(a) v \eta \rangle_H = \langle v\xi | \pi(a) v \eta \rangle_K = \langle 1_A \otimes \xi + N | a \otimes \eta + N \rangle_K = \langle \xi | \Phi(a) \eta \rangle_H,$$

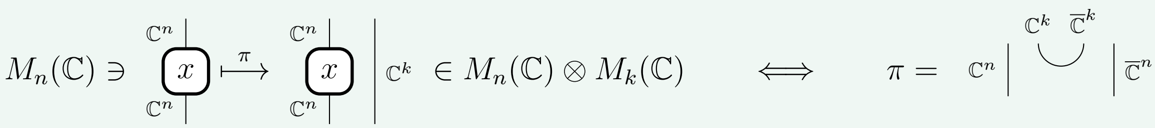
so $\Phi(a) = v^\dagger \pi(a) v$ by Lemma 1.4.19. $\square$

The next two examples are motivated by [Sel07, Cor. 4.13]. To the best of the authors' knowledge, the graphical representation of Stinespring in Example 2.6.7 below was first recognized in [Ver22].

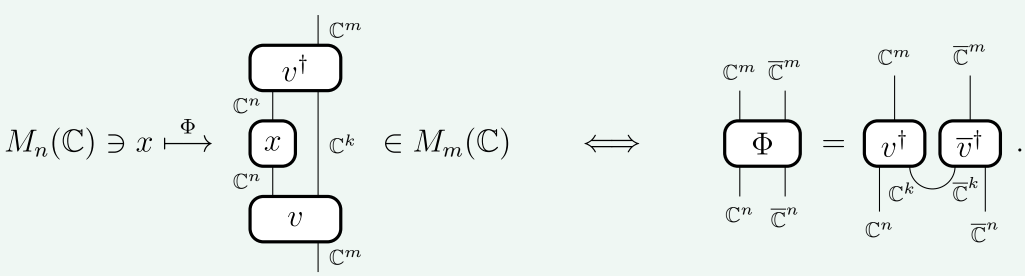
Example 2.6.7 ([Ver22]) — When $A = M_n(\mathbb{C})$, we can interpret Stinespring's Theorem 2.6.6 for completely positive maps $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ graphically as follows. First, using the unitary isomorphism $M_n(\mathbb{C}) \cong \mathbb{C}^n \otimes \overline{\mathbb{C}^n}$ from Corollary 1.6.18 and Exercise 1.6.19, we may diagrammatically view $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ as



Now by Exercise 2.4.19 based on Example 2.1.23, every unital $*$-homomorphism from $\pi : M_n(\mathbb{C}) \to B(K)$ is an amplification. Thus, without loss of generality, if $\Phi(x) = v^\dagger \pi(x) v$, then there is a $k \in \mathbb{N}$ such that $\pi : M_n(\mathbb{C}) \to M_{nk}(\mathbb{C})$, and $v : \mathbb{C}^m \to \mathbb{C}^n \otimes \mathbb{C}^k$. Graphically, we may represent amplification by drawing a strand to either the left or the right; we choose the right side for convenience. The isomorphism $M_n(\mathbb{C}) \cong \mathbb{C}^n \otimes \overline{\mathbb{C}^n}$ then bends one string up to the right.



Now taking the graphical expression for $\Phi(x) = v^\dagger \pi(x) v$ and turning up a strand to the right yields



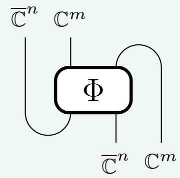
Conversely, any morphism of the form on the right hand side is visibly a unital $*$-homomorphism followed by a conjugation, which is manifestly completely positive.

Example 2.6.8 (Choi matrix, [Sel07, Cor. 4.13]) — Suppose $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$. The *Choi matrix* of Φ is the matrix

$$C_\Phi := \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{ij}) \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C}),$$

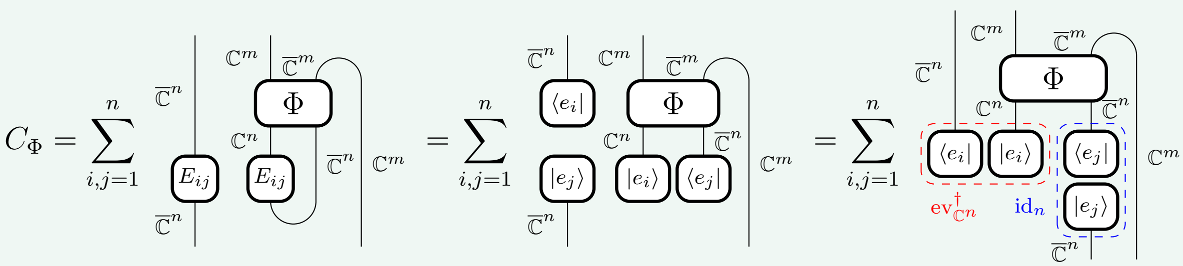
where $E_{ij} \in M_n(\mathbb{C})$ is the matrix whose ij-th entry is one and all other entries are zero. While this definition may appear unmotivated before reading the statement of Choi's Theorem 2.6.10 below, graphically, we may identify C_Φ with the *one click rotation* of Φ.

$C_\Phi =$



(2.6.9)

Again as in Example 2.6.7 above, we identify $M_n(\mathbb{C}) \cong \mathbb{C}^n \otimes \overline{\mathbb{C}^n}$, but we may also identify $M_n(\mathbb{C}) \cong \mathrm{End}(\overline{\mathbb{C}^n})$ via $E_{ij} \mapsto |e_j\rangle\langle e_i|$ acting on the ket space $\overline{\mathbb{C}^n}$ on the *right*. Under these identifications, the Choi matrix is given by



As we are separately summing over i, j and identifying $(\mathbb{C}^n)^\vee = \overline{\mathbb{C}}^n$, we may apply (1.6.8) in the dashed red and blue boxes locally for $\mathbb{C}^n$ and $\overline{\mathbb{C}}^n$ respectively to connect the strings, yielding the formula (2.6.9) as claimed.

The following theorem of Choi refines Stinespring's Theorem when $A = M_n(\mathbb{C})$.

Theorem 2.6.10 (Choi-Stinespring) — For a map $\Phi\colon M_n(\mathbb{C}) \to M_m(\mathbb{C})$, the following are equivalent:

(1) Φ is completely positive,

(2) $\mathrm{id} \otimes \Phi\colon M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ maps positive elements to positive elements,

(3) the Choi matrix $C_\Phi = \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{ij}) \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ is positive, and

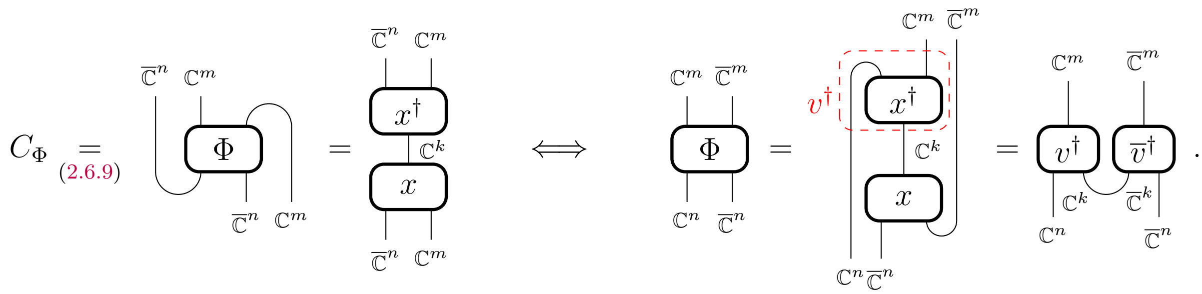
(4) (Stinespring) $\Phi(x) = v^\dagger(x \otimes \mathrm{id}_{\mathbb{C}^k})v$ for some map $v : \mathbb{C}^m \to \mathbb{C}^n \otimes \mathbb{C}^k$.

Proof.

(1)⇒(2): Immediate from the definition of completely positive.

(2)⇒(3): First, note that the matrix $C := \sum E_{ij} \otimes E_{ij}$ satisfies $C^\dagger = C$ and $C^2 = nC$. Thus $C = n^{-1}C^\dagger C \geq 0$. By (2), $C_\Phi = (\mathrm{id} \otimes \Phi)(C) \geq 0$.

(3)⇒(4): If $C_\Phi \geq 0$, then we may write $C_\Phi = x^\dagger x$ for some $x : \overline{\mathbb{C}^n} \otimes \mathbb{C}^m \to \mathbb{C}^k$.



Setting $v^\dagger : \mathbb{C}^n \otimes \mathbb{C}^k \to \mathbb{C}^m$ equal to $x^\dagger$ with one strand turned down, we visibly obtain a Stinespring expression for Φ as in Example 2.6.7 above. We conclude that Φ is completely positive.

(4)⇒(1): Immediate from Examples 2.6.3 and 2.6.4, together with the fact that the composite of completely positive maps is completely positive. □

We end this section with a more advanced result proving when a unital C*-algebra A is necessarily finite dimensional. This result depends on the notion of an *entropic index* for a unital completely positive map in the spirit of [PP86]. We freely use the following facts about (infinite dimensional) unital C*-algebras.

- Spectral permanence: If $B \subset A$ is a unital C*-subalgebra, then $\operatorname{spec}_B(b) = \operatorname{spec}_A(b)$ for all $b \in B$ (cf. 2.4.9).
- Gelfand's theorem: a commutative C*-algebra is always of the form $C(X)$ for a compact Hausdorff space X.
- The spectral theorem: If $a \in A$ is normal, then the unital C*-subalgebra $C^*(a) \subseteq A$ generated by a, a^* is isometrically isomorphic to $C(\operatorname{spec}(a))$ (cf. Theorem 1.7.13).

Finally, we also freely use standard topological techniques for compact Hausdorff spaces, e.g., normality and Urysohn's Lemma.

Exercise 2.6.11. Suppose A is a unitary algebra and φ is a faithful state on A. Prove there is a $\lambda > 0$ such that

$$\varphi(a) \geq \lambda \cdot \|a\| \qquad\qquad \forall\, a \geq 0.$$

Hint: First consider the case that $\varphi = \operatorname{tr}$ is a trace. Then use the Spectral Theorem 1.7.9 to show that any invertible density matrix is bounded below.

Lemma 2.6.12 — Suppose X is a compact Hausdorff space. Either X is finite or there are infinitely many norm one functions $\{f_i\} \subset C(X)$ with mutually disjoint supports.

Proof. Suppose X is not finite. By an induction argument, it suffices to prove that we can find disjoint closed subsets $F, G \subset X$ such that G is infinite. Indeed, since X is normal, by Urysohn's Lemma, there is a continuous function $f_1 : X \to [0,1]$ such that $f_1|_F = 1$ and $f|_G = 0$. We then replace X with G and use induction to construct the rest of our sequence.

Pick two distinct points $x, y \in X$. Since X is Hausdorff, there are disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$. Since $X = U^c \cup V^c$, without loss of generality, U^c is infinite. Then $F := \{x\}$ and $G := U^c$ are our desired disjoint closed subsets of X. □

Theorem 2.6.13 — Suppose A is a unital C*-algebra and $\varphi : A \to \mathbb{C}$ is a state and $\lambda > 0$ such that

$$\varphi(a) \geq \lambda \cdot \|a\| \qquad\qquad \forall\, a \geq 0.$$

Then $\dim(A) \leq \lambda^{-2}$. If A is commutative, then $\dim(A) \leq \lambda^{-1}$.

Proof communicated by Makoto Yamashita.

Step 1: There are at most λ^{-1} norm 1 positive elements in A which are mutually orthogonal, i.e., $x_i x_j = 0$ if $i \neq j$.

Proof. Since the x_i are mutually orthogonal, any finite sum of them satisfies $\sum_{i=1}^n x_i \leq 1$. (Indeed, one can compute this in the commutative C*-subalgebra of A generated by the x_i.) Since $\varphi(1) = 1$ and $\varphi(x_i) \geq \lambda \cdot \|x_i\| = \lambda$ for all i, we have

$$n \cdot \lambda \leq \sum_{i=1}^n \varphi(x_i) = \varphi\left(\sum_{i=1}^n x_i\right) \leq \varphi(1) = 1,$$

and thus $n \leq \lambda^{-1}$. □

Step 2: If $B \subseteq A$ is a commutative C*-subalgebra, then $\dim(B) \leq \lambda^{-1}$. In particular, for every $a \geq 0$ in A has $|\operatorname{spec}(a)| \leq \lambda^{-1}$.

Proof. By Gelfand's Theorem, $A \cong C(X)$ for some compact Hausdorff space X. By Lemma 2.6.12 and Step 1, we must have that X is finite with $|X| \leq \lambda^{-1}$. □

Step 3: Every $a \geq 0$ has $|\operatorname{spec}(a)| \leq \lambda^{-1}$.

Proof. Observe that such an a generates an abelian C*-subalgebra of A. By Gelfand's Theorem, $C^*(a) \cong C(\operatorname{spec}(a))$. Restricting φ to $C(\operatorname{spec}(a))$ and applying Step 2, we conclude that $|\operatorname{spec}(a)| \leq \lambda^{-1}$. □

Step 4: We can write $1 = \sum_{i=1}^n p_i$ as a sum of $n \leq \lambda^{-1}$ minimal orthogonal projections.

Proof. Write $1 = \sum_{i=1}^n p_i$ as a finite sum of mutually orthogonal projections (this sum may be the trivial sum). Observe that we must have $n \leq \lambda^{-1}$ by Step 1. If $a \in p_i A p_i$ is positive and not a scalar times p_i, then by Step 3, $\operatorname{spec}(a)$ is a finite set which is not a point. Hence there is a non-trivial sub-projection of p_i, and we can thus refine our partition of unity. We can only do this finitely many times, as $n \leq \lambda^{-1}$. Hence our final partition of unity must be by minimal projections. □

Finally, as $1 = \sum_{i=1}^n p_i$ is a sum of mutually orthogonal minimal projections with $n \leq \lambda^{-1}$, we see that

$$A = \bigoplus_{i,j=1}^n p_i A p_j \qquad \Longrightarrow \qquad \dim(A) \leq n^2 \leq \lambda^{-2}.$$

Indeed, $\dim(p_i A p_j) \leq 1$ for all i, j by Exercise 2.6.14 below, whence the result. □

Exercise 2.6.14. Suppose A is a unital C*-algebra, which is not assumed to be finite dimensional. Show that if p, q are minimal projections ($pAp = \mathbb{C}p$ and $qAq = \mathbb{C}q$), then $\dim(pAq) \leq 1$.
Hint: if there is an $a \in A$ with $x := paq \neq 0$, consider x^*x and xx^*.

Definition 2.6.15 — Suppose A, B are unital C*-algebras and $\Phi : B \to A$ is unital completely positive map. We say Φ has *finite index* if there is a $\lambda > 0$ such that

$$\Phi(b) \geq \lambda \cdot b \qquad \forall\, b \in B_+.$$

In this case, the *index* of Φ is determined by the formula

$$\operatorname{ind}(\Phi)^{-1} := \max\left\{\lambda > 0 \,\middle|\, \Phi(b) \geq \lambda \cdot b \text{ for all } b \in B_+\right\}.$$

Corollary 2.6.16 — Suppose A, B are unital C*-algebras and $\Phi : B \to A$ is a finite index unital completely positive map. If A is finite dimensional, then so is B.

Proof. Pick a faithful state φ on A, and observe that there is a $\mu > 0$ such that $\operatorname{tr}(a) \geq \mu \cdot \|a\|$ for all $a \geq 0$ in A by Exercise 2.6.11. Let $0 < \lambda \leq \operatorname{ind}(\Phi)^{-1}$. Then the composite map $\varphi \circ \Phi$ is a state on B such that

$$\varphi(\Phi(b)) \geq \mu \cdot \|\Phi(b)\| \geq \lambda\mu \cdot \|b\| \qquad \forall\, b \geq 0.$$

Thus B is finite dimensional by Theorem 2.6.13. □