Chapter 3

Morita equivalence

In this chapter, we study a form of equivalence for algebras called *Morita equivalence* which is weaker than the notion of isomorphism. We will see in [[Part II]] once we have introduced the notion of a category that a Morita equivalence between algebras A, B is the same as an equivalence between their associated categories of modules.

In order to discuss *unitary* Morita equivalence for unitary algebras, we should to know what a module for a unitary algebra is. This essential question admits several completely reasonable answers, each with their own features and drawbacks. One of the main themes of this book is that depending on the answer you choose, you get slightly different categories, which have profound differences in their higher categorical analogs, as well as the notion of a higher Hilbert space. (Recall Warning 0.0.1: choices have consequences!) In this chapter, we will explore 3 different notions.

- (§3.2) One can think of unitary algebras as finite dimensional C*-algebras, which naturally act on *Hilbert* C*-*modules* [Rie74], which are vector spaces equipped with right A-*valued inner products*.[1] As intertwiners, we only take the *adjointable operators*.
- (§3.3) One can think of unitary algebras as finite dimensional von Neumann algebras, also known as W*-algebras, which naturally act on Hilbert spaces.
- (§3.4) One can work with unitary algebras equipped with faithful tracial weights (A, Tr_A), also known as H*-algebras [Amb45], which also naturally act on Hilbert spaces. However, we also get a canonical commutant trace $\mathrm{Tr}_{A'}$ on $A' := \mathrm{End}(H_A)$.

Of course, since a unitary algebra A is a multimatrix algebra, by Corollary 2.2.6, every algebraic module is a direct sum of simple modules, which correspond to columns of the matrix summands of A. Thus each of these above three notions of unitary module forgets to this same algebraic notion.

We will see later in [[Part III]] that each of these unitary options adds additional structure to the collection of such algebras, their bimodules, and intertwiners. The first admits an

[1] For infinite dimensions, modules should be Banach spaces, where the Banach norm is compatible with the A-valued inner product and the norm on A.

adjoint for intertwiners, the second admits this adjoint along with a conjugate for bimodules, and the third admits an entire *homotopy $O(2)$-action*.

3.1 Algebraic Morita equivalence

Before we investigate the various notions of module for unitary algebras, we begin with a discussion of algebraic Morita equivalence to guide us later.

Definition 3.1.1 — Suppose A is an algebra, M_A is a right A-module, and ${}_AN$ is a left A-module. The *relative tensor product* $M \otimes_A N$ is the quotient of the tensor product $M \otimes N$ by the subspace generated by

$$\{ma \otimes n - m \otimes an \,|\, m \in M,\ a \in A,\ \text{and } n \in N\}.$$

For $m \in M$ and $n \in N$, we denote the image of $m \otimes n$ in $M \otimes_A N$ by $m \otimes_A n$.

Exercise 3.1.2. Show that $M \otimes_A N$ satisfies the following *universal property*. For every vector space P and every map $f : M \otimes N \to P$ which is A-*balanced*, i.e., $f(ma \otimes n) = f(m \otimes an)$ for all $m \in M$, $a \in A$, and $n \in N$, there is a unique linear map $\widetilde{f} : M \otimes_A N \to P$ such that the following diagram commutes

$$\begin{tikzcd}
M \otimes N \arrow[rr, "f"] \arrow[dr, "q"'] & & P \\
& M \otimes_A N \arrow[ur, dashed, "\widetilde{f}"'] &
\end{tikzcd} \tag{3.1.3}$$

where $q : M \otimes N \to M \otimes_A N$ is the canonical surjection.

Definition 3.1.4 — An $A - B$ *bimodule* ${}_AM_B$ is a B-module M_B equipped with a unital algebra map $A \to \operatorname{End}(M_B)$.

Exercise 3.1.5. Show that an $A - B$ bimodule ${}_AM_B$ is the same data as a left A-module structure ${}_AM$ and a right B-module structure M_B on M such that the following compatibility condition holds:

$$a(mb) = (am)b \quad \text{for all } a \in A, m \in M, b \in B.$$

Exercise 3.1.6 (Folding trick)**.** Show that an $A - B$ bimodule ${}_AM_B$ is the same data as an $A^{\mathrm{op}} \otimes B$-module $M_{A^{\mathrm{op}} \otimes B}$.

Example 3.1.7 — Suppose ${}_AM_B$ is an $A - B$ bimodule. We define the *dual bimodule* by

$$M^\vee := \operatorname{Hom}(M_B \to B_B).$$

Observe that $M^\vee$ carries the left B-action and a right $\operatorname{End}(M_B)$-action:

$$(bf)(m) := f(mb) \quad \text{and} \quad (fT)(m) := f(Tm) \qquad f \in M^\vee,\, b \in B,\, T \in \operatorname{End}(M_B),\, m \in M.$$

In particular, restricting the right $\operatorname{End}(M_B)$-action to A yields an organic $B-A$ bimodule structure on $M^\vee$.

Exercise 3.1.8. Suppose M_A is a right module and ${}_AN_B$ is an $A-B$ bimodule. Show that $(m \otimes n)b := m \otimes nb$ descends to the relative tensor product $M \otimes_A N$, which is thus a right B-module. Adapt this exercise to the case that M_A carries a left action as well.

Exercise 3.1.9. Suppose A, B are finite dimensional semisimple algebras and ${}_AM_B$ is a finite dimensional bimodule. Prove that the map

$${}_AM \otimes_B M^\vee_A \to {}_A\operatorname{End}(M_B)_A \qquad \text{given by} \qquad m \otimes f \mapsto [n \mapsto mf(n)]$$

is an $A-A$ bimodular isomorphism. Here, the $A-A$ bimodule structure on $\operatorname{End}(M_B)$ is induced from the inclusion $A \subseteq \operatorname{End}(M_B)$.
Hint: use an algebraic projective basis for M_B as in Corollary 2.2.22.

Exercise 3.1.10. Suppose M_A is a right module, ${}_AN_B$ is a bimodule, and ${}_BP$ is a left module. Construct an isomorphism $(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P)$. Do this in two ways: by hand and using Exercise 3.1.2.

Exercise 3.1.11. Prove that $M \otimes_A A_A \cong M_A$. Do this in two ways: by hand and using Exercise 3.1.2. Adapt this exercise for left A-modules.

Definition 3.1.12 — Given bimodule maps $x : {}_AK_B \to {}_AM_B$ and $y : {}_BL_C \to {}_BN_C$, the map

$$\begin{aligned} x \otimes_B y \colon K \otimes_B L &\to M \otimes_B N \\ k \otimes_B \ell &\mapsto xk \otimes_B y\ell \end{aligned}$$

is a well-defined $A-C$ bimodule map by the universal property (3.1.3).

Definition 3.1.13 — Suppose A, B are complex algebras. A *Morita equivalence* between A and B is a pair of bimodules ${}_AP_B$ and ${}_BQ_A$ and together with bimodule isomorphisms ${}_AP \otimes_B Q_A \cong {}_AA_A$ and ${}_BQ \otimes_A P_B \cong {}_BB_B$.

$$\begin{tikzcd} P\otimes_B Q & A \arrow[loop left, "\cong"'] \arrow[r, bend left, "P"] & B \arrow[l, bend left, "Q"] \arrow[loop right, "\cong"] & Q\otimes_A P \end{tikzcd}$$

We will see later in Part[[II]] §[[?]] that a Morita equivalence between A and B is the same data as an equivalence of categories between their categories of right modules. Indeed, give a right A-module M_A, we get a right B-module by $M \otimes_A P_B$, and we can recover M_A as

$$(M \otimes_A P) \otimes_B Q_A \underset{\text{(Exer. 3.1.10)}}{\cong} M \otimes_A (P \otimes_B Q)_A \cong M \otimes_A A_A \underset{\text{(Exer. 3.1.11)}}{\cong} M_A.$$

Exercise 3.1.14. Verify Morita equivalence is an equivalence relation on algebras. Then show that if ${}_AP_B, {}_BQ_A$ is a Morita equivalence, then ${}_BQ_A$ is determined by ${}_AP_B$ up to canonical $A-B$ bimodule isomorphism. That is, if ${}_AP_B, {}_BQ'_A$ is a Morita equivalence, there is a canonical $A-B$ bimodule isomorphism ${}_BQ_A \cong {}_BQ'_A$ built from the Morita equivalence isomorphisms.

Example 3.1.15 — Suppose A is semisimple and M_A is a faithful right module. Then $\mathrm{End}(M_A)$ and A are Morita equivalent via ${}_{\mathrm{End}(M_A)}M_A$ and ${}_AM^\vee_{\mathrm{End}(M_A)}$. Indeed, observe that an $m \in M$ can be identified with the map $L_m : a \mapsto ma$ in $\mathrm{Hom}(A_A \to M_A)$. One checks that:

- the map given by $m \otimes_A f \mapsto L_m f \in \mathrm{End}(M_A)$ is an $\mathrm{End}(M_A)$-bilinear isomorphism $M \otimes_A M^\vee \to \mathrm{End}(M_A)$.
- the map $f \otimes_{\mathrm{End}(M_A)} m \mapsto f(m)$ is an A-bilinear isomorphism $M^\vee \otimes_{\mathrm{End}(M_A)} M \to A$.

Exercise 3.1.16. Adapt the previous example to the setting where M_A is not faithful.

Given the above example, the next proposition extends Corollary 2.2.23.

Proposition 3.1.17 — Suppose ${}_AP_B, {}_BQ_A$ is a Morita equivalence. Then

$$\begin{aligned} A &\cong \mathrm{End}(P_B) & \qquad B &\cong \mathrm{End}(Q_A) \\ B^{\mathrm{op}} &\cong \mathrm{End}({}_AP) & \qquad A^{\mathrm{op}} &\cong \mathrm{End}({}_BQ) \end{aligned}$$

Proof. We prove $A = \mathrm{End}(P_B)$, and the other statements follow formally by swapping $A \leftrightarrow B$ and $P \leftrightarrow Q$ or by taking opposites.

We claim that the left action algebra map $\lambda\colon A \to \mathrm{End}(P_B)$ given by $\lambda_a(m) := am$ is an isomorphism. Since A acts faithfully on A_A and ${}_AP \otimes_B Q_A \cong {}_AA_A$, clearly λ is injective. Indeed, we have an explicit left inverse: for $x \in \mathrm{End}(P_B)$, $x \otimes_B \mathrm{id}_Q \in \mathrm{End}(P \otimes_B Q_A) \cong \mathrm{End}(P_A) = A$, so $x \mapsto x \otimes_B \mathrm{id}_Q$ gives an algebra map $\pi\colon \mathrm{End}(P_B) \to A$ such that $\pi \circ \lambda = \mathrm{id}_A$. Thus π is surjective, and to show that λ is an isomorphism, it suffices to prove that π is injective, i.e., π is an isomorphism. If $x \in \mathrm{End}(P_B)$ such that $x \otimes_B \mathrm{id}_Q = 0$, then observe that $x \otimes_B 1_Q \otimes_A 1_P = 0$ in $\mathrm{End}(P \otimes_B N \otimes_A Q_B) \cong \mathrm{End}(P_B)$. But under this isomorphism, $x \otimes_B 1_Q \otimes_A 1_P = x$, so $x = 0$. $\square$

Exercise 3.1.18. Suppose A is semisimple. Let M_A be a faithful right module, and identify A^{op} with its image in $\mathrm{End}(M)$ under ρ. Prove that $Z(A^{\mathrm{op}}) = A^{\mathrm{op}} \cap \mathrm{End}(M_A) = Z(\mathrm{End}(M_A))$. Then prove that two finite dimensional semisimple algebras are Morita equivalent if and only if their centers are isomorphic.

3.2 Hilbert C*-modules

In this section, we view unitary algebras as finite dimensional C*-algebras, and we explore the notion of a (right) Hilbert C*-module. The benefits of working with Hilbert C*-modules include the existence of canonical creation and annihilation operators (or bra-kets) and a well-defined notion of projective basis. However, even though the dual space of a right Hilbert C*-module is organically a *left* Hilbert C*-module, the notion of Hilbert C*-*correspondence* is manifestly asymmetric, which leads to a problem defining a conjugate/dual correspondence.

Definition 3.2.1 — Suppose X_A is an (algebraic) right A-module for a unitary algebra A. An *A-valued inner product* is a map $\langle\cdot|\cdot\rangle_A : X \times X \to A$ satisfying:

- (A-linear in second variable) $\langle\eta|\xi_1 a+\xi_2\rangle_A = \langle\eta|\xi_1\rangle_A a + \langle\eta|\xi_2\rangle_A$ for all $\eta, \xi_1, \xi_2 \in X$ and $a \in A$,
- (anti-symmetric) $\langle\eta|\xi\rangle_A^* = \langle\xi|\eta\rangle_A$ for all $\eta, \xi \in X$,
- (positive definite) $\langle\eta|\eta\rangle_A \geq 0$ in A for all $\eta \in X$ with equality if and only if $\eta = 0$.

Example 3.2.2 — When we consider $\mathbb{C}$ as a unitary algebra, a right $\mathbb{C}$-valued inner product is the same thing as an ordinary inner product.

Example 3.2.3 — Every unitary algebra A has the trivial A-valued inner product $\langle a|b\rangle_A := a^*b$.

Exercise 3.2.4. Prove that the map from $A_+^\times$ to A-valued inner products on A given by $\langle a|b\rangle_A^x := a^*xb$ for $x \in A_+^\times$ is a bijective correspondence.

Example 3.2.5 — Given an orthogonal projection $p \in M_n(A)$, the right A-module pA^n has A-valued inner product

$$\langle p(a_i)|p(b_i)\rangle_A := \sum_{i,j} a_i^* p_{ij} b_j.$$

We will see every Hilbert C*-module is of this form in Theorem 3.2.27 below.

Example 3.2.6 (Center valued trace) — Given a unitary algebra $A = \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$, consider the *center valued trace* given by

$$\mathrm{zTr}(a) := (\mathrm{Tr}_{n_i}(a_i))_{i=1}^k \in Z(A) \cong \mathbb{C}^k \qquad \text{where} \qquad a = (a_1, \ldots, a_k).$$

Setting $\langle a|b\rangle_{Z(A)} := \mathrm{zTr}(a^*b)$ is a $Z(A)$-valued inner product. The *normalized* center valued trace is given by $\mathrm{ztr}(a) := (\mathrm{tr}_{n_i}(a_i))_{i=1}^k$, which satisfies $\mathrm{ztr}(1) = 1$.

Example 3.2.7 — The standard right $M_n(\mathbb{C})$ module $\overline{\mathbb{C}^n}$ has $M_n(\mathbb{C})$-valued inner product given by

$$\langle \overline{\eta}|\overline{\xi}\rangle_{M_n(\mathbb{C})} := |\eta\rangle\langle\xi|.$$

Example 3.2.8 (Conjugate module) — If X_A is a right Hilbert C* module, then ${}_A\overline{X}$ is a left Hilbert C* module with left A-valued inner product

$${}_A\langle \overline{\eta}, \overline{\xi}\rangle := \langle \xi|\eta\rangle_A,$$

which is A-linear on the *left*.

Exercise 3.2.9. Suppose X_A and Y_A are right Hilbert C*-modules. Show $X_A \oplus Y_A$ is again a right Hilbert C*-module with A-valued inner product

$$\langle (\eta_1, \xi_1)|(\eta_2, \xi_2)\rangle_A := \langle \eta_1|\eta_2\rangle_A + \langle \xi_1|\xi_2\rangle_A.$$

Fact 3.2.10. Suppose X_A is equipped with an A-valued inner product. Observe

$$I := \mathrm{im}(\langle\cdot|\cdot\rangle_A) \subset A$$

is a 2-sided ideal. Since A is semisimple, $I = Az$ for some central projection $z \in Z(A)$ by Exercise 2.2.3 If $\xi \in X(1-z)$, then $\langle \xi|\xi\rangle_A = \langle \xi|\xi(1-z)\rangle_A = \langle \xi|\xi\rangle_A(1-z) \in I(1-z) = 0$, so $\xi = 0$. We see that z acts as 1 on X and $1 - z$ acts as zero. In particular, if X_A is faithful, then $\langle\cdot|\cdot\rangle_A$ is surjective.

Exercise 3.2.11. State and prove the polarization identity (1.3.4) for an A-valued inner product.

Exercise 3.2.12. Suppose X_A is an A-module equipped with an A-valued inner product $\langle\cdot|\cdot\rangle_A$. Show that

$$\langle\cdot|\cdot\rangle_\varphi^X := \varphi(\langle\cdot|\cdot\rangle_A) : X \times X \to \mathbb{C}$$

is an inner product for any faithful weight $\varphi : A \to \mathbb{C}$.

Definition 3.2.13 — Suppose X_A, Y_A are A-modules equipped with right A-valued inner products. A right A-linear map $x : X \to Y$ is called *adjointable* if there is a right A-linear map $x^\dagger : Y \to X$ called an *adjoint* such that

$$\langle x^\dagger\eta|\xi\rangle_A^Y = \langle \eta|x\xi\rangle_A^X \qquad \forall\, \xi \in X \text{ and } \eta \in Y.$$

Example 3.2.14 — The left A-action on A_A equipped with the A-valued inner product $\langle a|b\rangle_A^x = a^*xb$ for $x \in A_+^\times$ from Exercise 3.2.4 is a $*$-action by adjointable operators if and only if $x \in Z(A)$ as

$$\langle 1|a \cdot 1\rangle_A^x = xa \qquad \text{and} \qquad \langle a^* \cdot 1|1\rangle_A^x = ax.$$

Example 3.2.15 — Suppose X_A is an A-module equipped with a right A-valued inner product. Endow A_A with the A-valued inner product $\langle a|b\rangle_A := a^*b$. For each $\eta \in X$, the map $L_\eta : A_A \to X_A$ given by $a \mapsto \eta a$ is adjointable with adjoint $L_\eta^\dagger(\xi) := \langle \eta|\xi\rangle_A$.

We may employ bra-ket notation for $|\xi\rangle_A := L_\xi$ and ${}_A\langle\eta| := L_\eta^\dagger$, so that

$$\langle \eta|\xi\rangle_A = {}_A\langle\eta| \circ |\xi\rangle_A \in \operatorname{End}(A_A) \cong A.$$

The following trick lets us identify $X_A = \operatorname{Hom}(A_A \to X_A)$ as Hilbert C* modules.

Trick 3.2.16 (Realization) — The adjointable operators $\operatorname{Hom}(A_A \to X_A)$ form a right Hilbert C* module with right A-valued inner product

$$\langle f|g\rangle_A := f^\dagger \circ g \in \operatorname{End}(A_A) = A.$$

The map $X_A \to \operatorname{Hom}(A_A \to X_A)$ given by $\xi \mapsto |\xi\rangle_A$ is a unitary isomorphism of Hilbert C* modules.

The following essential exercise for finite dimensional A-modules equipped with A-valued inner products shows that they are not so different from Hilbert space modules.

Exercise 3.2.17. Suppose X_A, Y_A are two A-modules equipped with A-valued inner products. Show that every A-module map $x : X_A \to Y_A$ is adjointable, with adjoint given by the Hilbert space adjoint when X_A, Y_A are equipped with the inner products $\langle\cdot|\cdot\rangle_\varphi^X, \langle\cdot|\cdot\rangle_\varphi^Y$ respectively where $\varphi : A \to \mathbb{C}$ is any faithful weight as in Exercise 3.2.12.

Warning 3.2.18 — For infinite dimensional C*-algebras, there are bounded maps between Hilbert C* modules which are not adjointable. For example, the inclusion of a closed right ideal $I_A \hookrightarrow A_A$ which is not complemented does not admit an adjoint when I_A is endowed with the restricted A-valued inner product. For a concrete example, we can consider $A = C[0,1]$ and $I = \{f \in C[0,1] \mid f(0) = 0\}$.

Proposition 3.2.19 — Given a right A-module X_A equipped with an A-valued inner product, $\operatorname{End}(X_A)$ is a unitary algebra. In particular, adjoints are unique.

Proof. By Exercise 3.2.17, $\operatorname{End}(X_A)$ is isomorphic as a $*$-algebra to $B(X, \langle\cdot|\cdot\rangle_\varphi^X)$. Uniqueness of adjoints follows immediately. □

Exercise 3.2.20. Prove that $x \in \mathrm{End}(X_A)$ is positive if and only if $\langle \xi | x\xi \rangle_A$ is positive in A for all $\xi \in X$.

We give two proofs of the following important corollary. The first proof is a 'one-liner,' and the second introduces Roberts' 2×2 trick [GLR85, Lem. 2.6].

Corollary 3.2.21 — Suppose X_A, Y_A are two Hilbert C*-modules. For every $x \in \mathrm{Hom}(X_A \to Y_A)$, $x^\dagger x$ is positive in $\mathrm{End}(X_A)$.

Proof 1. For every $\xi \in X$, $\langle \xi | x^\dagger x \xi \rangle_A^X = \langle x\xi | x\xi \rangle_A^Y \geq 0$. Now apply Exercise 3.2.20. $\square$

Proof 2: Roberts' 2×2 trick. Using Exercise 3.2.9, we may view

$$x,\ x^\dagger,\ x^\dagger x \in \mathrm{End}(X_A \oplus Y_A) = \begin{pmatrix} \mathrm{Hom}(X_A \to X_A) & \mathrm{Hom}(Y_A \to X_A) \\ \mathrm{Hom}(X_A \to Y_A) & \mathrm{Hom}(Y_A \to Y_A) \end{pmatrix},$$

which is a unitary algebra. We thus have that $x^\dagger x$ is a positive operator, so it has a unique positive square root, which clearly lies in $\mathrm{End}(X_A)$. $\square$

Exercise 3.2.22. For $\eta_1, \ldots, \eta_n \in X_A$, show that the $n \times n$ matrix $(\langle \eta_i | \eta_j \rangle_A)$ is positive in $M_n(A)$.

Theorem 3.2.23 (Riesz Representation) — The adjointable operators $\mathrm{Hom}(X_A \to A_A)$ is a left Hilbert C* module with left A-valued inner product

$${}_A\langle f, g \rangle := f \circ g^\dagger \in \mathrm{End}(A_A) = A,$$

which is linear on the left. The map ${}_A\overline{X} \to \mathrm{Hom}(X_A \to A_A)$ given by $\overline{\eta} \mapsto {}_A\langle \eta |$ is a unitary isomorphism of left Hilbert C*-modules.

Proof. The first claim is left to the reader. The map $X_A \to \mathrm{Hom}(A_A \to X_A)$ given by $\eta \mapsto |\eta\rangle_A$ is a unitary isomorphism by Trick 3.2.16, and the map $\dagger : \mathrm{Hom}(A_A \to X_A) \to \mathrm{Hom}(X_A \to A_A)$ is an anti-linear unitary by Exercise 3.2.8. The result follows. $\square$

Next we extend Corollary 2.2.22, the existence of projective bases, to the unitary setting. We begin with Connes' trick.

Trick 3.2.24 (Connes [Con80, Proof of Prop. 3(b)]) — Suppose X_A is an A-module equipped with an A-valued inner product and $x \geq 0$ in $\mathrm{End}(X_A)$. We claim there is a finite subset $\{\xi_i\} \subset X$ such that $x = \sum |\xi_i\rangle_A \langle \xi_i|$.

By Corollary 2.2.22, we can write $x = \sum_i |\eta_i\rangle_A f_i$ for some $\eta_1, \ldots, \eta_n \in X$ and some right A-linear maps $f_1, \ldots, f_n : X_A \to A_A$. By the A-valued Riesz Representation Theorem 3.2.23, each $f_i = \langle \zeta_i|$ for some η_i, so $x = \sum |\eta_i\rangle_A \langle \zeta_i|$. Since x is self-adjoint,

$$x = \frac{1}{2}\left(\sum |\eta_i\rangle_A \langle \zeta_i| + \sum |\zeta_i\rangle_A \langle \eta_i|\right).$$

In particular, we see that

$$x \leq y := \sum |\eta_i + \zeta_i\rangle_A \langle \eta_i + \zeta_i|.$$

Assume for the time being that x is invertible. Then y is as well, as

$$0 = \langle \xi | y\xi \rangle_\varphi^X \geq \langle \xi | x\xi \rangle_\varphi^X = \langle x^{1/2}\xi | x^{1/2}\xi \rangle_\varphi^X \geq 0 \qquad \Longrightarrow \qquad x^{1/2}\xi = 0 \qquad \Longrightarrow \qquad \xi = 0.$$

We may thus conjugate y by $x^{1/2}y^{-1/2}$ to get x:

$$\begin{aligned} x &= x^{1/2}y^{-1/2}yy^{-1/2}x^{1/2} \\ &= x^{1/2}y^{-1/2}\left(\sum |\xi_i + \eta_i\rangle_A \langle \xi_i + \eta_i|\right) y^{-1/2}x^{1/2} \\ &= \sum |x^{1/2}y^{-1/2}(\xi_i + \eta_i)\rangle_A \langle x^{1/2}y^{-1/2}(\xi_i + \eta_i)|. \end{aligned}$$

When x is not invertible, we may first conjugate by $\operatorname{supp}(x)$ and restrict our attention to $\operatorname{supp}(x)X_A$ to reduce to the case that x is invertible.

Exercise 3.2.25. Suppose A is a unitary algebra and $0 \leq x \leq y$ in A. Use the reasoning in Connes' Trick 3.2.24 to find a $z \in A$ such that $x = z^*yz$.

Corollary 3.2.26 — Suppose X_A is an A-module equipped with a right A-valued inner product. There is a finite subset $\{\beta_i\} \subset X$ called an X_A-*basis* (or sometimes called a *projective basis*) such that $\xi = \sum_i \beta_i \langle \beta_i | \xi \rangle_A$ for all $\xi \in X$, i.e., $\sum |\beta_i\rangle_A \langle \beta_i| = 1_X$. In particular,

$$\begin{aligned} \operatorname{End}(X_A) &= \operatorname{span}\{|\eta\rangle_A\langle\xi| \,|\, \eta, \xi \in X\} \\ \operatorname{End}(X_A)_+ &= \operatorname{span}_{\mathbb{R}_+}\{|\eta\rangle_A\langle\eta| \,|\, \eta \in X\}. \end{aligned}$$

Proof. The final claim about $\operatorname{End}(X_A)_+$ is immediate from Connes' Trick 3.2.24. Since $1_X \in \operatorname{End}(X_A)_+$, X_A bases exist. Finally, since $\operatorname{span}\{|\eta\rangle_A\langle\xi| \,|\, \eta, \xi \in X\}$ is a 2-sided ideal of $\operatorname{End}(X_A)$ containing 1_X, $\operatorname{End}(X_A)$ is spanned by the A-rank one operators. □

Using Corollary 3.2.26, we get a complete classification of Hilbert C* A-modules similar to (SS6).

Theorem 3.2.27 (Classification of Hilbert C* modules) — For a Hilbert C*-module X_A, there is a projection $p \in M_n(A)$ such that $X_A \cong pA_A^n$ as right A-modules, where pA_A^n

has the A-valued inner product

$$\langle p(a_i)|p(b_i)\rangle_A = \sum_{i,j} a_i^* p_{ij} b_j.$$

Under conjugation by this unitary isomorphism, $\operatorname{End}(X_A) \cong pM_n(A)p$.

Proof. Let $\{\beta_i\}$ be an X_A-basis. The $1\times n$ matrix of A-linear maps $v := (|\beta_i\rangle_A)_{i=1}^n : A_A^n \to X_A$ given by $(a_i) \mapsto \sum \beta_i a_i$ is clearly right A-linear and surjective. Its adjoint $v^\dagger : X_A \to A_A^n$ is given by the $n \times 1$ matrix $({}_A\langle\beta_i|)_{i=1}^n$, which is injective. Observe that v is a coisometry, as

$$vv^\dagger = (|\beta_i\rangle_A)_{i=1}^n({}_A\langle\beta_i|)_{i=1}^n = \sum_i |\beta_i\rangle_A\langle\beta_i| = 1_X.$$

We conclude that $p := v^\dagger v \in \operatorname{End}(A_A^n) \cong M_n(A)$ is an orthogonal projection and that $v|_{pA^n} : pA_A^n \to X_A$ is a unitary A-module isomorphism. The final claim is immediate. $\square$

Definition 3.2.28 — Suppose A, B are unitary algebras. An $A-B$ *correspondence* is an $A-B$ bimodule ${}_AX_B$ equipped with a right B-valued inner product $\langle\cdot|\cdot\rangle_B$ such that the map $A \to \operatorname{End}(X_B)$ is a unital $*$-algebra map, so that

$$\langle a^*\eta|\xi\rangle_B = \langle\eta|a\xi\rangle_B \qquad \forall\, a \in A, \quad \eta,\xi \in X.$$

An isomorphism of $A-B$ correspondences is an isomorphism of $A-B$ bimodules which preserves the right B-valued inner products.

Example 3.2.29 — An $A-\mathbb{C}$ correspondence ${}_AH_{\mathbb{C}}$ is exactly a Hilbert space H together with a unital $*$-algebra map $A \to B(H)$.

Definition 3.2.30 (Relative tensor product) **—** Suppose A, B, C are unitary algebras and ${}_AX_B$ and ${}_BY_C$ are correspondences. The *relative tensor product* ${}_AX \boxtimes_B Y_C$ is the quotient space $X \otimes Y/N$ where

$$N = \operatorname{span}\{\eta b \otimes \xi - \eta \otimes b\xi \,|\, \eta \in X, \quad \xi \in Y, \text{ and } b \in B.\}$$

We denote the image of $\eta \otimes \xi$ in ${}_AX \boxtimes_B Y_C$ by $\eta \boxtimes \xi$. We equip this space with the C-valued inner product given by

$$\langle\eta_1 \boxtimes \xi_1|\eta_2 \boxtimes \xi_2\rangle_C := \langle\langle\eta_2|\eta_1\rangle_B\xi_1|\xi_2\rangle_C = \langle\xi_1|\langle\eta_1|\eta_2\rangle_B\xi_2\rangle_C, \tag{3.2.31}$$

which is well-defined by Lemma 3.2.32 below. Observe that $X \boxtimes_B Y$ carries a left A-action and a right C-action and is thus an $A-C$ correspondence.

As we are working in finite dimensions, the Connes fusion relative tensor product satisfies the same universal property (3.1.3) as in the algebraic setting. Thus given bimodule maps $x : {}_AX_B \to {}_AM_B$ and $y : {}_BY_C \to {}_BN_C$, the map

$$\begin{aligned} x \boxtimes_B y \colon X \boxtimes_B Y &\to M \boxtimes_B N \\ \eta \boxtimes \xi &\mapsto x\eta \boxtimes y\xi \end{aligned}$$

is a well-defined $A - C$ bimodule map by the universal property (3.1.3).

Lemma 3.2.32 — The formula (3.2.31) indeed defines a C-valued inner product.

Proof. The only interesting part is proving definiteness. Suppose that

$$\zeta = \sum \eta_i \boxtimes \xi_i \qquad \text{and} \qquad \langle \zeta|\zeta\rangle_C = \sum_{i,j} \langle \eta_i \boxtimes \xi_i|\eta_j \boxtimes \xi_j\rangle_C = \sum_{i,j} \langle \langle \eta_j|\eta_i\rangle_B \xi_i|\xi_j\rangle_C = 0.$$

Let $\{\beta_k\}_{k=1}^n$ be an X_B-basis as in Corollary 3.2.26 so that $\sum_{k=1}^n |\beta_k\rangle_B\langle\beta_k| = 1_X$. Observe that

$$\langle \eta_j|\eta_i\rangle_B = \sum_k \langle \beta_k \langle \beta_k|\eta_j\rangle_B|\eta_i\rangle_B = \sum_k \langle \eta_j|\beta_k\rangle_B\langle\beta_k|\eta_i\rangle_B,$$

so we see that

$$0 = \langle\zeta|\zeta\rangle_C = \sum_{i,j} \langle\langle\eta_j|\eta_i\rangle_B\xi_i|\xi_j\rangle_C = \sum_{i,j,k} \langle\langle\eta_j|\beta_k\rangle_B\langle\beta_k|\eta_i\rangle_B\xi_i|\xi_j\rangle_C = \sum_{i,j,k} \langle\langle\beta_k|\eta_i\rangle_B\xi_i|\langle\beta_k|\eta_j\rangle_B\xi_j\rangle_C.$$

By looking at the C-module Y_C^n with C-valued inner product $\langle(\sigma_k)|(\tau_\ell)\rangle_C^{Y^n} := \sum\langle\sigma_k|\tau_k\rangle_C$ and canonical left $M_n(B)$-action $(b_{ki})(\tau_i) = \sum_k (b_{ki}\tau_i)$, observe

$$0 = \sum_{i,j,k} \langle\langle\beta_k|\eta_i\rangle_B\xi_i|\langle\beta_k|\eta_j\rangle_B\xi_j\rangle_C = \langle((\langle\beta_k|\eta_i\rangle_B)(\xi_i)|(\langle\beta_k|\eta_i\rangle_B)(\xi_i)\rangle_C^{Y^n},$$

which implies $(\langle\beta_k|\eta_i\rangle_B)(\xi_i) = 0$ in Y_C^n by definiteness. Now the map $(\sigma_k) \mapsto \sum \beta_k \boxtimes \sigma_k$ is a well-defined C-linear map $Y_C^n \to X \boxtimes_B Y_C$. Applying this map to $(\langle\beta_k|\eta_i\rangle_B)(\xi_i) = 0$ yields

$$0 = \sum_{k,i} \beta_k \boxtimes \langle\beta_k|\eta_i\rangle_B\xi_i = \sum_{k,i} \beta_k\langle\beta_k|\eta_i\rangle_B \boxtimes \xi_i = \sum_i \eta_i \boxtimes \xi_i = \zeta$$

as desired. □

Exercise 3.2.33. Directly verify using (3.2.31) that $(x \boxtimes y)^\dagger = x^\dagger \boxtimes y^\dagger$ for bimodule maps $x : {}_AX_B \to {}_AM_B$ and $y : {}_BY_C \to {}_BN_C$.

Exercise 3.2.34. Show that the canonical isomorphisms

$$(X \boxtimes_A Y) \boxtimes_B L \cong X \boxtimes_A (Y \boxtimes_B L), \qquad X \boxtimes_A A \cong X, \qquad \text{and} \qquad A \boxtimes_A Y \cong Y$$

from Exercises 3.1.10 and 3.1.11 for the relative tensor product are unitary isomorphisms for the relative tensor product.

Exercise 3.2.35. Suppose $\varphi : A \to \mathbb{C}$ is a faithful weight and X_A is a right Hilbert C*-module. Prove that the map $X \to X \boxtimes_A L^2(A, \varphi)$ given by $\eta \mapsto \eta \boxtimes \Omega_\varphi$ is a unitary isomorphism of Hilbert spaces when the former is equipped with the inner product $\langle \cdot | \cdot \rangle_\varphi^X$. Prove this unitary intertwines the left $\mathrm{End}(X_A)$-actions. When does it intertwine the right A-actions? (See Warning 3.4.7 below.)

Definition 3.2.36 — A C* *Morita equivalence* (a.k.a. a *Rieffel-Morita equivalence* [Rie74]) between two unitary algebras A, B consists of $A - B$ correspondences ${}_AX_B$ and ${}_BY_A$ and correspondence unitary isomorphisms ${}_AX \boxtimes_B Y_A \cong {}_AA_A$ and ${}_BY \boxtimes_A X_B \cong {}_BB_B$.

$$\begin{tikzcd} A \arrow[loop left, "\cong" description, "X\boxtimes_B Y"'] \arrow[r, bend left, "X"] & B \arrow[l, bend left, "Y"] \arrow[loop right, "\cong" description, "Y\boxtimes_A X"] \end{tikzcd}$$

Remark 3.2.37. Since A, B were assumed to be finite dimensional, a C* Morita equivalence is an example of an ordinary Morita equivalence in the sense of Definition 3.1.13. Hence by Proposition 3.1.17,

$$\begin{aligned} A &\cong \mathrm{End}(X_B) & \qquad B &\cong \mathrm{End}(Y_A) \\ B^{\mathrm{op}} &\cong \mathrm{End}({}_AX) & \qquad A^{\mathrm{op}} &\cong \mathrm{End}({}_BY). \end{aligned}$$

Example 3.2.38 — Suppose X_A is a faithful right A-module with an A-valued inner product. We may promote X_A to an $\mathrm{End}(X_A) - A$ correspondence ${}_{\mathrm{End}(X_A)}X_A$. We define a *left* $\mathrm{End}(X_A)$-valued inner product (which is linear on the left) by

$${}_{\mathrm{End}(X_A)}\langle \xi, \eta \rangle := |\xi\rangle_A\langle \eta|.$$

Observe that for all $\eta, \xi, \zeta \in X$, we have

$$\eta\langle \xi|\zeta\rangle_A = |\eta\rangle_A\langle \xi|\zeta = {}_{\mathrm{End}(X_A)}\langle \eta, \xi\rangle\zeta.$$

This is the notion of an *imprimitivity bimodule*, which is the same as being part of a C* Morita equivalence by the next result.

Theorem 3.2.39 — For an $A - B$ correspondence ${}_AX_B$ which is faithful as both a left A and right B-module, the following are equivalent.

(1) ${}_AX_B$ can be equipped with a *left* A-valued inner product (which is linear on the left), under which ${}_AX_B$ it is an *imprimitivity bimodule*, i.e.,

$${}_A\langle \eta, \xi\rangle\zeta = \eta\langle \xi|\zeta\rangle_B \qquad\qquad \forall\, \eta, \xi, \zeta \in X.$$

(2) ${}_AX_B$ can be extended to a C* Morita equivalence, i.e., there is a $B-A$ correspondence ${}_BY_A$ and unitary bimodule isomorphisms ${}_AX \boxtimes_B Y_A \cong {}_AA_A$ and ${}_BY \boxtimes_A X_B \cong {}_BB_B$.

Proof.

(1)⇒(2): The other half of the unitary Morita equivalence is given by ${}_B\overline{X}_A$ with actions given by $b\overline{\eta}a := \overline{a^*\eta b^*}$. The algebra-valued inner products on $\overline{X}$ are given by ${}_B\langle\overline{\eta}, \overline{\xi}\rangle := \langle\xi|\eta\rangle_B$ and $\langle\overline{\eta}|\overline{\xi}\rangle_A := {}_A\langle\xi, \eta\rangle$. Using faithfulness of both actions, one now verifies that the maps

$$\begin{aligned} &\overline{X} \boxtimes_A X \longrightarrow B && \text{given by} && \overline{\eta} \boxtimes \xi \longmapsto \langle\eta|\xi\rangle_B \\ &X \boxtimes_B \overline{X} \longrightarrow A && \text{given by} && \eta \boxtimes \overline{\eta} \longmapsto {}_A\langle\eta, \xi\rangle \end{aligned}$$

are well-defined bilinear unitaries. Their adjoints/inverses are given by

$$\begin{aligned} &B \longrightarrow \overline{X} \boxtimes_A X && \text{given by} && 1_B \longmapsto \sum \overline{\beta_i} \boxtimes \beta_i \\ &A \longrightarrow X \boxtimes_B \overline{X} && \text{given by} && 1_A \longmapsto \sum \gamma_j \boxtimes \overline{\gamma_j} \end{aligned}$$

where $\{\beta_i\}$ is an X_B-basis and $\{\overline{\gamma_j}\}$ is an $\overline{X}_A$-basis as in Corollary 3.2.26.

(2)⇒(1): The $*$-algebra map $\pi : A \to \operatorname{End}(X_B)$ is an isomorphism by Proposition 3.1.17, so the result follows by Example 3.2.38. □

Exercise 3.2.40. Verify the claim that the maps in (1) ⇒ (2) in the above proof are unitary as claimed.

Exercise 3.2.41. Directly verify that for an X_B-basis $\{\beta_i\}$, $\sum \overline{\beta_i} \boxtimes \beta_i \in \overline{X} \boxtimes_A X$ is a B-central vector.
Hint: First show that this element is independent of the choice of $\{\beta_i\}$. *Then show that* $\sum \overline{\beta_i} \boxtimes \beta_i u = \sum u\overline{\beta_i} \boxtimes \beta_i$ *for every unitary* $u \in B$.

Now that we have a working notion of C* Morita equivalence between unitary algebras, we can ask how one might extend the conjugate operation to all $A-B$ correspondences ${}_AX_B$, and not just imprimitivity bimodules. Unfortunately, there is no way to canonically endow $\overline{X}$ with a right A-valued inner product, and we must again make a choice. This brings us to our next attempt using Hilbert space bimodules.

3.3 Hilbert space modules for von Neumann algebras

As unitary algebras are also finite dimensional von Neumann algebras, they naturally act on Hilbert spaces. Thus one might reasonably say that a unitary right A-module should be a Hilbert space H with a right A-action, i.e., a unital $*$-homomorphism $A^{\mathrm{op}} \to B(H)$, where A^{op} is the opposite algebra of A as in Examplre 2.1.5. It follows immediately that the right A-module maps are closed under taking adjoints.

Lemma 3.3.1 — Suppose H_A and K_A are two Hilbert space right A-modules where A is a unitary algebra. If $x : H \to K$ is a right A-linear map, i.e.,

$$x(\eta \lhd a) = (x\eta) \lhd a \qquad \forall \eta \in H, a \in A,$$

then $x^\dagger$ is also right A-linear.

Proof. For all $\eta \in H$, $\xi \in K$, and $a \in A$,

$$\langle x^\dagger(\xi a)|\eta\rangle_H = \langle \xi a|x\eta\rangle_K = \langle \xi|(x\eta)a^\dagger\rangle_K = \langle \xi|x(\eta a^\dagger)\rangle_K = \langle x^\dagger\xi|\eta a^\dagger\rangle_H = \langle (x^\dagger\xi)a|\eta\rangle_H. \qquad \square$$

As the space of standard forms for a unitary algebra A is contractible, we simply denote the standard form by L^2A. When we worked with Hilbert C*-modules X_A, we started with A-valued inner products, from which we defined the notion of an adjointable A-linear map. (It turns out that all A-linear maps are adjointable in finite dimensions, but this is beside the point.) Now when working with Hilbert space modules H_A, this process is inverted. The standard form L^2A allows us to work with all linear maps $f : L^2A_A \to H_A$ and their Hilbert space adjoints. We then observe that $\mathrm{Hom}(L^2A_A \to H_A)$ is canonically a Hilbert C*-module with A-valued inner product

$$\langle f|g\rangle_A := f^\dagger g \in \mathrm{End}(L^2A_A) = A. \tag{3.3.2}$$

Observe that for all A-linear $x : H_A \to K_A$, the Hilbert space adjoint $x^\dagger$ agrees with the A-valued adjoint of the corresponding operator

$$x \circ - : \mathrm{Hom}(L^A_A \to H_A) \to \mathrm{Hom}(L^2A \to K_A)$$

as for all $f : L^2A_A \to H_A$ and $g : L^2A_A \to K_A$, we have

$$\langle xf|g\rangle_A = f^\dagger x^\dagger g = \langle f^\dagger|x^\dagger g\rangle.$$

Thus there is no ambiguity regarding adjoints.

Warning 3.3.3 — There is no canonical isomorphism between the Hilbert space H_A and the Hilbert C*-module $\mathrm{Hom}(L^2A_A \to H_A)$. In particular, the Hilbert space H_A does *not* have a canonical A-valued inner product (unless $A = \mathbb{C}$). In particular, even though L^2A and A are isomorphic as vector spaces, A_A has a canonical A-valued inner product $\langle a|b\rangle_A = a^*b$, but L^2A_A does not. This is another manifestation of the fact that we cannot identify L^2A with A as discussed in Warning 2.5.22.

The following theorem quantifies the above warning.

Theorem 3.3.4 — For a unitary algebra A, the following are all torsors for $A_+^\times$:

(1) right A-module isomorphisms $A_A \cong L^2A_A$ sending 1_A into the positive cone L^2A_+,

(2) faithful weights on A, and

(3) A-valued inner products on L^2A.

The following structures are all torsors for $Z(A)_+^\times$:

(Z1) $A - A$ bimodule isomorphisms ${}_AA_A \cong {}_AL^2A_A$ sending 1_A into the positive cone L^2A_+,

(Z2) faithful traces on A, and

(Z3) A-valued inner products on L^2A such that the left A-action is a $*$-algebra map into the adjointable operators.

Proof.

(1) Choosing a faithful trace Tr on A to model $L^2A = L^2(A, \mathrm{Tr})$, such a map $A_A \to L^2(A, \mathrm{Tr})_A$ is uniquely determined by where 1_A goes, which must be of the form $x\Omega_{\mathrm{Tr}}$ for some $x \in A_+^\times$ by Exercise 2.5.3.

(2) Immediate by Proposition 2.3.16.

(3) This is Exercise 3.2.4.

(Z1) A maps from (1) which is also left A-linear requires $x \in Z(A)_+^\times$.

(Z2) This is Exercise 2.3.8.

(Z3) This is Example 3.2.14. □

The following trick gets around the above ambiguity and allows us to bootstrap the classification of Hilbert C*-modules to a classification of Hilbert space modules. It also affords an elegant definition of fusion following [Was98].

Trick 3.3.5 — We may identify the underlying vectors space of a Hilbert space module H_A as $\mathrm{Hom}(L^2A_A \to H_A)$, which has a canonical A-valued inner product (3.3.2). We can recover the Hilbert space H under the relative tensor product of this Hilbert C*-module with ${}_AL^2A_{\mathbb{C}}$ via the unitary isomorphism

$$\rho_H : \mathrm{Hom}(L^2A_A \to H_A) \boxtimes_A L^2A_{\mathbb{C}} \longrightarrow H \qquad\qquad f \otimes \xi \longmapsto f(\xi).$$

However, L^2A also has a right A-action, and the map ρ_H is clearly right A-linear, which promotes ρ_H to a right A-linear unitary of Hilbert space A-modules.

Observe that ρ_{L^2A} is the canonical left A-action $a \boxtimes \xi \mapsto a\xi$. Moreover, for every A-module map $T : H_A \to K_A$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}(L^2A_A \to H_A) \boxtimes_A L^2A_A \arrow[r, "\rho_H"] \arrow[d, "(T\circ -)\boxtimes_A \mathrm{id}_{L^2A}"'] & H_A \arrow[d, "T"] \\
\mathrm{Hom}(L^2A_A \to K_A) \boxtimes_A L^2A_A \arrow[r, "\rho_K"] & H_A
\end{tikzcd}$$

This has two immediate consequences.

- Setting $H = L^2A$, ρ_K is uniquely determined by ρ_{L^2A}. (To see this, model $L^2A = L^2(A, \mathrm{Tr})$ for a faithful tracial weight and consider $T = |\eta\rangle_{\mathrm{Tr}} : a\Omega_{\mathrm{Tr}} \mapsto \eta a$.)
- By setting $K = H$, we see that ρ_H commutes with the left action of $A' = \mathrm{End}(H_A)$.

We have a similar trick for left Hilbert space A-modules ${}_AK$.

Theorem 3.3.6 (Classification of Hilbert space modules) — Suppose H_A is a Hilbert space module. There is a projection $p \in M_n(A)$ and an A-linear unitary isomorphism $H_A \cong pL^2A^{\oplus n}$ for some $n \in \mathbb{N}$. Conjugating by this unitary gives a $*$-algebra isomorphism $\mathrm{End}(H_A) \cong pM_n(A)p$.

Proof. By the Classification Theorem for Hilbert C* modules 3.2.27, there is a unitary Hilbert C* module isomorphism $v : pA_A^n \to \mathrm{Hom}(L^2A_A \to H_A)$ for some projection $p \in M_n(A)$. We then get a unitary isomorphism

$$pL^2A_A^{\oplus n} \cong pA_A^n \boxtimes_A L^2A \underset{v\boxtimes \mathrm{id}}{\cong} \mathrm{Hom}(L^2A_A \to H_A) \boxtimes_A L^2A_{\mathbb{C}} \underset{\rho_H}{\cong} H$$

of Hilbert spaces which intertwines the right A-action. The result follows. □

Definition 3.3.7 (Connes fusion) — Given a right B-module H_B and a left B-module ${}_BK$ the *Connes fusion* relative tensor product Hilbert space $H \boxtimes_B K$ can be defined in three equivalent ways using the relative tensor product for C*-Hilbert modules.

- $\mathrm{Hom}(L^2B_B \to H_B) \boxtimes_B K$ with $\mathbb{C}$-valued inner product $\langle f_1 \boxtimes \xi_1 | f_2 \boxtimes \xi_2\rangle := \langle \langle f_2|f_1\rangle_B \xi_1|\xi_2\rangle_K$ where as in (3.3.2),
$$\langle f_2|f_1\rangle_B = f_2^\dagger f_1 \in \mathrm{End}(L^2B_B) = B,$$
- $H \boxtimes_B \mathrm{Hom}({}_BL^2B \to {}_BK)$ with $\mathbb{C}$-valued inner product $\langle \eta_1 \boxtimes g_1|\eta_2 \boxtimes g_2\rangle := \langle \eta_1|\eta_2 {}_B\langle g_1, g_2\rangle\rangle_H$ where
$${}_B\langle g_1, g_2\rangle = Jg_1^\dagger g_2 J \in J\,\mathrm{End}({}_BL^2B)J = JB'J = B,$$
where $J : L^2B \to L^2B$ is the canonical standard form conjugate-linear unitary from §2.5, or
- $\mathrm{Hom}(L^2B_B \to H_B) \boxtimes_B L^2B \boxtimes_B \mathrm{Hom}({}_BL^2B \to {}_BK)$ with $\mathbb{C}$-valued inner product
$$\langle f_1 \boxtimes \zeta_1 \boxtimes g_1 | f_2 \boxtimes \zeta_2 \boxtimes g_2\rangle := \langle \langle f_2|f_1\rangle_B \zeta_1|\zeta_2 {}_B\langle g_1, g_2\rangle\rangle_{L^2B}.$$

Given $x : H_B \to M_B$ and $y : {}_BK \to {}_BN$, we get an operator $x \boxtimes_B y : H \boxtimes_B K \to M \boxtimes_B N$ by $f \boxtimes \xi \mapsto xf \boxtimes y\xi$, where $f \in \mathrm{Hom}(L^2B_B \to H_B)$ and $\xi \in K$. There are analogous definitions for the other equivalent definitions of $H \boxtimes_B K$.

We postpone the discussion of universal property of Connes fusion and the associator and unitor unitary isomorphisms to the next section after we introduce the notion of an H*-*algebra* (a unitary algebra equipped with a faithful tracial weight).

Exercise 3.3.8. In this exercise, we derive the unintended consequences for the most naive definition of the relative tensor product for Hilbert space modules that we can think of.

First, define $H \boxed{?}_B K$[2] to be $L^\perp \subset H \otimes K$ where

$$L := \mathrm{span}\left\{\eta b \otimes \xi - \eta \otimes b\xi \,\middle|\, \eta \in H, \quad \xi \in K, \text{ and } b \in B\right\}.$$

Denote the image of $\eta \otimes \xi$ in $L^\perp$ under $p_{L^\perp} : H \otimes K = L \oplus L^\perp \to L^\perp$ by $\eta \boxed{?} \xi$, for which

$$\eta b \boxed{?} \xi = \eta \boxed{?} b\xi \qquad \forall b \in B.$$

(1) When H is equipped with a commuting left A-action and K is equipped with a commuting right C-action, show these actions preserve $L^\perp$ and thus descend to $H \boxed{?}_B K$.

(2) Show that when $B = M_n(\mathbb{C})$, $H_B = \overline{\mathbb{C}^n}$, and ${}_BK = \mathbb{C}^n$, the subspace $L^\perp \subseteq H \otimes K$ is one dimensional and spanned by the vector $\sum \langle e_i| \otimes |e_i\rangle$.

(3) We would like L^2B to behave as an 'identity bimodule' for the naive $\boxed{?}$ fusion operation. Prove that if φ is a faithful weight on $B = M_n(\mathbb{C})$ so that the map $\overline{\mathbb{C}^n} \boxed{?}_B L^2(B, \varphi) \to \overline{\mathbb{C}^n}$ given by $\xi \boxed{?} b\Omega_\varphi \mapsto \xi b$ is unitary, then $\varphi = n \cdot \mathrm{Tr}$.

(4) Extend the above result to the case B is an arbitrary unitary algebra and H_B is an arbitrary faithful B-module to show that the $H \boxed{?}_B L^2(B, \varphi) \to H$ map given by $\eta \boxed{?} b\Omega_\varphi \mapsto \eta b$ is unitary for exactly one tracial weight on B.

We are now in the position to define the Hilbert space/von Neumann version of Morita equivalence.

Definition 3.3.9 — A W*/*von Neumann Morita equivalence* between unitary algebras A, B consists of Hilbert space bimodules ${}_AH_B$ and ${}_BK_A$ together with bimodule unitary isomorphisms ${}_AH \boxtimes_B K_A \cong {}_AL^2A_A$ and ${}_BK \boxtimes_A H_B \cong {}_BL^2B_B$.

As in Remark 3.2.37 above, ${}_AH_B, {}_BK_A$ being a W* Morita equivalence means that

$$\begin{aligned} A &\cong \mathrm{End}(H_B) & \qquad B &\cong \mathrm{End}(K_A) \\ B^{\mathrm{op}} &\cong \mathrm{End}({}_AH) & \qquad A^{\mathrm{op}} &\cong \mathrm{End}({}_BK). \end{aligned}$$

One advantage of viewing unitary algebras as von Neumann/W*-algebras and working with Hilbert space modules is that we may always define a conjugate bimodule, not just for imprimativity bimodules.

[2]Here, we have chosen the deliberately terrible notation of a question mark box so that no one is tempted to propagate this notion as correct.

Definition 3.3.10 — The *conjugate* Hilbert space bimodule $\overline{{}_AH_B} = {}_B\overline{H}_A$ of ${}_AH_B$ is the conjugate Hilbert space $\overline{H}$ with left and right actions given by

$$b \triangleright \overline{\eta} \triangleleft a := \overline{a^*\eta b^*}.$$

We now show that in a W* Morita equivalence, we may take ${}_BK_A = {}_B\overline{H}_A$ as the other half of the equivalence.

Theorem 3.3.11 (Sauvageot splitting [Sau83, Prop. 3.1]) — Suppose H_B is a faithful Hilbert space B-module. The space $H \boxtimes_B \overline{H}$ with

- the anti-linear unitary J given by $\eta \boxtimes \overline{\xi} \mapsto \xi \boxtimes \overline{\eta}$, and
- the positive cone $P := \operatorname{span}\left\{\xi \boxtimes \overline{\xi} \,\middle|\, \xi \in H\right\}$

is a standard form for $B' = \operatorname{End}(H_B)$.

We defer the proof of Sauvageot Splitting Theorem 3.3.11 to the next section after we introduce the *canonical commutant trace* Tr' for an H*-algebra (B, Tr). Indeed, the construction of the unitary isomorphism of standard forms $H \boxtimes_B \overline{H} \cong L^2(B', \operatorname{Tr}')$ will be short once we model $L^2B = L^2(B, \operatorname{Tr})$.

The following corollary is the Hilbert space bimodule/W* Morita equivalence version of Theorem 3.2.39.

Corollary 3.3.12 — A faithful Hilbert space B-module H_B can be canonically promoted to a W* Morita equivalence with $B' = \operatorname{End}(H_B)$.

3.4 H*-modules for H*-algebras

Working with Hilbert modules H_A for unitary algebras allowed us to define a canonical conjugate module ${}_A\overline{H}$, but we had no way of constructing a C* projective basis, as the dual module ${}_AH^\vee = (H_A)^\vee = \operatorname{Hom}(H_A \to A_A)$ is not canonically a Hilbert space. We avoided this complication in the last section by use of Trick 3.3.5 which identifies

$$\operatorname{Hom}(L^2A_A \to H_A) \boxtimes_A L^2A \cong H_A.$$

Another way to avoid this complication and avoid making arbitrary choices is to change our objects of study to H*-*algebras*, which are unitary algebras equipped with a fixed choice of faithful tracial weight. By considering the collection of H*-*algebras*, we can now 'follow our noses' to define fusion and conjugates for bimodules, and we obtain a canonical unitary isomorphism $(H_A)^\vee \cong \overline{H_A} = {}_A\overline{H}$. Thus our objects of study are not just unitary algebras, but also Hilbert spaces under the GNS construction $L^2(A, \operatorname{Tr})$.

Definition 3.4.1 ([Amb45]) — An H^*-*algebra* is a unitary algebra A equipped with a faithful tracial weight $\mathrm{Tr} : A \to \mathbb{C}$.

Remark 3.4.2. An infinite dimensional analog of an H*-algebra is a *tracial von Neumann algebra*, i.e., a von Neumann algebra equipped with a faithful tracial state.

Exercise 3.4.3. A (finite dimensional) *Hilbert algebra* is a Hilbert space H equipped with a multiplication and an involution $*$ such that

$$\langle ab|c\rangle = \langle b|a^*c\rangle = \langle a|cb^*\rangle \qquad\qquad \forall\, a, b, c \in H.$$

Show that there is a bijective correspondence between Hilbert algebras and H*-algebras. *Hint: Given an* H^**-algebra* (A, Tr)*, consider the Hilbert algebra* $H = L^2(A, \mathrm{Tr})$*. For the converse, recover an* H^**-algebra by* $\mathrm{Tr}_H(a) := \langle 1|a\rangle$*.*

We saw in Theorem 3.3.4 that faithful tracial weights on A are in bijection with A-valued inner products on L^2A, as both are torsors for $Z(A)_+^\times$. In fact, the following explicit construction uses our trace Tr to endow *every* Hilbert space module H_A with a canonical A-valued inner product.

Construction 3.4.4 — Let Tr be a faithful tracial weight on A so that we may model $L^2A = L^2(A, \mathrm{Tr})$. Every map in $\mathrm{Hom}(L^2A_A \to H_A)$ is of the form $|\eta\rangle_A : a\Omega_{\mathrm{Tr}} \mapsto \eta a$ for some $\eta \in H$. Writing ${}_A\langle\eta| = |\eta\rangle_A^\dagger$, observe that

$$\langle\eta|\xi\rangle_A := {}_A\langle\eta| \circ |\xi\rangle_A \tag{3.4.5}$$

is an A-valued product on H_A which is completely determined by the formula

$$\mathrm{Tr}(\langle\eta|\xi\rangle_A) = \langle\eta|\xi\rangle_H \qquad\qquad \forall\, \eta, \xi \in H. \tag{3.4.6}$$

Warning 3.4.7 — For non-tracial faithful weights φ on A, the situation is far more subtle, as $a\Omega_\varphi \mapsto \eta a$ for $\eta \in H$ is no longer right A-linear.

Exercise 3.4.8. Show that if (A, Tr) is an H*-algebra and ${}_AH$ is a left module, every map in $\mathrm{Hom}({}_AL^2A \to {}_AH)$ is of the form $R_\eta : a\Omega_{\mathrm{Tr}} \mapsto a\eta$ for some $\eta \in H$. Then prove that $R_\eta^\dagger R_\xi \in \mathrm{End}({}_AL^2A) = A' = JAJ$, and

$${}_A\langle\eta, \xi\rangle := JR_\eta^\dagger R_\xi J \in JA'J = A \tag{3.4.9}$$

is a well-defined A-valued inner product on H, which is linear on the *left*.

Just as von Neumann algebras come in pairs A, A' when acting on Hilbert spaces, so do H*-algebras.

Definition 3.4.10 — An H*-*module* for an H*-algebra (A, Tr_A) is a right Hilbert space module H_A, whose space of endomorphisms $A' = \mathrm{End}(H_A)$ is equipped with its canonical *commutant trace*. In more detail, since

$$A' = \mathrm{End}(H_A) = \mathrm{span}\{|\xi\rangle_A\langle\eta| \mid \eta, \xi \in H\}$$

by Corollary 3.2.26, the formula

$$\mathrm{Tr}'(|\xi\rangle_A\langle\eta|) = \mathrm{Tr}(\langle\eta|\xi\rangle_A) \underset{(3.4.6)}{=} \langle\eta|\xi\rangle_H \tag{3.4.11}$$

uniquely determines a faithful tracial weight $\mathrm{Tr}' : A' \to \mathbb{C}$.

Exercise 3.4.12. Verify the commutant trace is indeed a trace on $\mathrm{End}(H_A)$. Then prove that

$$|\eta\rangle_A\langle\xi|\zeta = \eta\langle\xi|\zeta\rangle_A \qquad\qquad \forall\, \eta, \xi, \zeta \in H,$$

and that distinct traces on A yield distinct A-valued inner products $\langle\eta|\xi\rangle_A$ and distinct rank one operators $|\xi\rangle_A\langle\eta| \in \mathrm{End}(H_A)$.

Exercise 3.4.13. Show that the commutant trace Tr' on $A' = \mathrm{End}(H_A)$ is given by the formula

$$\mathrm{Tr}'(x) = \sum \langle\beta_i|x\beta_i\rangle$$

where $\{\beta_i\}$ is any H_A-basis as in Corollary 3.2.26.

Exercise 3.4.14. Consider $A = M_n(\mathbb{C})$ and $H_A = \mathbb{C}^k \otimes \overline{\mathbb{C}^n}$. Show that Tr_n' for the standard trace Tr_n on $M_n(\mathbb{C})$ is the standard trace Tr_k on $M_k(\mathbb{C})$. What happens for tr on $M_n(\mathbb{C})$?

Exercise 3.4.15. Suppose Tr_k for $k = 1, 2$ are two faithful tracial weights on A. Let $z = \frac{d\,\mathrm{Tr}_2}{d\,\mathrm{Tr}_1} \in Z(A)_+^\times$ be the density of Tr_2 with respect to Tr_1, so that $\mathrm{Tr}_2(a) = \mathrm{Tr}_1(za)$ for all $a \in A$. Suppose H_A is a Hilbert space module, and denote the A-valued inner product on H induced by Tr_k by $\langle\,\cdot\,|\,\cdot\,\rangle_A^k$. Show that $\langle\eta|\xi\rangle_A^2 = z^{-1}\langle\eta|\xi\rangle_A^1$ for all $\eta, \xi \in H$. Using $Z(A') = Z(A)$, deduce that $|\xi\rangle_A^2\langle\eta| = z^{-1}|\xi\rangle_A^1\langle\eta|$ and $\mathrm{Tr}_2'(x) = \mathrm{Tr}_1'(zx)$ for all $x \in A' = \mathrm{End}(H_A)$.

Example 3.4.16 (Classification of H*-modules) — Suppose H_A is an H*-module H_A for an H*-algebra (A, Tr_A). Since H is just a Hilbert space module for the underlying unitary algebra A, the Classification of Hilbert Space Modules 3.3.6 applies. Thus there is a unitary isomorphism $H_A \cong pL^2A_A^{\oplus n}$ for some projection $p \in M_n(A)$, and under this isomorphism, $\mathrm{End}(H_A) \cong pM_n(A)p$. It remains to identify the commutant trace Tr_A' on $A' = \mathrm{End}(H_A)$. For the remainder of this proof, we identify $H_A = pL^2A_A^{\oplus n}$ and $A' = \mathrm{End}(H_A) = pM_n(A)p$.

Write $A = \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$ and denote the minimal central projections by $p_1, \ldots, p_n \in Z(A)$. Set $d_i := n_i^{-1}\,\mathrm{Tr}_A(p_i)$ so that $d = \sum d_i p_i \in Z(A)$ is the density of Tr_A with respect to Tr, the direct sum of the canonical unnormalized traces on A, i.e., $\mathrm{Tr}_A(x) = \mathrm{Tr}(dx)$ for all $x \in A$.

We now observe that the corresponding minimal central projections of $Z(A')$ are given by $q_i := pp_i$. (Note that some of the q_i may be zero.) For each i, there is an $m_i \geq 0$ such that $q_i L^2 A^{\oplus n} p_i \cong \mathbb{C}^{\operatorname{rank}(q_i)} \otimes \overline{\mathbb{C}^{n_i}}$ as an $q_i A' - p_i A$ bimodule; this m_i is the integer such that $q_i A' \cong M_{m_i}(\mathbb{C})$. (Exercise: Prove that $m_i = \operatorname{rank}(q_i)$ where we view $q_i \in M_n(A)$ acting on $\mathbb{C}^n \otimes \mathbb{C}^{\sum n_i}$.) By Exercise 3.4.14, Tr' is the direct sum of the canonical commutant traces on $pL^2 A_A^{\oplus n}$, i.e.,

$$\operatorname{Tr}'(x) = \sum_{i=1}^{n} \operatorname{Tr}_{m_i}(x) \qquad\qquad \forall\, x \in M_{m_i}(\mathbb{C}) = q_i A'.$$

Applying Exercise 3.4.15, we see that $\operatorname{Tr}'_A(x) = \operatorname{Tr}'(d'x)$ where $d' = \sum d_i q_i \in Z(A')$. Explicitly, d_i equals both Tr_A of a minimal projection in $p_i A$ and Tr'_A of a minimal projection in $q_i A'$.

Remark 3.4.17. When (B, Tr_B) is an H*-algebra, any Hilbert space $A - B$ bimodule can be canonically promoted to an $A - B$ correspondence using the B-valued inner product from (3.4.5).

Definition 3.4.18 — A bimodule ${}_AH_B$ for H*-algebras (A, Tr_A) and (B, Tr_B) is a Hilbert space module for (B, Tr_B) equipped with a left action

$$A \to \operatorname{End}(H_B).$$

Note that we do not require this map to be isometric/trace preserving.

Warning 3.4.19 — Given a bimodule ${}_AH_B$ for H*-algebras (A, Tr_A) and (B, Tr_B), Tr_A induces the commutant trace $\operatorname{Tr}_{{}_AH}$ on $\operatorname{End}({}_AH)$ and Tr_B induces the commutant trace Tr_{H_B} on $\operatorname{End}(H_B)$. In general, these traces do not agree on $\operatorname{End}({}_AH_B) = \operatorname{End}({}_AH) \cap \operatorname{End}(H_B)$. Thus one should only view $\operatorname{End}(H_B)$ and $\operatorname{End}({}_AH)$ as equipped with traces, and not $\operatorname{End}({}_AH_B)$. This makes more sense from a categorical perspective, which we will study in Part[[III]] §[[]].

Fusion of bimodules is the same as in the last section, viewing our H*-algebras as W*/von Neumann algebras by forgetting their traces. However, H*-algebras offer two main advantages. First, we can simplify the construction considerably using our distinguished traces; see Exercise 3.4.20 below. Second, the algebra valued inner products induced by our traces allow us to define projective bases, yielding canonical evaluation and coevaluation maps. We will describe this in detail in §3.6 below, once we have introduced the graphical calculus for bimodules and their intertwiners.

Exercise 3.4.20. Suppose (B, Tr_B) is an H*-algebra. Show that for a right module H_B and a left module ${}_BK$, the Connes fusion Hilbert space $H \boxtimes_B K$ is canonically isomorphic to the relative tensor product vector space $H \otimes_B K$ endowed with the inner product

$$\langle \eta_1 \boxtimes \xi_1 | \eta_2 \boxtimes \xi_2 \rangle := \operatorname{Tr}_B(\langle \eta_2 | \eta_1 \rangle_B \cdot {}_B\langle \xi_1, \xi_2 \rangle)$$

where we have used the B-valued inner products (3.4.5) on H_B and (3.4.9) on ${}_BK$.

Exercise 3.4.21. Use Exercise 3.4.20 to prove that $H \boxtimes_B K$ satisfies the a modified universal property (3.1.3) with respect to maps into a Hilbert space L.[3] Use this universal property to construct associators and unitors for $\boxtimes$ as in 3.1.10 and 3.1.11 for the relative tensor product for the Connes fusion.

Equipped with this easier definition of Connes fusion, we now return to the proof of the Sauvageot Splitting Theorem 3.3.11. To recall the setup, H_B is a faithful Hilbert space B-module.

Proof of Sauvageot Splitting Theorem 3.3.11. By Example 2.5.3 and Theorem 2.5.19, we may uniquely identify $L^2B = L^2(B, \operatorname{Tr})$ for a faithful tracial weight. The map $u : H \boxtimes_B \overline{H} \to L^2(B', \operatorname{Tr}')$ given by $\eta \boxtimes \overline{\xi} \mapsto |\eta\rangle_B\langle \xi|\Omega_{\operatorname{Tr}'}$ is a B'-linear unitary; it is surjective by Corollary 3.2.26 and injective as it is isometric:

$$\begin{aligned}
\left\| \sum_i \eta_i \boxtimes \overline{\xi}_i \right\|^2 &= \sum_{i,j} \langle \langle \eta_j | \eta_i \rangle_B \overline{\xi}_i | \overline{\xi}_j \rangle_{\overline{H}} = \sum_{i,j} \langle \xi_j | \xi_i \langle \eta_i | \eta_j \rangle_B \rangle_H \\
&\underset{(3.4.6)}{=} \sum_{i,j} \operatorname{Tr}_B(\langle \xi_j | \xi_i \rangle_B \langle \eta_i | \eta_j \rangle_B) \underset{(3.4.11)}{=} \sum_{i,j} \operatorname{Tr}'_B(|\xi_i\rangle_B \langle \eta_i | \eta_j \rangle_B \langle \xi_j|) \\
&= \left\| \sum_i |\eta_i\rangle_B \langle \xi_i | \Omega_{\operatorname{Tr}'} \right\|^2 .
\end{aligned}$$

One directly verifies that $uJ = J_{\operatorname{Tr}'}u$, and $uP = P_{\operatorname{Tr}'}$ by Connes' Trick 3.2.24. The result now follows by Theorem 2.5.19. □

Remark 3.4.22. There is a stronger version of H*/isometric Morita equivalence for H*-algebras beyond W* Morita equivalence, where our canonical unitary isomorphisms $\overline{H} \boxtimes_A H \to L^2B$ and $L^2A \to H \boxtimes_B \overline{H}$ are given by the canonical evaluation and coevaluation maps coming from the B-valued inner product induced by Tr_B. We see this implicitly in our proof of the Sauvageot Splitting Theorem 3.3.11 above, where we chose a faithful tracial weight Tr on B and equipped B' with the canonical commutant trace Tr'. We postpone further discussion until Example 3.6.8 in §3.6 below, once we have introduced the graphical calculus for bimodules and their intertwiners.

[3] This universal property for Connes fusion relies on the finite dimensionality of H, K and B.

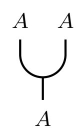
3.5 Separable algebras and Frobenius algebras

In this section, we introduce the notion of separability for algebras and the definition of a Frobenius algebra. We then discuss the unitary versions of these notions, connecting them to unitary algebras and H*-algebras.

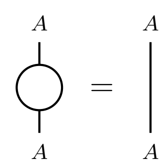
Definition 3.5.1 — A complex algebra is called *separable* if the multiplication $m : A \otimes A \to A$ admits a *section* $s : A \to A \otimes A$ which an $A - A$ bimodule map, i.e., $ms = \mathrm{id}_A$ and s satisfies

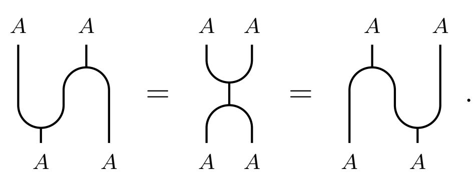
$$(\mathrm{id}_A \otimes m)(s \otimes \mathrm{id}_A) = sm = (m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes s).$$

Remark 3.5.2. In the same fashion as Remark 2.1.2, we represent a choice of section $s\colon A \to A \otimes A$ as in Definition 3.5.1 by a trivalent vertex going in the opposite direction.

$s =$ 

Finally, we describe s being an $A - A$ bimodule section to m diagrammatically.





Exercise 3.5.3. Prove that a section s is automatically coassociative, i.e.,

$$(s \otimes \mathrm{id}_A)s = (\mathrm{id}_A \otimes s)s.$$

Draw the corresponding diagrams for this coassociativity property.

Definition 3.5.4 — A *Frobenius algebra* is a complex algebra equipped with a section for the multiplication, i.e., an $A - A$ bimodule map $\Delta : A \to A \otimes A$ satisfying

$$(\mathrm{id}_A \otimes m)(\Delta \otimes \mathrm{id}_A) = \Delta m = (m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta)$$

and a *counit* $\varepsilon : A \to \mathbb{C}$ satisfying

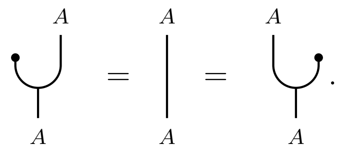
$$(\varepsilon \otimes \mathrm{id}_A)\Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \varepsilon)\Delta.$$

Remark 3.5.5. We represent the counit $\varepsilon\colon A \to \mathbb{C}$ diagrammatically by

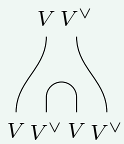
$\varepsilon =$

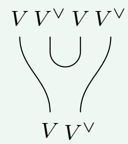
,

which we require to satisfy



Example 3.5.6 — Let V be a finite dimensional vector space, and consider the algebra $V \otimes V^\vee \cong \mathrm{End}(V) \cong M_{\dim(V)}(\mathbb{C})$ with multiplication and section given by

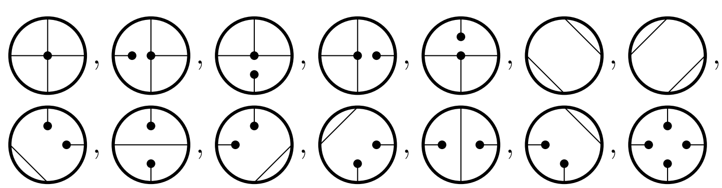
$m :=$ 

$s := \dfrac{1}{\dim(V)}$ 

Setting $\Delta := \dim(V)s$ and $\varepsilon := (v \otimes f \mapsto f(v))$ endows $V \otimes V^\vee$ with the structure of a Frobenius algebra.

Remark 3.5.7. A separable algebra can be augmented to a Frobenius algebra with $\Delta = s$ if and only if s admits a counit.

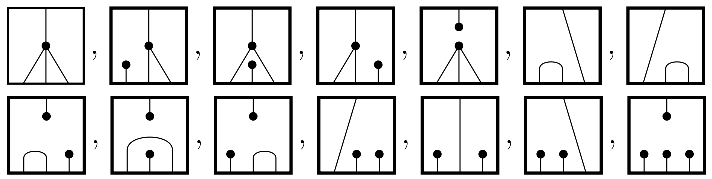
Exercise 3.5.8. A partition of n points drawn on the unit circle S^1 is called *non-crossing* if any time x_1, x_2 and y_1, y_2 are each chosen from the same subset of the partition, the chords from x_1 to x_2 and from y_1 to y_2 do not cross. For example, the non-crossing partitions of 4 points on S^1 may be graphically represented by



where boundary points are in the same subset if and only if they are connected in the above diagram.

Show that the number of non-crossing partitions of n points on a circle is the n-th *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Exercise 3.5.9. Suppose A is a Frobenius algebra. Find a bijective correspondence between morphisms in $\mathrm{Hom}(A^{\otimes m} \to A^{\otimes n})$ generated by $m, i, \Delta, \varepsilon$ and non-crossing partitions of $m+n$ points drawn on a rectangle with m lower boundary points and n upper boundary points. For example, when $m = 3$ and $n = 1$, the following diagrams are in bijective correspondence with the non-crossing partitions.



Deduce that all connected trivalent graphs drawn as a morphisms $A^{\otimes m} \to A^{\otimes n}$ define the same morphism. (Also deduce that this last result also holds for separable, possibly non-unital algebras.)

Theorem 3.5.10 — The following are equivalent for a finite dimensional $\mathbb{C}$-algebra A.

(1) A is semisimple.

(2) A admits the structure of a Frobenius algebra.

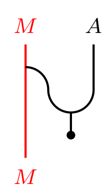
(3) A is separable.

Proof.

(1)⇒(2): Immediate from Example 3.5.6 as A is a multimatrix algebra by the Artin-Wedderburn Theorem 2.2.5.

(2)⇒(3): Trivial.

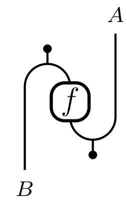
(3)⇒(1): By (SS7), it suffices to prove every A-module M_A is a summand of a finitely generated free module. Consider the free module $M \otimes A \cong A^{\oplus \dim(M)}$. Observe that the map

$$s_M :=$$ 

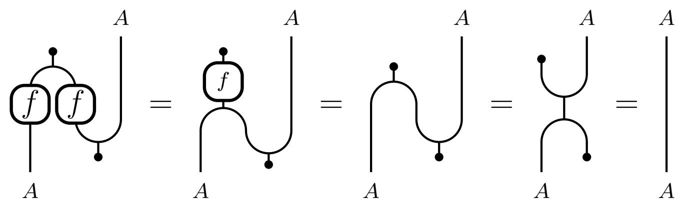
is an A-module map (exercise!) which splits the A-module map action map $r_M : M \otimes A \to M$. (One can choose the map $A \to A \otimes A$ above to be either s or Δ depending on whether A is separable or Frobenius.) Thus $s_M \circ r_M : M \otimes A \to M \otimes A$ is an idempotent whose image is isomorphic to M_A as an A-module. □

Proposition 3.5.11 — Suppose A, B are two Frobenius algebras. Every map $f : A \to B$ which is compatible with the (co)multiplications and (co)units is invertible.

Proof. We claim that the inverse of f is given by

$f^\vee :=$  .

Observe

$f^\vee \circ f =$  $= \mathrm{id}_A,$

and a similar computation reveals $f \circ f^\vee = \mathrm{id}_B$. □

Now in order to define a unitary version of a separable algebra or a Frobenius algebra, we must have adjoints of linear maps. We must then endow our unitary algebra with a faithful weight φ in order to view it as a Hilbert space. When φ is a faithful trace, we are then in the setting of an H*-algebra as in Definition 3.4.1 above.

Definition 3.5.12 — Suppose (A, φ) is a unitary algebra equipped with a faithful weight, and identify $A = L^2(A, \varphi)$. We call A:

- a *unitary Frobenius algebra* if $m^\dagger$ is an $A - A$ bimodule map.
- *unitarily separable* if the adjoint $m^\dagger : A \to A \otimes A$ of the multiplication map $m : A \otimes A \to A$ splits m as an $A - A$ bimodule map.

Remark 3.5.13. Observe that a unitarily separable algebra is always counital with counit $i^\dagger$. Thus every unitarily separable algebra is always a unitary Frobenius algebra.

Example 3.5.14 — Let φ be a faithful weight on $M_n(\mathbb{C})$, and let $H_\varphi := L^2(M_n(\mathbb{C}), \varphi)$. We compute the adjoint of the multiplication map $m : H_\varphi \otimes H_\varphi \to H_\varphi$ given by $a\Omega_\varphi \otimes b\Omega_\varphi \mapsto ab\Omega_\varphi$. By Proposition 2.3.16, $\varphi = \mathrm{Tr}(d\,\cdot)$ for an invertible, positive density matrix d. We work in an ONB of $\mathbb{C}^n$ under which d is diagonal. Let (e_{ij}) denote a system of matrix units for $M_n(\mathbb{C})$ (see Example 1.4.15) with respect to this ONB. Observe that $\{e_{ij}\Omega_\varphi\}$ is an orthogonal basis for H_φ with

$$\|e_{ij}\Omega_\varphi\|^2 = \langle e_{ij}\Omega_\varphi | e_{ij}\Omega_\varphi\rangle = \varphi(e_{ji}e_{ij}) = \varphi(e_{jj}) = \mathrm{Tr}(de_{jj}) = d_{jj}.$$

Thus $\{d_{jj}^{-1/2} e_{ij}\Omega_\varphi\}$ is an ONB for H_φ. We now compute $m^\dagger(d_{jj}^{-1/2} e_{ij}\Omega_\varphi)$.

$$\begin{aligned}
\langle m^\dagger(d_{jj}^{-1/2} e_{ij}\Omega_\varphi) | d_{\ell\ell}^{-1/2} e_{k\ell}\Omega_\varphi \otimes d_{ss}^{-1/2} e_{rs}\Omega_\varphi\rangle_{H_\varphi} &= \frac{1}{\sqrt{d_{jj}d_{\ell\ell}d_{ss}}}\langle e_{ij}\Omega_\varphi | m(e_{k\ell}\Omega_\varphi \otimes e_{rs}\Omega_\varphi)\rangle_{H_\varphi} \\
&= \frac{\delta_{\ell=r}}{\sqrt{d_{jj}d_{\ell\ell}d_{ss}}}\langle e_{ij}\Omega_\varphi | e_{ks}\Omega_\varphi\rangle_{H_\varphi} \\
&= \frac{\delta_{\ell=r}}{\sqrt{d_{jj}d_{\ell\ell}d_{ss}}}\varphi(e_{ji}e_{ks}) \\
&= \frac{\delta_{\ell=r}\delta_{i=k}}{\sqrt{d_{jj}d_{\ell\ell}d_{ss}}}\operatorname{Tr}(de_{js}) \\
&= \frac{\delta_{\ell=r}\delta_{i=k}\delta_{j=s}}{\sqrt{d_{jj}d_{\ell\ell}d_{ss}}}d_{jj} \\
&= \frac{\delta_{\ell=r}\delta_{i=k}\delta_{j=s}}{\sqrt{d_{\ell\ell}}}.
\end{aligned}$$

We thus have the following formula for $m^\dagger$:

$$m^\dagger(d_{jj}^{-1/2} e_{ij}\Omega_\varphi) = \sum_\ell \frac{1}{\sqrt{d_{\ell\ell}}} \cdot d_{\ell\ell}^{-1/2} e_{i\ell}\Omega_\varphi \otimes d_{jj}^{-1/2} e_{\ell j}\Omega_\varphi.$$

One verifies by direct computation that $m^\dagger$ is a bimodule map, so H_φ is always a unitary Frobenius algebra. We now calculate

$$mm^\dagger(d_{jj}^{-1/2} e_{ij}\Omega_\varphi) = m\left(\sum_\ell \frac{1}{\sqrt{d_{\ell\ell}}} \cdot d_{\ell\ell}^{-1/2} e_{i\ell}\Omega_\varphi \otimes d_{jj}^{-1/2} e_{\ell j}\Omega_\varphi\right) = \left(\sum_\ell \frac{1}{d_{\ell\ell}}\right) \cdot d_{jj}^{-1/2} e_{ij}\Omega_\varphi.$$

So H_φ is unitarily separable ($mm^\dagger = \mathrm{id}_{H_\varphi}$) if and only if $\sum_\ell d_{\ell\ell}^{-1} = 1$. Thus for every faithful weight φ on $M_n(\mathbb{C})$, there is a unique scaling of φ under which $L^2(M_n(\mathbb{C}), \varphi)$ is unitarily separable.

Theorem 3.5.15 — For a unitary algebra A, there is a unique faithful trace Tr_A such that the H*-algebra (A, Tr_A) is unitarily separable.

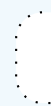
Proof. As A is multimatrix by the Artin-Wedderburn Theorem 2.2.5, and as the multiplication on A is a direct sum of the multiplications on each simple summand, it suffices to consider the case $A = M_n(\mathbb{C})$. If (A, Tr_A) is unitarily separable, then by Example 3.5.14, we must have

$$1 = \sum_{i=1}^n \frac{1}{\operatorname{Tr}_A(e_{ii})} \qquad \Longleftrightarrow \qquad \operatorname{Tr}_A(e_{ii}) = n \quad \forall\, i. \qquad \square$$

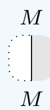
3.6 Diagrammatic calculus for algebras and bimodules

Just as we had a powerful graphical calculus for Hilbert spaces and their morphisms in §1.6, there is also a graphical calculus for algebras, their bimodules, and intertwiners.

Notation 3.6.1 — Algebras are denoted by shaded regions, e.g.,

 $= A$ $= B$ $= C,$

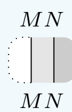
and bimodules are denoted by labeled strands whose shading on either side denotes which algebras act where, e.g.,



$= {}_AM_B$

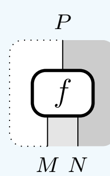
$= {}_BN_C$

The relative tensor product of bimodules is denoted by horizontal juxtaposition, e.g.,



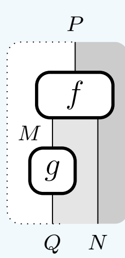
$= {}_AM \otimes_B N_C.$

We denote intertwiners between bimodules by coupons on strings, e.g., $f : {}_AM \otimes_B N_C \to {}_AP_C$ could be denoted



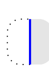
.

The identity map is just the string for the bimodule itself. Relative tensor product of intertwiners is horizontal juxtaposition, and composition of intertwiners is vertical stacking. For example, if $g : {}_AQ_B \to {}_AM_B$, then

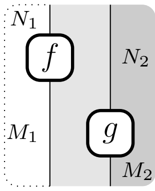


$= f \circ (g \otimes_B \mathrm{id}_P) : {}_AQ \otimes_B N_C \to {}_AP_C.$

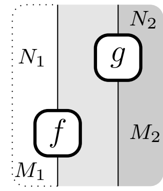
Remark 3.6.2. We sometimes denote bimodules with colored strands so we can visibly tell the difference between two different bimodules. For example, ${}_AM_B$ and ${}_BN_C$ could be denoted

 $= {}_AM_B$ $= {}_BN_C.$

As before, we have the interchange law

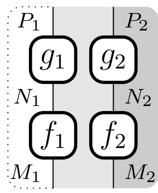


$$= (f \otimes \mathrm{id}_{N_2}) \circ (\mathrm{id}_{M_1} \otimes g) = (\mathrm{id}_{N_1} \otimes g) \circ (f \otimes \mathrm{id}_{M_2}) =$$



.

which implies that the following diagram is not ambiguous:

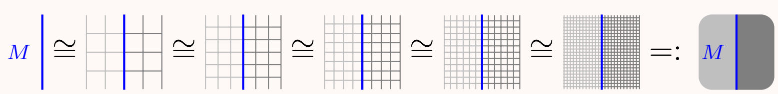


.

Again, we will rely on Exercise 3.1.10 to completely ignore the difference between ${}_AM \otimes_B (N \otimes_C P)_D$ and ${}_A(M \otimes_B N) \otimes_C P_D$ and Exercise 3.1.11 to complete ignore the identity bimodules ${}_AA_A$ and simply represent them by empty space, or sometimes by a dotted line for pedagogical reasons. We will discuss why this is OK later in Part[[II]] §[[]].

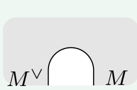
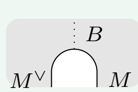
Ethos 3.6.3 — Although the shaded graphical calculus works for bimodules over algebras which are not separable, we may expand Ethos 1.6.6 by interpreting the shading for a separable algebra in terms of *condensing a string net from the vacuum*.

Indeed, given a separable algebra A, one can now proliferate A-strands in a 2-dimensional bulk region in the graphical calculus for vector spaces and linear maps. As long as the network is connected, separability ensures that the resulting morphism is an idempotent whose range is isomorphic to ${}_AA_A$. Again taking a limit, we can think of the shading for A as an arbitrarily fine connected mesh made from A-strands. One can perform this procedure for an $A - B$ bimodule M on either side.

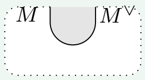


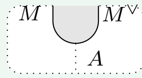
One resolves the 4-valent vertices using associativity and Exercise 3.5.9. We will explore this notion in detail in Part[[III]] §[[]].

Example 3.6.4 — Suppose A, B are semisimple and ${}_AM_B$ is a bimodule. Recall that the dual $M^\vee = \mathrm{Hom}(M_B \to B_B)$ is a $B - A$ bimodule. We have a canonical evaluation map

$\mathrm{ev}_M =$  $=$  $: {}_BM^\vee \otimes_A M_B \to {}_BB_B$ given by $f \otimes m \mapsto f(m).$

Similarly, there is a coevaluation map

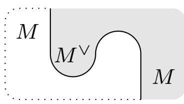
$\mathrm{coev}_M =$ 

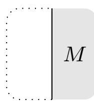
$=$ 

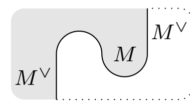
$: {}_AA_A \to {}_AM \otimes_B M^\vee_A$ given by $1_A \mapsto \sum m_i \otimes f_i$

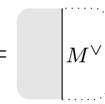
where (m_i, f_i) is an algebraic projective basis for M_B as in Corollary 2.2.22. Observe that coev_M is independent of the choice of algebraic projective basis, as $\sum m_i \otimes f_i$ corresponds to id_M under the canonical isomorphism $M \otimes_B M^\vee \cong \mathrm{End}(M_B)$ from Exercise 3.1.9.

Exercise 3.6.5. Prove that $\mathrm{ev}_M, \mathrm{coev}_M$ satisfy the zig-zag/snake equations



$=$ 

$= \mathrm{id}_M$ and 

$=$ 

$= \mathrm{id}_{M^\vee}$.

Warning: these diagrams suppress the unitor isomorphisms.

Remark 3.6.6. When A, B are unitary algebras and ${}_AX_B$ is a Hilbert C* bimodule, we have similar evaluation and coevaluation maps

$$\begin{aligned} \mathrm{ev}_X : {}_BX^\vee \boxtimes_A X_B &\to {}_BB_B & \qquad \mathrm{coev}_X : {}_AA_A &\to {}_AX \boxtimes_B X^\vee_A. \\ {}_B\langle \eta| \boxtimes |\xi\rangle_B &\mapsto \langle \eta|\xi\rangle_B & 1_A &\mapsto \sum |\beta_i\rangle_B \boxtimes {}_B\langle \beta_i| \end{aligned}$$

where $\{\beta_i\}$ is an X_B-basis as in Corollary 3.2.26. There is no canonical choice for a Hilbert space bimodule, as the dual module ${}_BH^\vee_A = \mathrm{Hom}(H_B \to B_B)$ is not a Hilbert space. When A, B are equipped with faithful tracial weights $\mathrm{Tr}_A, \mathrm{Tr}_B$ and are thus H*-algebras and ${}_AH_B$ is a Hilbert space bimodule, we again get a B-valued inner product by (3.4.5), which yields a canonical evaluation and coevaluation. In this case, we can canonically (unitarily) identify $H^\vee \cong \overline{H}$ as $B - A$ bimodules, and we have canonical choices for $\mathrm{ev}_H, \mathrm{coev}_H$ as

$$\begin{aligned} \mathrm{ev}_H : {}_B\overline{H} \boxtimes_A H_B &\to {}_BL^2(B, \mathrm{Tr}_B)_B & \qquad \mathrm{coev}_H : {}_AL^2(A, \mathrm{Tr}_A)_A &\to {}_AH \boxtimes_B \overline{H}_A \\ \overline{\eta} \boxtimes \xi &\mapsto \langle \eta|\xi\rangle_B \Omega_{\mathrm{Tr}_B} & \Omega_{\mathrm{Tr}_A} &\mapsto \sum \beta_i \boxtimes \overline{\beta_i} \end{aligned}$$

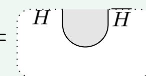
where $\{\beta_i\}$ is an H_B-basis.

Warning 3.6.7 — The formula for ev_H only depends on Tr_B, whereas the formula for coev_H depends on both Tr_A and Tr_B. One might think this is a mistake, as changing the trace on A can change the norm of Ω_{Tr_A}. However, for any choice of Tr_A, the unitor unitary isomorphism $L^2(A, \mathrm{Tr}_A) \boxtimes_A H \to H$ is given by $\Omega_{\mathrm{Tr}_A} \boxtimes \eta \mapsto \eta$ as

$$\|\Omega_{\mathrm{Tr}_A} \boxtimes \eta\|^2_{L^2(A,\mathrm{Tr}_A)\boxtimes_A H} = \langle \Omega_{\mathrm{Tr}_A}, \Omega_{\mathrm{Tr}_A} {}_A\langle \eta, \eta\rangle\rangle_{L^2(A,\mathrm{Tr}_A)} = \mathrm{Tr}_A({}_A\langle \eta, \eta\rangle) = \langle \eta|\eta\rangle_H = \|\eta\|^2_H.$$

Thus our formula for coev_H is correct, as the zig-zag/snake equations hold. A different trace on A yields both a different formula for coev_H and a different formula for the unitor, which cancel.

Example 3.6.8 (H* Morita equivalence) — Let (B, Tr_B) be an H*-algebra, and suppose H_B is a Hilbert space B-module. Equip $B' := \mathrm{End}(H_B)$ with the commutant trace Tr'_B. Let $\{\beta_i\}$ be an H_B-basis, and observe that under the canonical Sauvageot splitting unitary isomorphism $u : H \boxtimes_B \overline{H} \to L^2(B', \mathrm{Tr}'_B)$ from Theorem 3.3.11,

$$u \sum \beta_i \boxtimes \overline{\beta_i} = \sum |\beta_i\rangle_B \langle \beta_i | \Omega_{\mathrm{Tr}'_B} = \Omega_{\mathrm{Tr}'_B} \qquad \Longleftrightarrow \qquad u^\dagger = \mathrm{coev}_H =$$  .

When H_B is faithful, we call ${}_{B'}H_B$ an H* *Morita equivalence*, where our bimodular unitary isomorphisms $\overline{H} \boxtimes_{B'} H \to L^2 B$ and $L^2 B' \to H \boxtimes_B \overline{H}$ are given by the canonical evaluation and coevaluation maps coming from the B-valued inner product induced by Tr_B.

Construction 3.6.9 (Conditional expectation) — Suppose $A \subset B$ is a unital inclusion of unitary algebras, and let $\mathrm{Tr}_A, \mathrm{Tr}_B$ be faithful tracial weights on A, B respectively. For each $b \in B$, we consider the right A-linear map $|b\Omega_{\mathrm{Tr}_B}\rangle_A : L^2(A, \mathrm{Tr}_A) \to L^2(B, \mathrm{Tr}_B)$, which yields an A-valued inner product on $L^2(B, \mathrm{Tr}_B)$ by

$$\langle b_1\Omega_{\mathrm{Tr}_B} | b_2\Omega_{\mathrm{Tr}_B}\rangle_A := {}_A\langle b_1\Omega_{\mathrm{Tr}_B}| \circ |b_2\Omega_{\mathrm{Tr}_B}\rangle_A \in \mathrm{End}(L^2A_A) = A.$$

The canonical *trace preserving conditional expectation* $E : (B, \mathrm{Tr}_B) \to (A, \mathrm{Tr}_A)$ is given by $E(b) := \langle \Omega_{\mathrm{Tr}_B} | b\Omega_{\mathrm{Tr}_B}\rangle_A$ and satisfies

$$\mathrm{Tr}_A(E(b)) = \mathrm{Tr}_A(\langle \Omega_{\mathrm{Tr}_B} | b\Omega_{\mathrm{Tr}_B}\rangle_A) \underset{(3.4.6)}{=} \langle \Omega_{\mathrm{Tr}_B} | b\Omega_{\mathrm{Tr}_B}\rangle_{L^2(B,\mathrm{Tr}_B)} = \mathrm{Tr}_B(b).$$

Observe that E is clearly $A-A$ bimodular. Moreover, it is manifestly completely positive as it is conjugation by the operator $|\Omega_{\mathrm{Tr}_B}\rangle_A : L^2(A, \mathrm{Tr}_A) \to L^2(B, \mathrm{Tr}_B)$.

Exercise 3.6.10. Prove that $E : B \to A$ is unital and thus $E|_A = \mathrm{id}_A$ exactly when $\mathrm{Tr}_B|_A = \mathrm{Tr}_A$.

Proposition 3.6.11 — Suppose $(A, \mathrm{Tr}_A), (B, \mathrm{Tr}_B)$ are H*-algebras, ${}_AH_B$ is a Hilbert space bimodule, and let Tr'_B be the canonical commutant trace on $B' = \mathrm{End}(H_B)$. The following diagram commutes

$$\begin{tikzcd}
H \boxtimes_B \overline{H} \arrow[rr, "u"] \arrow[dr, "\mathrm{coev}_H^\dagger"'] & & L^2(B', \mathrm{Tr}'_B) \arrow[dl, "L^2E"] \\
& L^2(A, \mathrm{Tr}_A) &
\end{tikzcd}$$

where $u : H \boxtimes_B \overline{H} \to L^2(B', \mathrm{Tr}'_B)$ is the canonical Sauvageot Splitting isomorphism $\eta \boxtimes \overline{\xi} \mapsto |\eta\rangle_B\langle\xi|$ and $L^2E : L^2(B', \mathrm{Tr}'_B) \to L^2(A, \mathrm{Tr}_A)$ is the map $x\Omega_{\mathrm{Tr}'_B} \mapsto E(x)\Omega_{\mathrm{Tr}_A}$.

Proof. The formula for $E : B' \to A$ is determined by

$$\mathrm{Tr}_A(a^* \cdot E(|\eta\rangle_B\langle\xi|)) = \mathrm{Tr}'_B(a^*|\eta\rangle_B\langle\xi|) \underset{(3.4.11)}{=} \mathrm{Tr}_B(\langle\xi|a^*\eta\rangle_B) \underset{(3.4.6)}{=} \langle\xi|a^*\eta\rangle_H \qquad \forall\, a \in A.$$

We compute that

$$\begin{aligned}\langle a\Omega_{\mathrm{Tr}_A} |\, \mathrm{coev}_H^\dagger(\eta \boxtimes \overline{\xi})\rangle_{L^2(A,\mathrm{Tr}_A)} &= \langle \mathrm{coev}_H\, a\Omega_{\mathrm{Tr}_A} | \eta \boxtimes \overline{\xi}\rangle = \sum \langle a\beta_i \boxtimes \overline{\beta_i} | \eta \boxtimes \overline{\xi}\rangle \\ &= \sum \langle\langle \eta | a\beta_i\rangle_B \overline{\beta_i} | \overline{\xi}\rangle_{\overline{H}} = \sum \langle \xi | \beta_i \langle \beta_i | a^*\eta\rangle_B\rangle_H = \langle\xi|a^*\eta\rangle_H.\end{aligned}$$

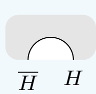
The result follows. □

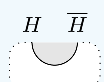
Corollary 3.6.12 — Suppose $(A, \mathrm{Tr}_A), (B, \mathrm{Tr}_B)$ are H*-algebras, and let Tr'_B be the canonical commutant trace on $B' = \mathrm{End}(H_B)$. Identifying ${}_A\overline{\overline{H}}_B = {}_AH_B$, $\mathrm{coev}_{\overline{H}}^\dagger = \mathrm{ev}_H : \overline{H} \boxtimes_A H \to L^2(B, \mathrm{Tr}_B)$.

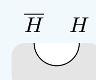
Proof. By swapping ${}_AH_B$ with ${}_B\overline{H}_A$ in the proof of the previous proposition at the equality marked (!) below, we see that for all $b \in B$,

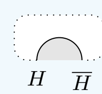
$$\begin{aligned}\langle b\Omega_{\mathrm{Tr}_B} |\, \mathrm{ev}_H(\overline{\eta} \boxtimes \xi)\rangle_{L^2(B,\mathrm{Tr}_B)} &= \langle b\Omega_{\mathrm{Tr}_B} | \langle\eta|\xi\rangle_B\Omega_{\mathrm{Tr}_B}\rangle_{L^2(B,\mathrm{Tr}_B)} = \mathrm{Tr}_B(b^*\langle\eta|\xi\rangle_B\Omega_{\mathrm{Tr}_B}) \\ &= \langle\eta b|\xi\rangle_H = \langle\overline{\xi}|b^*\overline{\eta}\rangle_{\overline{H}} \underset{(!)}{=} \langle b\Omega_{\mathrm{Tr}_B} |\, \mathrm{coev}_{\overline{H}}^\dagger(\overline{\eta} \boxtimes \xi)\rangle_{L^2(B,\mathrm{Tr}_B)}.\end{aligned}$$ □

Notation 3.6.13 — The shadings for the regions represented by H*-algebras allow us to unambiguously represent $\mathrm{ev}_H, \mathrm{coev}_H, \mathrm{ev}_H^\dagger, \mathrm{coev}_H^\dagger$ by



$= \mathrm{ev}_H$


$= \mathrm{coev}_H$


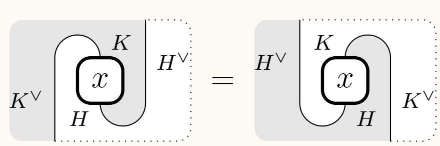
$= \mathrm{ev}_H^\dagger$


$= \mathrm{coev}_H^\dagger$.

When working with $A - A$ bimodules in the unitary setting for an H*-algebra, we again use framings as in Notation 1.6.13, as we will sometimes be in the position where $\overline{H} \cong H$, but we cannot directly identify the two without inducing subtle errors. For more details, see Part[[II]] §[[FS indicators]].

Using evaluations and coevaluations, we can define the transpose of an $A - B$ bimodule map ${}_AH_B \to {}_AK_B$.

Proposition 3.6.14 — Suppose $(A, \mathrm{Tr}_A), (B, \mathrm{Tr}_B)$ are H*-algebras. For $x : {}_AH_B \to {}_AK_B$,



This common morphism is called the *transpose* and denoted by $x^\vee : {}_B K_A^\vee \to {}_B H_A^\vee$.

Similar to the compatibility of the three involutions $(\cdot)^\dagger, \overline{(\cdot)}, (\cdot)^\vee$ on operators from Exercise 1.4.2, we have a similar compatibility for bimodular operators.

Exercise 3.6.15. Suppose A, B are H*-algebras and $x : {}_A H_B \to {}_A K_B$. Identifying ${}_B H^\vee = {}_B \overline{H}_A, {}_B K^\vee = {}_B \overline{K}_A$, as well as ${}_A \overline{\overline{H}}_B = {}_A H_B, {}_A \overline{\overline{K}}_B = {}_A K_B$, show that the operations $(\cdot)^\dagger, \overline{(\cdot)}, (\cdot)^\vee$ each have period 2. Then show that the composite of any two of these operations equals the third.

3.7 CP maps

Warning 3.7.1 — This highly technical section may be skipped on a first read.

In the Choi-Stinespring Theorem 2.6.10, we saw that one could characterize a completely positive map $M_n(\mathbb{C}) \to M_m(\mathbb{C})$ in terms of a map of Hilbert spaces $\mathbb{C}^n \otimes \overline{\mathbb{C}^n} \to \mathbb{C}^m \otimes \overline{\mathbb{C}^m}$ after the identification $L^2(M_k(\mathbb{C}), \mathrm{Tr}_k) \cong \mathbb{C}^k \otimes \overline{\mathbb{C}^k}$ by Corollary 1.6.18 and Exercise 1.6.19. In this section, we define the notion of a *cp map* between bimodules of the form ${}_A H \boxtimes_B \overline{H}_A$ and ${}_A K \boxtimes_C \overline{K}_A$. To draw the parallel with the Choi-Stinespring Theorem 2.6.10, one can use the Sauvageot Splitting Theorem 3.3.11, which allows us to canonically write $H \boxtimes_B \overline{H} = L^2 B'$ for $B' = \mathrm{End}(H_B)$.

Exercise 3.7.2. Suppose ${}_A H_B$ is a Hilbert space bimodule over unitary algebras A, B. Use the Sauvageot Splitting Theorem 3.3.11 to write $H \boxtimes_B \overline{H} = L^2 B'$ for $B' = \mathrm{End}(H_B)$. Show that

$$\mathbb{C}^n \otimes H \boxtimes_B \overline{H} \otimes \overline{\mathbb{C}^n} \cong H^{\oplus n} \boxtimes_B \overline{H}^{\oplus n} \tag{3.7.3}$$

is a standard form for $M_n(B') \cong \mathrm{End}(H_B^{\oplus n})$ via operators of the form $x \boxtimes 1$ with positive cone

$$P_{H,n} := \mathrm{span}_{\mathbb{R}_{\geq 0}} \left\{ f \boxtimes \xi \boxtimes f^* \,\middle|\, \xi \in L^2 B_+ \text{ and } f \in \mathrm{Hom}(L^2 B_B \to H_B^{\oplus n}) \right\}$$

where $f^* := \overline{f} \circ J_B : {}_B L^2 B \to {}_B \overline{H}$ and $J_{H,n}(f \boxtimes \eta \boxtimes g^*) := g \boxtimes J_B \eta \boxtimes f^*$ for $\eta \in L^2 B$ and $f, g \in \mathrm{Hom}(L^2 B_B \to H_B^{\oplus n})$.

Exercise 3.7.4. Picking a faithful tracial weight Tr_B on B to use the H*-algebra model for Connes fusion from Exercise 3.4.20, show that

$$P_{H,n} := \mathrm{span}_{\mathbb{R}_{\geq 0}} \left\{ (\eta_i) \boxtimes \overline{(\eta_i)} \,\middle|\, (\eta_i) \in H_B^{\oplus n} \right\}$$

and $J_{H,n}((\eta_i) \boxtimes \overline{(\xi_i)} := (\xi_i) \boxtimes \overline{(\eta_i)}$ gives a standard form for $M_n(B')$ acting on $H^{\oplus n} \boxtimes_B \overline{H^{\oplus n}}$.

Definition 3.7.5 ([HP23, Defn. 5.5]) — Suppose A, B, C are unitary algebras and ${}_AH_B$ and ${}_AK_C$ are Hilbert space bimodules. An $A-A$ bimodule map $x : {}_AH \boxtimes_B \overline{H}_A \to {}_AK \boxtimes_C \overline{K}_A$ is called *cp* if

$$(\mathrm{id}_{\mathbb{C}^n} \otimes x \otimes \mathrm{id}_{\overline{\mathbb{C}^n}})P_{H,n} \subseteq P_{K,n} \qquad \forall n \in \mathbb{N},$$

where $P_{H,n}, P_{K,n}$ are the amplified positive cones from Exercise 3.7.2. The set of cp maps ${}_AH \boxtimes_B \overline{H}_A \to {}_AK \boxtimes_C \overline{K}_A$ is denoted by $\mathcal{P}_{H,K}$.

Example 3.7.6 — Suppose ${}_AH_B, {}_AK_B$ are Hilbert space bimodules. For any map $y : {}_AH_B \to {}_AK_B$, the map

$$y \boxtimes_B \overline{y} : {}_AH \boxtimes_B \overline{H}_A \to {}_AK \boxtimes_B \overline{K}_A$$

is cp. Indeed, under the isomorphism (3.7.3), for all $f \in \mathrm{Hom}(L^2B_B \to H_B^{\oplus n})$ and $\xi \in L^2B_+$,

$$(\mathrm{id}_{\mathbb{C}^n} \otimes y \boxtimes_B \overline{y} \otimes \mathrm{id}_{\overline{\mathbb{C}^n}})(f \boxtimes \xi \boxtimes f^*) = g \otimes \xi \otimes g^* \in P_{K,n}$$

where $g := (\mathrm{id}_{\mathbb{C}^n} \otimes y) \circ f : L^2B_B \to K_B^{\oplus n}$.

Example 3.7.7 — Whenever $x : {}_AH \boxtimes_B \overline{H}_A \to {}_AK \boxtimes_C \overline{K}_A$ is cp and ${}_DL_A$ is an arbitrary Hilbert space bimodule,

$$\mathrm{id}_L \boxtimes x \boxtimes \mathrm{id}_{\overline{L}} : {}_DL \boxtimes_A H \boxtimes_B \overline{H} \boxtimes_A \overline{L}_D \to {}_DL \boxtimes_A K \boxtimes_C \overline{K} \boxtimes_A \overline{L}_D$$

is again cp. Indeed, this is trivial when $L = L^2A^{\oplus n}$ for any $n \in \mathbb{N}$ as then $L \boxtimes_A H \cong H^{\oplus n}$. Now an arbitrary L is of the form $pL^2A^{\oplus n}$ for some projection $p \in \mathrm{End}(L^2A_A^{\oplus n})$ by the Classification of Hilbert Space Modules 3.3.6. Hence

$$\mathrm{id}_L \boxtimes x \boxtimes \mathrm{id}_{\overline{L}} = \underbrace{(p \boxtimes \mathrm{id}_K) \boxtimes_C (\mathrm{id}_K \boxtimes p)}_{\text{cp by Ex. 3.7.6}} \circ (\mathrm{id}_{L^2A^{\oplus n}} \boxtimes x \boxtimes \mathrm{id}_{L^2A^{\oplus n}})$$

is manifsetly cp as a composite of cp maps

Example 3.7.8 — If $x : {}_AH \boxtimes_B \overline{H}_A \to {}_AK \boxtimes_C \overline{K}_A$ is cp, then so is $x^\dagger$. Indeed, for all $\eta \in P_{H,n}$ and $\xi \in P_{K,n}$,

$$\langle x^\dagger \xi | \eta \rangle_{H^{\oplus n} \boxtimes_B \overline{H}^{\oplus n}} = \langle \xi | x\eta \rangle_{K^{\oplus n} \boxtimes_B \overline{K}^{\oplus n}} \geq 0 \qquad \Longrightarrow \qquad x^\dagger \xi \in P_{H,n}$$

by self-duality of the positive cone $P_{H,n}$.

Lemma 3.7.9 — All cp maps are self-conjugate, i.e., if $x : {}_AH \boxtimes_B \overline{H}_A \to {}_AK \boxtimes_C \overline{K}_A$ is cp, then the following diagram commutes.

$$\begin{tikzcd} {}_AH \boxtimes_B \overline{H}_A \arrow[r, "x"] \arrow[d, "J"] & {}_AK \boxtimes_C \overline{K}_A \arrow[d, "J"] \\ \overline{{}_AH \boxtimes_B \overline{H}_A} \arrow[r, "\overline{x}"] & \overline{{}_AK \boxtimes_C \overline{K}_A} \end{tikzcd}$$

Proof. Recall that $J_{H,1}P_{H,1} = P_{H,1}$ and $J_{K,1}P_{K,1} = P_{K,1}$. Since $xP_{H,1} \subseteq P_{K,1}$, for all $\xi \in P_{H,1}$, we have that

$$\overline{x}\xi = \overline{x}J_{H,1}\xi = J_{K,1}x\xi = x\xi.$$

Since an operator acting on a standard form is completely determined by its action on the positive cone by Exercise 2.5.13, we conclude $x = \overline{x}$. □

Lemma 3.7.10 — Suppose B, C are unitary algebras equipped with unital $*$-algebra homomorphisms from A, which equips them with the structure of $A - A$ bimodules. For an $A - A$ bimodular map $\Phi : B \to C$, the following are equivalent.

(1) Φ is completely positive.

(2) For any choices of faithful tracial weights $\mathrm{Tr}_B, \mathrm{Tr}_C$ on B, C respectively, the map

$$ {}_AL^2B \boxtimes_B L^2B_A \cong {}_AL^2(B, \mathrm{Tr}_B)_A \to {}_AL^2(C, \mathrm{Tr}_C)_A \cong {}_AL^2C \boxtimes_C L^2C_A$$

given by $x\Omega_{\mathrm{Tr}_B} \mapsto \Phi(x)\Omega_{\mathrm{Tr}_C}$ (which depends on the traces!) is cp.

Proof. The commutant trace Tr'_B on $M_n(\mathbb{C}) \otimes B \cong M_n(B) = \mathrm{End}(L^2B_B^{\oplus n})$ is given by $\mathrm{Tr}_n \otimes \mathrm{Tr}_B$, and similarly for C. Recall from Example 2.5.3 that

$$P_{\mathrm{Tr}_n \otimes \mathrm{Tr}_B} = \{x^*x\Omega_{\mathrm{Tr}_n \otimes \mathrm{Tr}_A} \,|\, x \in M_n(B)\},$$

and similarly for C. Thus for all $N \in \mathbb{N}$ and $x \in M_n(B)$,

$$(\mathrm{id}_{\mathbb{C}^n} \otimes L^2\Phi \otimes \mathrm{id}_{\overline{\mathbb{C}^n}})(x^*x\Omega_{\mathrm{Tr}_n \otimes \mathrm{Tr}_B}) = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \Phi)(x^*x)\Omega_{\mathrm{Tr}_n \otimes \mathrm{Tr}_B} \in P_{\mathrm{Tr}_n \otimes \mathrm{Tr}_C}$$

if and only if $(\mathrm{id}_{M_n(\mathbb{C})} \otimes \Phi)(x^*x) \geq 0$. The result follows. □

Exercise 3.7.11. Suppose φ_B, φ_C are two faithful weights on the unitary algebras B, C respectively. If $\Phi : B \to C$ is completely positive, when is the map $x\Omega_{\varphi_B} \mapsto \Phi(x)\Omega_{\varphi_C}$ cp?

Example 3.7.12 — Suppose $(A, \mathrm{Tr}_A), (B, \mathrm{Tr}_B)$ are H*-algebras and ${}_AH_B$ is a Hilbert space $A - B$ bimodule. By Proposition 3.6.11 $\mathrm{coev}_H^\dagger : {}_AH \boxtimes_B H_A \to L^2(A, \mathrm{Tr}_A)$ is equal to L^2E, where $E : B' \to A$ is the canonical trace preserving conditional expectation, which is completely positive. By Lemma 3.7.10, $L^2E = \mathrm{coev}_H^\dagger$ is cp, so coev_H is cp by

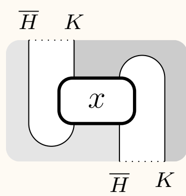
Example 3.7.8. By Corollary 3.6.12, $\mathrm{ev}_H = \mathrm{coev}_{\overline{H}}^\dagger : {}_B\overline{H} \boxtimes_A H_B \to L^2(B, \mathrm{Tr}_B)$ is also cp.

The following theorem is the unitary algebra bimodule analog of the Choi-Stinespring Theorem 2.6.10.

Theorem 3.7.13 (Bimodule Choi-Stinespring) — Suppose A, B, C are unitary algebras and ${}_AH_B, {}_AK_C$ are Hilbert space bimodules. The following are equivalent for a map $x : {}_AH \boxtimes_B \overline{H}_A \to {}_AK \boxtimes_C \overline{K}_A$.

(1) x is cp.

(2) For any choice of faithful tracial weights $\mathrm{Tr}_B, \mathrm{Tr}_C$ on B, C, the *one click rotation/Choi matrix* of x is a positive operator in $\mathrm{End}({}_B\overline{H} \boxtimes_A K_C)$.



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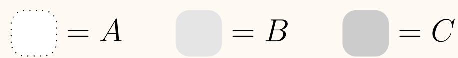
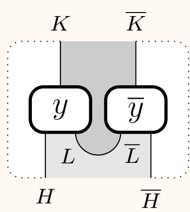


(3) For any choice of faithful tracial weights $\mathrm{Tr}_B, \mathrm{Tr}_C$ on B, C, we can write x in a *Stinespring representation* as

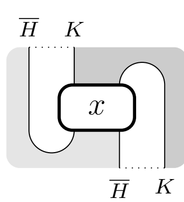
$x =$


$$: H \boxtimes_B \overline{H} \xrightarrow{\mathrm{id}_H \boxtimes_B \mathrm{coev}_L \boxtimes_B \mathrm{id}_{\overline{H}}} H \boxtimes_B L \boxtimes_C \overline{L} \boxtimes_B \overline{H} \xrightarrow{y \boxtimes_C \overline{y}} K \boxtimes_C \overline{K}$$

for some $y \in \mathrm{Hom}({}_AH \boxtimes_B L_C \to {}_AK_C)$ and some Hilbert space bimodule ${}_BL_C$.

Proof.

(1)⇒(2): For simplicity, we also pick a faithful tracial weight Tr_A for A to use the easy version of Connes fusion from Exercise 3.4.20. Suppose $\sum_{i=1}^n \overline{\eta_i} \boxtimes \xi_i \in \overline{H} \boxtimes_A K$. By expanding the definition of the one-click rotation of x in terms of projective bases, one calculates that

$$\left\langle \sum_{i=1}^n \overline{\eta_i} \boxtimes \xi_i \right|$$


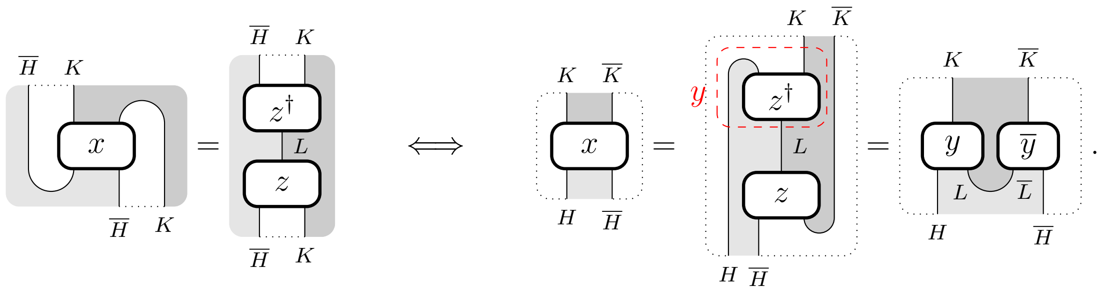
$$\left. \sum_{j=1}^n \overline{\eta_j} \boxtimes \xi_j \right\rangle = \sum_{i,j=1}^n \langle \xi_i \boxtimes \overline{\xi_j} | x(\eta_i \boxtimes \overline{\eta_j}) \rangle_{K \boxtimes_C \overline{K}}.$$

Considering $(\eta_\ell) \in H^{\oplus n}$ and $(\xi_k) \in K^{\oplus n}$, we see that the above sum is equal to

$$\left\langle (\xi_k) \boxtimes \overline{(\xi_k)} \middle| (\mathrm{id}_{\mathbb{C}^n} \otimes x \otimes \mathrm{id}_{\overline{\mathbb{C}^n}})((\eta_\ell) \boxtimes \overline{(\eta_\ell)}) \right\rangle_{K^{\oplus n} \boxtimes_C \overline{K}^{\oplus n}} \geq 0$$

as x is cp, $(\eta_\ell) \boxtimes \overline{(\eta_\ell)} \in P_{H,n}$, and $(\xi_k) \boxtimes \overline{(\xi_k)} \in P_{K,n}$.

(2)⇒(3): The proof is identical to that of (3)⇒(4) in the Choi-Stinespring Theorem 2.6.10 after adding shadings. In more detail,



Setting $y : {}_A H \boxtimes_B L_C \to {}_A K_C$ equal to $z^\dagger$ with one strand turned down and noting that $\overline{y} = y^{\dagger\vee}$, we visibly obtain a Stinespring representation for x.

(3)⇒(1): By Example 3.7.12, coev_L is cp, and by Example 3.7.6, so is $y \boxtimes_C \overline{y}$. Since composites of cp maps are manifestly cp, the result follows. □

Corollary 3.7.14 — The collection of cp maps between Hilbert/H*-algebras is the smallest collection of maps containing coevaluations which is closed under composites, adjoints, and conjugations.

Proof. We already know the cp maps contain the coevaluations by Example 3.7.12, and that cp maps are closed under composites, adjoints by Example 3.7.8, and conjugates by Examples 3.7.6 and 3.7.7. It remains to prove that any cp map is generated by the coevaluations under these operations. This follows directly from Theorem 3.7.13. □