Chapter 4

The Temperley-Lieb-Jones algebras

The Temperley-Lieb-Jones (TLJ) algebras are fundamental examples diagrammatic quantum algebras. These algebras first arose in the study of 'ice-type' lattice models [TL71], and they were discovered independently in the context of subfactor theory in operator algebras [Jon83]. This operator algebra approach led to Jones' discovery of his famous knot polynomial [Jon85] (see §4.8 below). The graphical representation of the TLJ algebras is due to Kauffman [Kau87], although hints certainly appeared in [TL71, p. 265].

4.1 The algebraic and diagrammatic TLJ algebras

The following abstract algebras were defined in [Jon83]. We denote the groups of units of the fields $\mathbb{R}, \mathbb{C}$ by $\mathbb{R}^\times, \mathbb{C}^\times$ respectively.

Definition 4.1.1 — For $n \geq 0$ and $d \in \mathbb{C}^\times$, we define the *algebraic Temperley-Lieb-Jones algebra* $TLJ_n(d)$ as the unital algebra generated by $1, e_1, \ldots, e_{n-1}$ subject to the following relations:

(J1) $e_i^2 = e_i$ for all $i = 1, \ldots, n-1$

(J2) $e_i e_j = e_j e_i$ for all $|i-j| > 1$, and

(J3) $e_i e_{i\pm 1} e_i = d^{-2} e_i$.

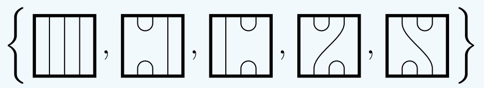
When $d \in \mathbb{R}^\times$, we may endow $TLJ_n(d)$ with a $*$-structure by imposing that each $e_i = e_i^*$.

Exercise 4.1.2. Use the relations (J1) – (J3) to prove that any word in $e_1, \ldots, e_n$ is equal to a word with at most one e_n.

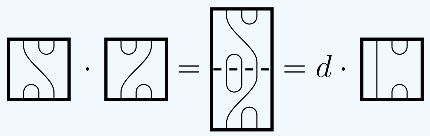
Exercise 4.1.3. Prove that $\dim(TLJ_n(d)) \leq \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.
Hint: Use Exercise 4.1.2.

In his skein-theoretic description of the Jones polynomial [Kau87], Kauffman provided a diagrammatic description of the Temperley-Lieb-Jones algebras.

Definition 4.1.4 — For $n \geq 0$ and $d \in \mathbb{C}^\times$, we define the *diagrammatic Temperley-Lieb-Kaufmann algebra* $TLK_n(d)$ to be the complex vector space whose standard basis is the set of non-intersecting string diagrams (up to isotopy) on a rectangle with n boundary points on the top and bottom. For example, the basis for $TLK_3(d)$ is given by

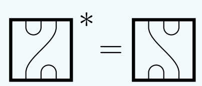


On $TLK_n(d)$, we define a multiplication by (the bilinear extension of) stacking boxes, removing the middle line segment and removing any closed loops at a cost of multiplying by a factor of d, e.g.



(4.1.5)

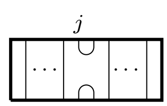
When $d \in \mathbb{R}^\times$, we define an involution by (the anti-linear extension of) reflection about a horizontal line, e.g.



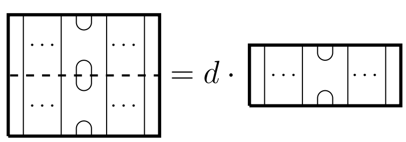
(4.1.6)

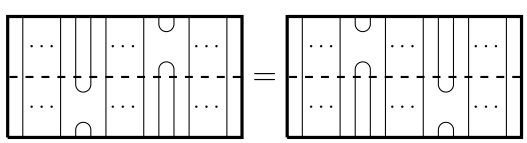
Exercise 4.1.7. Prove that $\dim(TLK_n(d)) = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.
Hint: Research how Catalan numbers are related to parenthesizations.

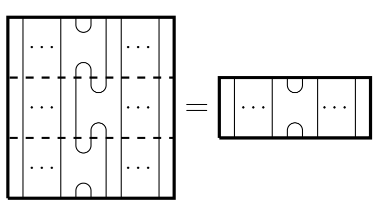
Exercise 4.1.8. Prove that for $j = 1, \ldots, n-1$, the elements

$E_j :=$  $\in TLK_n(d)$

satisfy the following relations:

(K1) $E_j^2 =$  $= dE_j$,

(K2) $E_jE_k =$  $= E_kE_j$ if $|j-k| > 1$, and

(K3) $E_jE_{j\pm 1}E_j =$  $= E_j$.

Compare these relations with (J1) – (J3). Show that $E_j^* = E_j$ when $d \in \mathbb{R}^\times$.

We now construct an algebra isomorphism $TLJ_n(d) \cong TLK_n(d)$ for $n \geq 0$ and $d \in \mathbb{C}^\times$, which is a $*$-algebra isomorphism when $d \in \mathbb{R}^\times$. By Exercise 4.1.8, the map $e_i \mapsto d^{-1}E_i$ extends to a well-defined unital ($*$-)algebra homomorphism $\Phi_n : TLJ_n(d) \to TLK_n(d)$, since the relations (J1) – (J3) are satisfied by dE_i for $i = 1, \ldots, n-1$.

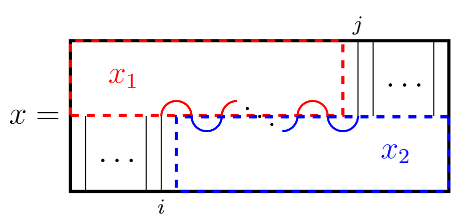
Proposition 4.1.9 — For every $n \geq 0$ and $d \in \mathbb{C}^\times$, the map

$$\begin{aligned} \Phi_n \colon TLJ_n(d) &\to TLK_n(d) \\ e_i &\mapsto d^{-1}E_i \end{aligned}$$

is a unital algebra isomorphism which is $*$-preserving whenever $d \in \mathbb{R}^\times$.

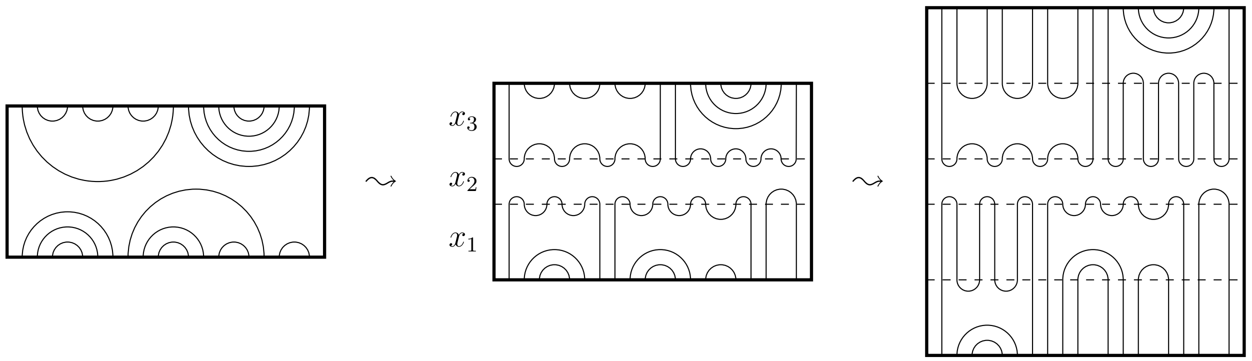
Proof. By Exercises 4.1.3 and 4.1.7, we have $\dim(TLJ_n(d)) \leq \dim(TLK_n(d)) = \frac{1}{n+1}\binom{2n}{n}$. By the Rank-Nullity theorem, it suffices to show that Φ_n is surjective. We proceed by strong induction on n. The base case $n = 0$ is trivial. Suppose that Φ_k is surjective for all $0 \leq k < n$. Let $x \in TLK_n(d)$ be a standard basis element.

Case 1: Suppose x has a *through string*, i.e., a string which connects the i-th lower boundary point to the j-th upper boundary point. Notice that $i \equiv j (\mod 2)$; without loss of generality, we assume $i < j$. Performing isotopy on the diagram x, we divide it up as follows:



Notice that the part in red, denoted x_1, is a diagrammatic basis element in $TLK_{j-1}(d)$, and the part in blue, denoted x_2, is a diagrammatic basis element in $TLK_{n-i}(d)$. Since $i, j \geq 1$, by the induction hypothesis, both x_1 and x_2 can be expressed as products of the E_k. Observe that x_2 shifted by i strings to the left is exactly the product of those E_k shifted by i, i.e., E_{k+i}. Since x is the product of x_1 with $n - j + 1$ strings added to the right together with the shift of x_2 by i strings to the left, and both of these latter diagrammatic basis elements are products of the E_k, so is x. Thus $x \in \operatorname{im}(\Phi_n)$.

Case 2: Suppose x has no through strings. By applying isotopy, we pull all the *outermost cups and caps* on the top and bottom toward the center of the diagram, and wiggle the strings as in Case 1 to divide the diagram into the product of three diagrams x_1, x_2, and x_3, where x_1 and x_3 are each a horizontal concatenation of basis elements in $TLK_i(d)$ for i strictly smaller than n, and x_2 is visibly a product of the odd E_k. We provide an explicit example below:



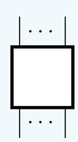
Applying isotopy again to shift the smaller basis elements which are horizontally concatenated within x_1 and x_3 up and down as in the rightmost diagram above, we can express each of x_1 and x_3 as products of the E_k, and thus x is a product of the E_k. We conclude $x \in \mathrm{im}(\Phi_n)$, and we are finished. $\square$

Notation 4.1.10 — From this point forward, we simply write $TLJ_n(d)$ to denote $TLJ_n(d)$ and $TLK_n(d)$, which we identify under the unital ($*$-)algebra isomorphisms Φ_n.

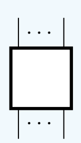
4.2 Graphical calculus for the TLJ-algebras

As the TLJ algebras afford a diagrammatic description, we get a powerful planar calculus. Linear maps are represented by various *planar tangles*, which are string diagrams with input rectangles and an output rectangle. Typically, we omit the external boundary rectangle, which is always assumed to be large.

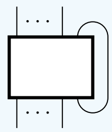
Definition 4.2.1 (Linear operations) — The *identity* map $\mathrm{id}_n : TLJ_n(d) \to TLJ_n(d)$ is given by the following diagram:



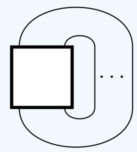
The *right inclusion* tangle is a unital, injective ($*$-)algebra homomorphism

$i_n :=$  $: TLJ_n(d) \to TLJ_{n+1}(d).$

The *conditional expectation/partial trace* tangle is a surjective ($*$-)map of $\mathbb{C}$-vector spaces

$\mathcal{E}_{n+1} :=$  $: TLJ_{n+1}(d) \to TLJ_n(d).$

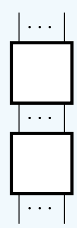
The *trace* tangle is a linear ($*$-)map of $\mathbb{C}$-vector spaces

$\mathrm{Tr}_n :=$  $: TLJ_n(d) \to TLJ_0(d).$ (4.2.2)

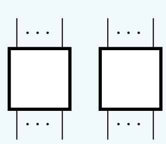
Note that $TLJ_0(d) \cong \mathbb{C}$ as a ($*$-)algebra via the map which sends the empty diagram to the complex number $1_{\mathbb{C}}$. Using Tr_n, we can define a bilinear form on $TLJ_n(d)$ by $(x, y)_n := \mathrm{Tr}_n(xy)$. When $d \in \mathbb{R}^\times$, we define a sesquilinear form on $TLJ_n(d)$ by

$$\langle x|y\rangle_n := \begin{cases} \mathrm{Tr}_n(x^*y) & \text{if } d > 0 \\ (-1)^n \,\mathrm{Tr}_n(x^*y) & \text{if } d < 0. \end{cases} \tag{4.2.3}$$

Definition 4.2.4 (Quadratic operations) — We already saw that multiplication was given by vertically stacking diagrams. We can also draw a tangle for multiplication as follows:

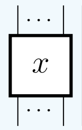
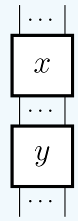
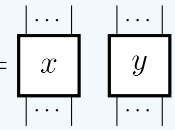
 $: TLJ_n(d) \times TLJ_n(d) \to TLJ_n(d).$

The tensor product tangle takes elements which may live in distinct TL algebras and horizontally concatenates them

 $: TLJ_m(d) \times TLJ_n(d) \to TLJ_{m+n}(d).$

Notation 4.2.5 — When we apply one of these operations to an $x \in TLJ_n(d)$ (or possibly two elements from two distinct TL algebras), we denote the output by labeling

the tangle with the input(s). For example,

$i_n(x) =$  $\qquad xy =$  $\qquad x \otimes y =$ 

Exercise 4.2.6. Prove the following relations amongst the maps $i_n, \mathcal{E}_{n+1}$, Tr_n, and id_n by drawing diagrams.

(1) $\mathcal{E}_{n+1} \circ i_n = d\, \mathrm{id}_n$,

(2) $\mathrm{Tr}_{n+1} = \mathrm{Tr}_n \circ \mathcal{E}_{n+1}$,

(3) $(i_n \circ i_{n-1} \circ \mathcal{E}_n(x))E_n = E_n i_n(x) E_n$ for all $x \in TLJ_n(d)$,

(4) (Traciality) $\mathrm{Tr}_n(xy) = \mathrm{Tr}_n(yx)$ for all $x, y \in TLJ_n(d)$,

(5) (Markov property) $\mathrm{Tr}_{n+1}(i_n(x) \cdot E_n) = \mathrm{Tr}_n(x)$ for all $x \in TLJ_n(d)$, and

(6) $\mathrm{Tr}_n(\mathcal{E}_{n+1}(x) \cdot y) = \mathrm{Tr}_{n+1}(x \cdot i_n(y))$ for all $x \in TLJ_{n+1}(d)$ and $y \in TLJ_n(d)$.

4.3 Quantum integers and Jones-Wenzl idempotents

In this section, we study the structure of the TLJ algebras by analyzing the *Jones-Wenzl idempotents*. To do so, we use *quantum integers* $[n]_q$, which are functions of $q \in \mathbb{C}^\times$. The number q and the number d from the TLJ algebra $TLJ_n(d)$ are related by $d = q + q^{-1}$, and we show in Lemma 4.3.4 below that $[n]$ only depends on d, not q.

Definition 4.3.1 (Quantum integers) — For $q \in \mathbb{C}^\times$ and $n \geq 2$, we define *quantum* n by

$$[n] = [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

When $q = \pm 1$, we define

$$[n]_{\pm 1} := \lim_{q \to \pm 1} [n]_q = (\pm 1)^n n.$$

We set $[0] := 0$ by convention.

Remark 4.3.2. In our definition of quantum integers $[n]_q$, we have allowed $q = \pm i$. In what follows, we often explicitly exclude these cases from study as $[2]_{\pm i} = 0$.

Exercise 4.3.3. Prove that the quantum integers satisfy the following relations:

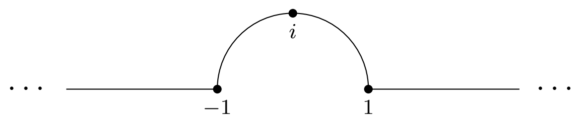
(1) For all $n \in \mathbb{N}$, $[2][n] = [n+1] + [n-1]$.

(2) For all $m, n, p \in \mathbb{N}$, $[m+p][n+p] = [m][n] + [m+n+p][p]$.
Note: Setting $p = n-1$ and $m = 2$ gives the above identity.

Lemma 4.3.4 — The quantum integer $[n]_q$ is always polynomial in $d := [2] = q + q^{-1}$, and does not rely on the choice of q beyond $d = q + q^{-1}$.

Proof. Clearly $[1] = 1$ and $[2] = d$. The result now follows by the relation $[2][n] = [n-1] + [n+1]$ from Exercise 4.3.3 and induction. $\square$

Exercise 4.3.5. Prove that the map $q \mapsto [2] = q + q^{-1}$ is a bijection



$\longrightarrow \mathbb{R}.$

Exercise 4.3.6. Show that the map $q \mapsto -q$ fixes all odd quantum integers $[2n+1]$ and negates all even quantum integers $[2n]$.

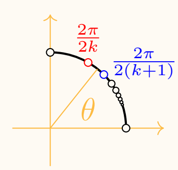
Exercise 4.3.7. Show that:

(1) If $q = e^{i\theta}$, then $[2]_q = 2\cos(\theta)$.

(2) $[n]_q = 0$ if and only if q is a $(2n)$-th root of unity.

The following lemma is essential for the Jones Modulus Rigidity Theorem 4.3.17.

Lemma 4.3.8 — Suppose $q = e^{i\theta}$ for some $\theta \in (0, \frac{\pi}{2})$, where $\theta \neq \frac{2\pi}{2n}$ i.e., q is not a primitive $(2n)$-th root of unity. Choose $k \geq 2$ such that $\frac{2\pi}{2k} > \theta > \frac{2\pi}{2(k+1)}$.



Then $[2], \ldots, [k] > 0$, but $[k+1] < 0$.

Proof. Note that since $q = e^{i\theta}$,

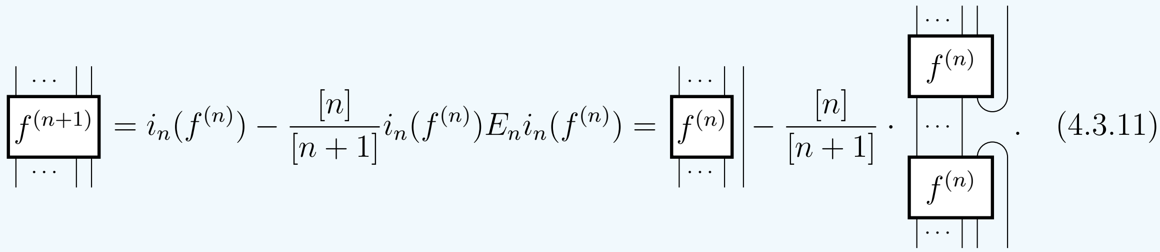
$$[j] = \frac{e^{ij\theta} - e^{-ij\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(j\theta)}{\sin(\theta)}.$$

Since $\sin(\theta) > 0$, we only care about the sign of $\sin(j\theta)$. Since $\frac{\pi}{k} > \theta > \frac{\pi}{k+1}$, we know that each of $\sin(\theta)$, $\sin(2\theta)$, $\ldots$, $\sin(k\theta)$ are strictly positive, but $\sin((k+1)\theta) < 0$. $\square$

Exercise 4.3.9. Adapt Lemma 4.3.8 for $q = e^{i\phi}$ where $\phi \in (\frac{\pi}{2}, \pi)$ and $\phi \neq \frac{2(n-1)\pi}{2n}$ for some $n > 2$. Find k minimal such that $(-1)^{n+1}[n] > 0$ for $n = 1, \ldots, k$, but $(-1)^{k+2}[k+1] < 0$.
Hint: Write $\phi = \pi - \theta$ where θ and k are as in Lemma 4.3.8.

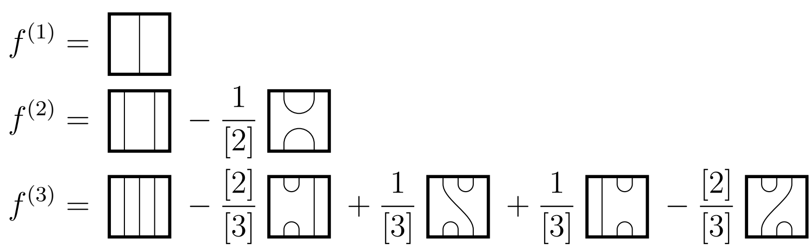
The following idempotents were first defined in [Jon83]. The recurrence relation first appeared in [Wen87]. Recall that by Lemma 4.3.4, $[n]$ is always a polynomial in d.

Definition 4.3.10 (Jones-Wenzl Idempotents) — Let $f^{(0)} \in TLJ_0(d)$ be the empty diagram. Let $f^{(1)} \in TLJ_1(d)$ be the strand, i.e., $f^{(1)} =$. If $[2], \ldots, [n+1] \neq 0$, we inductively define the $(n+1)$-th *Jones-Wenzl idempotent*



$$f^{(n+1)} = i_n(f^{(n)}) - \frac{[n]}{[n+1]} i_n(f^{(n)}) E_n i_n(f^{(n)}) \qquad (4.3.11)$$

Exercise 4.3.12. Compute the first three Jones-Wenzl idempotents, assuming $[2], [3]$ are non-zero. Namely, verify the following expressions for $f^{(1)}, f^{(2)}, f^{(3)}$.

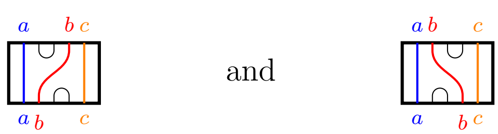


If you are feeling brave, compute $f^{(4)}$ additionally assuming $[4] \neq 0$.

Exercise 4.3.13. Suppose $[2], \ldots, [n+1] \neq 0$ so that $f^{(n)}$ exists. Prove the following statements by induction on n.

(1) The coefficient of the identity diagram id_n in $f^{(n)}$ is equal to 1.

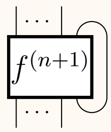
(2) Using thicker strings with color and labels to denote multiple parallel strings, the coefficients of the diagrams

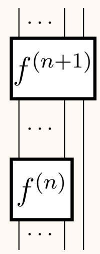
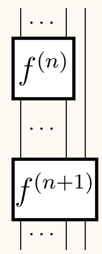


in $f^{(n)}$ are both equal to $(-1)^{b+1}\frac{[a+1][c+1]}{[n]}$.

Proposition 4.3.14 — Suppose $n \geq 0$ and $[2], \cdots, [n+1] \neq 0$ so that $f^{(0)}, f^{(1)}, \ldots, f^{(n+1)}$ are well-defined. Then $f^{(n+1)}$ satisfies the following properties:

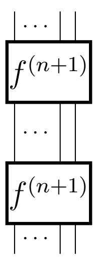
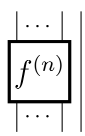
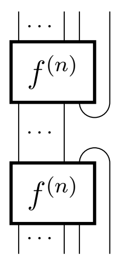
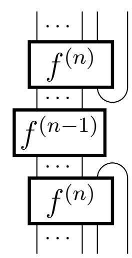
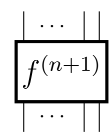
(JW1) $f^{(n+1)}$ is an idempotent, i.e., $(f^{(n+1)})^2 = f^{(n+1)}$. When $d \in \mathbb{R}^\times$, $(f^{(n+1)})^* = f^{(n+1)}$ is an orthogonal projection.

(JW2) $\mathcal{E}_{n+1}(f^{(n+1)}) =$  $= \dfrac{[n+2]}{[n+1]} f^{(n)}$.

(JW3) $(i_n(f^{(n)}))f^{(n+1)} =$  $= f^{(n+1)}(i_n(f^{(n)}) =$  $= f^{(n+1)}$.

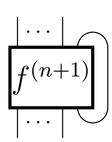
Proof. We proceed by induction on n. The base case $n = 0$ is straightforward as $f^{(1)}$ is the strand and $f^{(0)}$ is the empty diagram. Suppose the result holds for $f^{(n)}$.

(JW1): We calculate $f^{(n+1)} = (f^{(n+1)})^2$ using (JW1), (JW2), and (JW3) for $f^{(n)}$, which hold by the induction hypothesis. Indeed,

 $\underset{\text{(JW1)(JW2)}}{=}$  $- 2 \cdot \dfrac{[n]}{[n+1]} \cdot$  $+ \dfrac{[n]^2}{[n+1]^2} \dfrac{[n+1]}{[n]} \cdot$  $\underset{\text{(JW3)}}{=}$  .

When $d \in \mathbb{R}^\times$, one sees that $(f^{(n+1)})^* = f^{(n+1)}$ from the definition of $f^{(n+1)}$ along with the fact that $(f^{(n)})^* = f^{(n)}$ by (JW1) for $f^{(n)}$, which holds by the induction hypothesis.

(JW2): By (JW1) for $f^{(n)}$, which holds by the induction hypothesis, we see that

$\mathcal{E}_{n+1}(f^{(n+1)}) =$  $= \left([2] - \dfrac{[n]}{[n+1]}\right) f^{(n)} = \dfrac{[n+2]}{[n+1]} f^{(n)}$.

(JW3): By the definition of $f^{(n+1)}$, this property follows directly from $(f^{(n)})^2 = f^{(n)}$ by (JW1) for $f^{(n)}$, which holds by the induction hypothesis. □

Exercise 4.3.15. Deduce that when $f^{(0)}, \ldots, f^{(n)}$ are well-defined, $\mathrm{tr}_n(f^{(n)}) = [n+1]$.

Exercise 4.3.16. Suppose $[2], \ldots, [n] \neq 0$ so that $f^{(1)}, \ldots, f^{(n)}$ are well-defined.

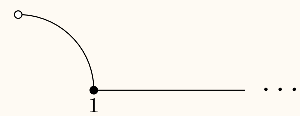
(1) Prove that the diagrams in $TLJ_n(d)$ with strictly fewer than n through strings span a maximal 2-sided ideal M in $TLJ_n(d)$.

(2) Prove that $f^{(n)}$ is orthogonal to M in the sense that $f^{(n)}m = mf^{(n)} = 0$ for all $m \in M$. Deduce that $f^{(n)}$ is the unique idempotent in $TLJ_n(d)$ which is orthogonal to M. *Hint: First use induction on n to prove that $f^{(n)}E_k = E_k f^{(n)} = 0$ for every k. Then show that the E_k generate M.*

(3) Deduce that $\mathbb{C} \cdot f^{(n)}$ is a minimal 2-sided ideal in $TLJ_n(d)$ complementing M.

Theorem 4.3.17 (Jones' modulus restriction) — Suppose $d > 0$, so we may assume

$$q \in \left\{ e^{i\theta} \,\middle|\, \theta \in (0, \frac{\pi}{2}) \right\} \cup [1, \infty).$$



If $\langle x|y\rangle_j = \mathrm{Tr}_j(x^*y)$ is positive semidefinite for all $j \geq 0$, then either $q \geq 1$, or $q = \exp(\pi i/n)$ for some $n \geq 3$. Hence

$$d = [2] = q + q^{-1} \in \left\{ 2\cos\left(\frac{\pi}{n}\right) \,\middle|\, n \geq 3 \right\} \cup [2, \infty).$$

Proof. We now prove the contrapositive. If $q \in Q$ is not of this form, then let k be as in Lemma 4.3.8 so that $[2], \ldots, [k] > 0$, but $[k+1] < 0$. Since $[2], \ldots, [k] \neq 0$, $f^{(k)}$ is well-defined. However,

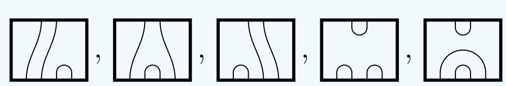
$$\langle f^{(k)}|f^{(k)}\rangle_k = \mathrm{Tr}_k(f^{(k)}) \underset{\text{(Exer. 4.3.15)}}{=} [k+1] < 0. \qquad \square$$

Exercise 4.3.18. State and prove a Jones Modulus Restriction Theorem for $d < 0$.

4.4 Tree bases and semisimplicity

In this section, we prove that when $[2], \ldots, [n] \neq 0$, so that $f^{(1)}, \ldots, f^{(n)}$ exist, the TLJ-algebra $TLJ_n(d)$ is semisimple. To do so, we construct an explicit *tree basis* our of distinguished *trivalent vertices*, which allows us to give matrix units for the simple summands.

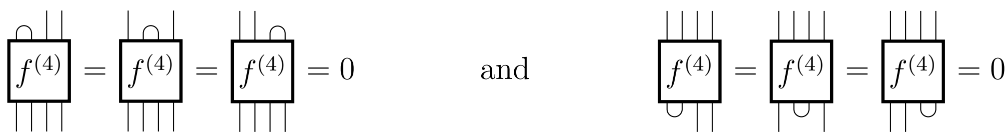
Definition 4.4.1 — Even though they are not strictly elements in TLJ algebras, we can write linear combinations of diagrams with *different* numbers of strings on the top and bottom. We may also 'compose' such diagrams by vertically stacking them, provided we again swap closed loops for multiplicative factors of d. We will denote the linear span of TLJ diagrams with m strings on the top and n strings on the bottom by $TLJ_n^m(d)$, and obviously $TLJ_n^n(d) = TLJ_n(d)$. For example,

$$TLJ_4^2(d) = \mathrm{span}\Bigg\{$$

$$\Bigg\}$$

Observe that $TLJ_n^m(d)$ is only non-zero if $m \equiv n \mod 2$.

When $d \in \mathbb{R}^\times$, there is an anti-linear involution $* : TLJ_n^m(d) \to TLJ_m^n(d)$ for which $x^{**} = x$ and $(xy)^* = y^* x^*$ for composable x, y. When $d > 0$, $*$ is again reflection about a horizontal line, but when $d < 0$, you reflect and multiply by a factor of -1 for every cup or cap in the diagram. For example,

Exercise 4.4.2. Show by induction that for all $k \leq n$, $f^{(k)}$ is *rectangularly uncappable*, that is, capping any two strings on the top or bottom of $f^{(k)}$ gives zero. For example,



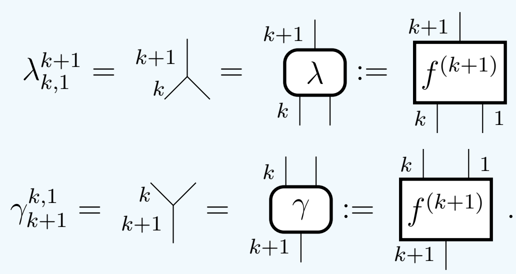
Hint: Use Exercise 4.3.16.

The above exercise gives the following immediate corollary.

Corollary 4.4.3 — Suppose $[2], \ldots, [n]$ are non-zero so that $f^{(1)}, \ldots, f^{(n)}$ exist. If $j, k \leq n$ are distinct, then $f^{(k)} \cdot TLJ_j^k(d) \cdot f^{(j)} = 0$.

Exercise 4.4.4. Show that if $f \in TLJ_j^k(d)$ and $g \in TLJ_k^j(d)$, then $\mathrm{Tr}_k(fg) = \mathrm{Tr}_j(gf)$.

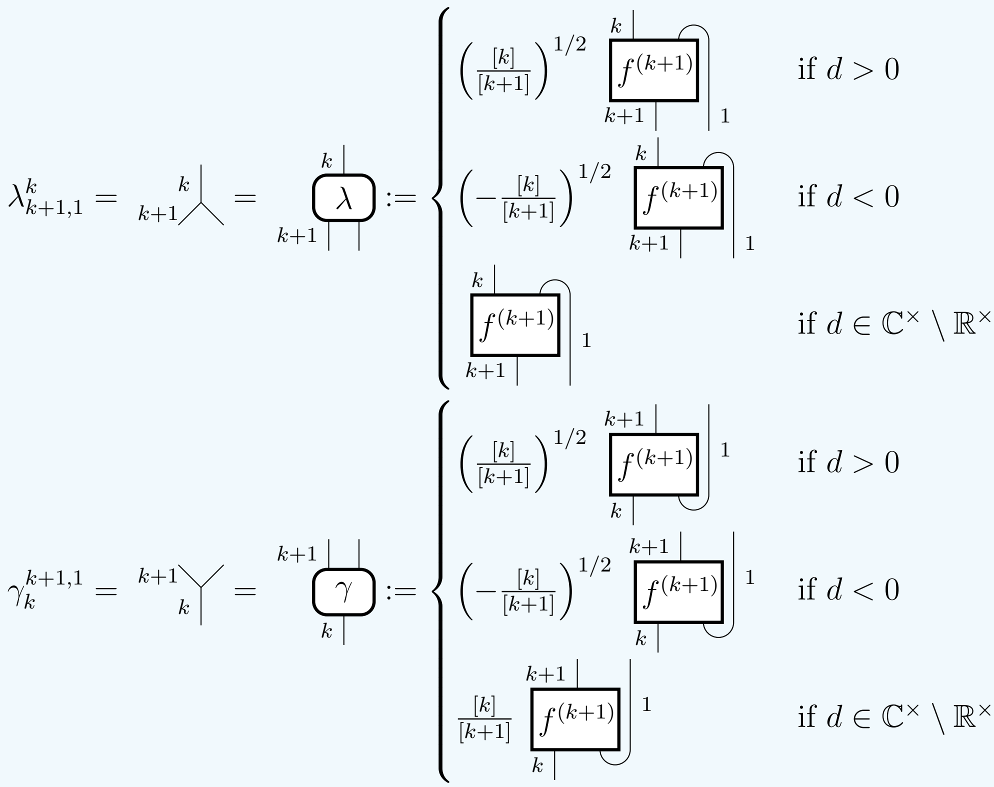
Definition 4.4.5 — For $k < n$, we pick distinguished *trivalent vertices* which are given by the following diagrams:



Observe that $\lambda^{k+1}_{k,1}\gamma^{k,1}_{k+1} = f^{(k+1)} = \gamma^{k,1}_{k+1}\lambda^{k+1}_{k,1}$, and when $d \in \mathbb{R}$, $(\gamma^{k,1}_{k+1})^* = \lambda^{k+1}_{k,1}$.

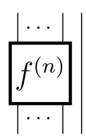
For $k < n-1$, we pick distinguished trivalent vertices depending on whether $d > 0$,

$d < 0$, or $d \in \mathbb{C}^\times \setminus \mathbb{R}^\times$:



(When $d < 0$, $[k]/[k+1] < 0$; since $k < n-1$, we know $[k+1] \neq 0$.) Again, observe that $\lambda^k_{k+1,1}\gamma^{k+1,1}_k = f^{(k)}$ by (JW2), and when $d \in \mathbb{R}^\times$, $(\gamma^{k+1,1}_k)^* = \lambda^k_{k+1,1}$. (We avoid choosing square roots of non-positive numbers whenever possible.)

Exercise 4.4.6. When $[2], \ldots, [n] \neq 0$, use Wenzl's Recurrence Relation (4.3.11) to prove that



$$= \gamma^{k,1}_{k+1}\lambda^{k+1}_{k,1} + \gamma^{k,1}_{k-1}\lambda^{k-1}_{k,1}. \tag{4.4.7}$$

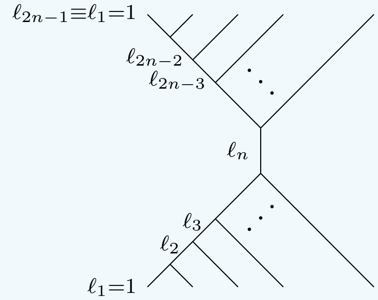
Construction 4.4.8 — Consider a *loop* of length $2n-2$

$$\ell := (\ell_1 = 1, \ell_2, \ldots, \ell_n, \ldots, \ell_{2n-2}, \ell_{2n-1} \equiv \ell_1 = 1)$$

starting at the vertex $\star = 1$ on the so-called *Coxeter-Dynkin diagram*

$A_{n+1} :=$  (4.4.9) so that $\ell_j = \ell_{j-1} \pm 1$ for all j. We define the *tree basis diagram* $T_\ell \in TLJ_n(d)$ as

$T_\ell :=$



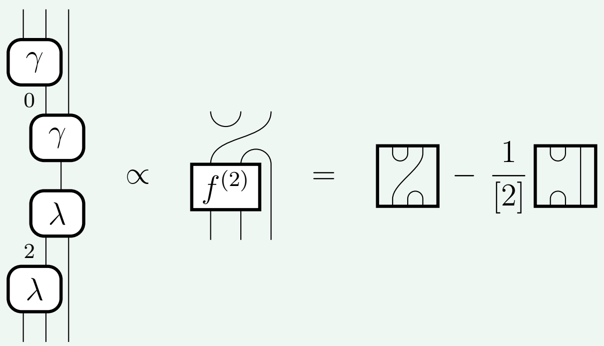
$\in TLJ_n(d)$

where each trivalent vertex on the top half is $\gamma^{\ell_j+1}_{\ell_j,1}$ or $\gamma^{\ell_j}_{\ell_j+1,1}$, and each trivalent vertex on the bottom half is $\lambda^{\ell_j+1}_{\ell_j,1}$ or $\lambda^{\ell_j}_{\ell_j+1,1}$, depending on ℓ_j and ℓ_{j+1}. The edges on the left hand side, commonly called the *spine* of the diagram, are labeled by the loop ℓ.

The *tree basis* for $TLJ_k(d)$ consists of all tree basis diagrams.

Example 4.4.10 — For $\ell = (1, 2, 1, 0, 1)$,

$T_\ell =$



Exercise 4.4.11. Draw all tree basis diagrams (ignoring scalars if you like) for $n = 2, 3, 4$.

Exercise 4.4.12. There is only one loop of length $2n-2$ on A_{n+1} based at $\star = 1$ which passes through n, namely

$$\ell = (1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1).$$

Show that $T_\ell = f^{(n)}$ for this loop. Deduce that $f^{(n)}$ does not appear in any other tree basis diagram for $TLJ_n(d)$.

Exercise 4.4.13.

(1) Prove that the number of loops of length $2n$ on the Coxeter-Dynkin diagram A_{n+1} starting at the vertex 0 is the n-th Catalan number C_n.

(2) Consider the map from loops ℓ of length $2n$ on the Coxeter-Dynkin diagram A_{n+1} starting at the vertex 0 to loops of length $2n-2$ on the Coxeter-Dynkin diagram A_{n+1} starting at the vertex 1 given by deleting the based vertex $\ell_{2n-1} \equiv \ell_1 = 0$. Prove this map is bijective.

Facts 4.4.14. We have the following facts about tree basis diagrams in $TLJ_n(d)$ when $[2], \ldots, [n]$ are non-zero. For a loop of length $2n-2$ on A_{n+1} based at $\star = 1$

$$\ell = (1 = \ell_1, \ell_2, \ldots, \ell_n, \ldots, \ell_{2n-2}, \ell_{2n-1} = 1),$$

we denote the loop in the reverse order by

$$\ell^* := (1 = \ell_{2n-1}, \ell_{2n-2}, \ldots, \ell_n, \ldots, \ell_2, \ell_1 = 1).$$

(TB1) Consider a second loop

$$\ell' := (1 = \ell'_1, \ell'_2, \ldots, \ell'_n, \ldots, \ell'_{2n-2}, \ell'_{2n-1} = 1).$$

Then

$$T_{\ell'} T_\ell = \delta_{\ell'_2 = \ell_{2n-2}}\, \delta_{\ell'_3 = \ell_{2n-3}} \cdots \delta_{\ell'_n = \ell_n} \cdot T_{\ell''}$$

for the loop

$$\ell'' = (1 = \ell_1,\, \ell_2, \ldots,\, \ell_n = \ell'_n, \ldots, \ell'_{2n-2}, \ell'_{2n-1} = 1).$$

If $d \in \mathbb{R}^\times$, then $T_\ell^* = T_{\ell^*}$.

Proof. Stacking $T_{\ell'}$ on top of T_ℓ, we get zero unless $\ell'_2 = \ell_{2n-2}$, $\ell'_3 = \ell_{2n-3}$, ..., and $\ell'_n = \ell_n$ by Corollary 4.4.3. If we have $\ell'_2 = \ell_{2n-2}$, $\ell'_3 = \ell_{2n-3}$, ..., and $\ell'_n = \ell_n$, then since $\lambda_{k,1}^{k+1}\gamma_{k+1}^{k,1} = f^{(k+1)}$ when $k < n$ and $\lambda_{k+1,1}^{k}\gamma_k^{k+1,1} = f^{(k)}$ when $k < n-1$, these Jones-Wenzl idempotents that are left behind are absorbed into the next trivalent vertex, yielding $T_{\ell''}$ as claimed.

The last claim is immediate. $\square$

(TB2) If $\ell = \ell^*$, then T_ℓ is an idempotent with trace $[\ell_n + 1]$. In particular, T_ℓ is non-zero. If $d \in \mathbb{R}^\times$, then T_ℓ is an orthogonal projection.

Proof. That $T_\ell^2 = T_\ell$ when $\ell = \ell^*$ is immediate from (TB1). The fact that $\mathrm{Tr}_n(T_\ell) = \mathrm{Tr}_n(f^{(\ell_n)}) = [\ell_n + 1]$ follows from Exercise 4.4.4 and then Exercise 4.3.15 by stacking the bottom half of T_ℓ on the top half and then taking the trace.

We immediately see $T_\ell \neq 0$ unless $\ell_n = n$ and $[n+1] = 0$. In this case, $T_\ell = f^{(n)}$ by Exercise 4.4.12, as ℓ is the unique loop of length $2n$ with midpoint n. However, we know the coefficient of id_n in $f^{(n)}$ is 1 by Exercise 4.3.13, so $f^{(n)} \neq 0$.

The last claim is immediate. $\square$

(TB3) Every tree basis diagram T_ℓ is non-zero.

Proof. For an arbitrary loop ℓ of length $2n-2$, observe that $T_{\ell^*}T_\ell$ is necessarily an idempotent by (TB1). Thus $T_{\ell^*}T_\ell \neq 0$ by (TB2), so $T_\ell \neq 0$. $\square$

(TB4) The set of tree basis diagrams T_ℓ which the same midpoint ℓ_n are a system of matrix units for a full matrix algebra indexed by paths of length $n-1$ on A_{n+1} starting at $\star$ and ending at ℓ_n. When $d \in \mathbb{R}^\times$, $*$ on $TLJ_n(d)$ restricted to this matrix algebra is the usual adjoint.

Proof. Immediate from (TB1) and (TB3). But in more detail, a loop of length $2n-2$ starting at $\star$ and midpoint ℓ_n can be viewed as a pair of paths p, q of length $n-1$ from $\star$ to ℓ_n. The loop ℓ is then pq^*. We can thus think of $T_\ell = T_{p,q^*}$, and the multiplication and adjoint rules from (TB1) and (TB3) are those of a system of matrix units. $\square$

(TB5) Tree basis elements with distinct midpoints ℓ_n are orthogonal (multiply to zero). In particular, the tree basis is linearly independent.

Proof. Immediate from Corollary 4.4.3 and (TB4). $\square$

(TB6) The tree basis spans $TLJ_n(d)$ and is thus a basis. In particular, $TLJ_n(d)$ is semisimple. If $d \in \mathbb{R}^\times$, $TLJ_n(d)$ is a unitary algebra. If $d > 0$, Tr_n is positive, and if $d < 0$, $(-1)^n \mathrm{Tr}_n$ is positive.

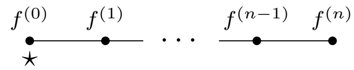
Proof. By Exercise 4.4.13, there are C_n loops of length $2n-2$ based at $\star = 1$, so there are C_n distinct tree basis elements. Since $\dim(TLJ_n(d)) = C_n$ by Exercise 4.1.7, the tree basis is indeed a basis by (TB6). When $d \in \mathbb{R}^\times$, unitarity and positivity of Tr_n or $(-1)^n \mathrm{Tr}_n$ follow by (TB1) and (TB2). $\square$

(TB7) If $[n+1] \neq 0$, then Tr_n is non-degenerate on $TLJ_n(d)$. If $[n+1] = 0$, then Tr_n is degenerate with negligible ideal $N_n = N(\mathrm{Tr}_n) = \mathbb{C} \cdot f^{(n)}$ as in Definition 2.2.28.

Proof. By (TB6), the trace on $TLJ_n(d)$ is a linear combination of the traces on the matrix blocks corresponding to the midpoints ℓ_n, in which the trace of a minimal idempotent T_ℓ for $\ell = \ell^*$ is $[\ell_n + 1]$. We know $[2], \ldots, [n] \neq 0$, so the only summand on which Tr_n can vanish is $\mathbb{C} \cdot f^{(n)}$. This happens if and only if $[n+1] = 0$ by Exercise 4.3.15, in which case $\mathbb{C} \cdot f^{(n)}$ is the negligible ideal. $\square$

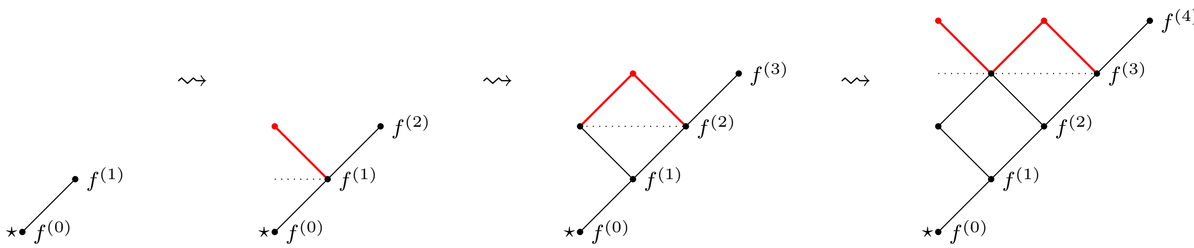
Remark 4.4.15. One can use a graphical trick in terms of paths on A_{n+1} to visualize the matrix algebra summands of TLJ_k for $k \leq n$ when $[2], \dots, [n] \neq 0$ (so that $f^{(1)}, \dots, f^{(n)}$ exist). We give an informal discussion here which will be formalized in §4.6 below.

We relabel the nodes $0, \dots, n$ on A_{n+1} as $f^{(0)}, \dots, f^{(n)}$.

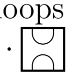


Instead of loops of length $2n-2$ based at 1, we work with the corresponding loop of length $2n$ on A_{n+1} based at $\star = 0$ (recall Exercise 4.4.13). There is one loop of length 0 starting at $\star = 0$ and one loop of length 2 starting at $\star = 0$. These correspond to the unique non-zero projections $f^{(0)} \in TLJ_0$ and $f^{(1)} \in TLJ_1$.

We visualize these algebras by drawing nodes corresponding to the vertices $f^{(k)}$ at different *heights* corresponding to the k. Below, we describe a *reflection* procedure which allows us to visualize loops of length $2n$ starting at $\star$ on A_{n+1} as loops which start at $\star$, ascend upward to height n to its midpoint, and then descend back to $\star$.



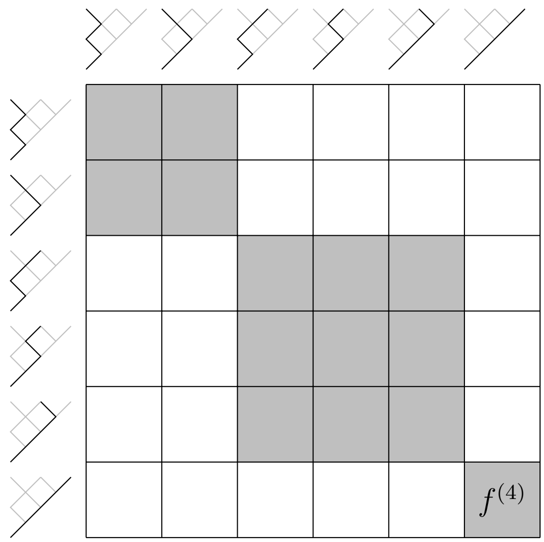
(1) All the data for $f^{(0)}$ and $f^{(1)}$ can be seen from looking at the initial A_2 chain, corresponding to the first picture above.

(2) In the second picture above, we draw the A_3 graph with nodes $f^{(0)}, f^{(1)}, f^{(2)}$ at different heights. As a visual aide, we draw a second node corresponding to $f^{(0)}$ at height 2 above the node for $f^{(0)}$ at height 0. We add an extra edge which is the reflection of the edge from $f^{(0)}$ to $f^{(1)}$ about the horizontal dotted line. Loops of length 4 on A_3 starting at $\star$ now can be viewed as loops of length 4 starting at $\star$ which first travel upward to height 2 and then descend back to $\star$. There are two distinct loops of length 4; the one which passes through the copy of $f^{(0)}$ at height 2 is $e_1 = d^{-1} \cdot$ , and the one which passes through the vertex corresponding to $f^{(2)}$ on A_3 is $f^{(2)}$. Observe that $1^2 + 1^2 = 2 = C_2$.

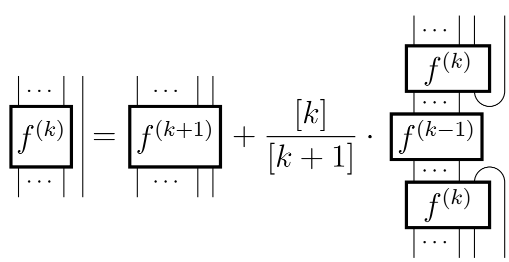
(3) In the third picture above, we draw the A_4 graph with nodes $f^{(0)}, f^{(1)}, f^{(2)}, f^{(3)}$ at different heights. We draw a second vertex corresponding to $f^{(1)}$ at height 3 above the $f^{(1)}$ node at height 1. The extra edges drawn are reflected from the edges between heights 1 and 2 about the dotted line. Loops of length 6 based at $\star$ ascend to height 3 and descend back to $\star$. There are 4 loops passing through the vertex corresponding to $f^{(1)}$, which are a system of matrix units for a copy of $M_2(\mathbb{C})$. Tree basis diagrams in

this copy of $M_2(\mathbb{C})$ have waist $f^{(1)}$. The unique loop of length 6 passing through $f^{(3)}$ is $f^{(3)}$ itself. Observe that $2^2 + 1^2 = 5 = C_3$.

(4) Finally, in the fourth picture, we again add nodes at height 4 and edges by reflecting the nodes at height 2 and edges between heights 2 and 3 about the dotted line. The 4 loops of length 8 passing through the vertex corresponding to $f^{(0)}$ are the tree basis diagrams with waist $f^{(0)}$ which span a copy of $M_2(\mathbb{C})$; the 9 loops which pass through the vertex corresponding to $f^{(2)}$ are the tree basis diagrams with waist $f^{(2)}$ which span a copy of $M_3(\mathbb{C})$. The final loop passing through $f^{(4)}$ is $f^{(4)}$ itself. Observe that $2^2 + 3^2 + 1^2 = 14 = C_4$. Below is a cartoon to visualize the matrix units of TLJ_3



The above graphs are called *Bratteli diagrams*, which we make rigorous in §4.6 below. This visualization makes sense given Wenzl's recurrence relation (4.3.11) for the Jones-Wenzl idempotents. Adding a strand to the right is the algebra inclusion $TLJ_k \hookrightarrow TLJ_{k+1}$. Wenzl's relation says that the way you obtain $f^{(k+1)}$ is by including $f^{(k)}$ into TLJ_{k+1} and observing it decomposes as the sum of two idempotents: $f^{(k+1)}$ itself and an idempotent with waist $f^{(k-1)}$.



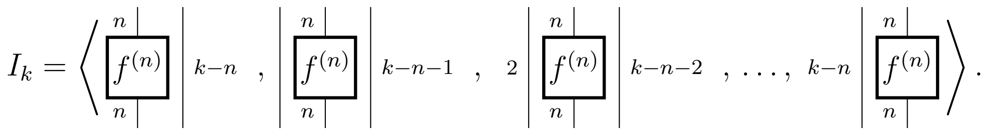
Visually, we see that the node in A_n labelled k includes into level $k+1$ along two edges, one going to the right corresponding to $f^{(k+1)}$ and one going to the left, which is above the vertex corresponding to $f^{(k-1)}$.

4.5 The semisimple quotient at a root of unity

We now study further when $[2], \ldots, [n] \neq 0$, but $[n+1] = 0$. By (TB6), $TLJ_k(d)$ is semisimple and Tr_k is nondegenerate for all $1 \leq k \leq n$, but $f^{(n)} \in TLJ_n(d)$ is negligible for Tr_n by (TB7). The basic idea is that we want to 'quotient out' $TLJ_k(d)$ for $k \geq n$ by the negligible ideal, which should generated by $f^{(n)}$. To do so, we should view the relation

$$f^{(n)} = 0$$

as a *local skein relation*. That is, for $k \geq n$, we define $I_k \subset TLJ_k(d)$ as the 2-sided ideal of linear combinations of diagrams which include a $f^{(n)}$ as a sub-diagram. More precisely, we have the following generating set for I_k:



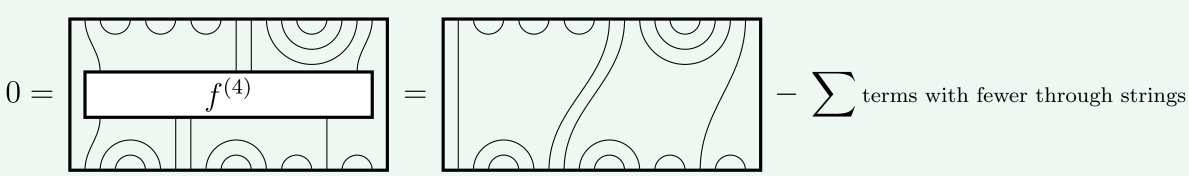
By convention, for $k < n$, we set $I_k = 0$.

Definition 4.5.1 — We define $\mathcal{TLJ}_k(d) := TLJ_k(d)/I_k$, the quotient of TLJ by the negligible ideal. To simplify notation in the results below, we will overload notation and draw string diagrams for their images in $\mathcal{TLJ}_k(d)$.

Observe that the maps $i_k, \mathcal{E}_k, \mathrm{Tr}_k$ along with horizontal and vertical concatenation preserve diagrams in the ideals I_k and thus descend to the quotients $\mathcal{TLJ}_k(d)$. We again overload notation and denote the maps descended by the quotients by the same names.

The following is the essential trick about working in the quotient $\mathcal{TLJ}_k(d)$.

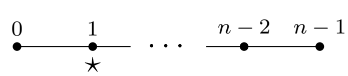
Trick 4.5.2 — As $f^{(n)} = 0$ in $\mathcal{TLJ}_n(d)$ and the coefficient of id_n in $f^{(n)}$ is 1, any n parallel through strings may be replaced by a linear combination with fewer than n through strings. For example, if $n = 4$, then



By iteratively applying Trick 4.5.2, we have the following result.

Proposition 4.5.3 — Every string diagram in $\mathcal{TLJ}_k(d)$ is a linear combination of diagrams with at most $n-1$ through strings.

Exercise 4.5.4. Show that the number of string diagrams in $TLJ_k(d)$ with at most $n-1$ through strings is equal to the number of loops of length $2k-2$ on the Coxeter-Dynkin diagram A_n (not A_{n+1}!) based at $\star = 1$.



Facts (TB1)–(TB7) have straightforward adaptations to $\mathcal{TLJ}_k(d)$ for $k \geq n$ using loops of length $2k$ on A_n based at $\star = 1$; we leave the proofs of the following facts to the reader.

Facts 4.5.5. We have the following facts about tree basis diagrams in $\mathcal{TLJ}_k(d)$ for $k \geq n$ when $[2], \ldots, [n] \neq 0$, but $[n+1] = 0$. For a loop of length $2k-2$ on A_n based at $\star = 1$, we denote the corresponding tree basis element by $\mathcal{T}_\ell \in \mathcal{TLJ}_k(d)$, and as before, ℓ^* is the reversed loop.

($\mathcal{TB}$1) For a second loop ℓ' of length $2k-2$ based at $\star = 1$ on A_n, we have the multiplication formula

$$\mathcal{T}_{\ell'}\mathcal{T}_\ell = \delta_{\ell'_2=\ell_{2k-2}}\delta_{\ell'_3=\ell_{2k-3}}\cdots\delta_{\ell'_k=\ell_k}\cdot\mathcal{T}_{\ell''}$$

for the loop

$$\ell'' = (1 = \ell_1, \ell_2, \ldots, \ell_k = \ell'_k, \ldots, \ell'_{2k-2}, \ell'_{2k-1} = 1).$$

If $d \in \mathbb{R}^\times$, then $T_\ell^* = T_{\ell^*}$.

($\mathcal{TB}$2) If $\ell = \ell^*$, then $\mathcal{T}_\ell$ is an idempotent with trace $[\ell_k + 1] \neq 0$, so $\mathcal{T}_\ell \neq 0$. If $d \in \mathbb{R}^\times$, $\mathcal{T}_\ell$ is an orthogonal projection.

($\mathcal{TB}$3) Every tree basis diagram $\mathcal{T}_\ell$ is non-zero.

($\mathcal{TB}$4) The set of tree basis diagrams $\mathcal{T}_\ell$ with the same midpoint ℓ_k are a system of matrix units for a full matrix algebra whose dimension is a function of k, ℓ_k. When $d \in \mathbb{R}^\times$, $*$ on $\mathcal{TLJ}_k(d)$ restricted to this matrix algebra is the usual adjoint.

($\mathcal{TB}$5) Tree basis diagrams with distinct midpoints ℓ_k are orthogonal. In particular, the tree basis is linearly independent.

($\mathcal{TB}$6) The tree basis diagrams spans $\mathcal{TLJ}_k(d)$ and is thus a basis. In particular, $\mathcal{TLK}_k(d)$ is semisimple, and Tr_k is non-degenerate. If $d \in \mathbb{R}^\times$, $\mathcal{TLJ}_k(d)$ is a unitary algebra. If $d > 0$, Tr_k is positive, and if $d < 0$, $(-1)^k \mathrm{Tr}_k$ is positive.

4.6 Bratteli diagrams

We now formalize *Bratteli diagrams*, which are a combinatorial tool to calculate the size of the full matrix summands of $\mathcal{TLJ}_k(d)$. We gave an informal discussion for the TLJ algebras in Remark 4.4.15 above. The main idea is that one can completely characterize an inclusion of multimatrix algebras $A \subset B$ up to conjugacy in B by a bipartite graph. This is a straightforward exercise if A, B are full matrix algebras.

Example 4.6.1 — Suppose we have a non-zero unital algebra homomorphism $\iota : M_k(\mathbb{C}) \hookrightarrow M_n(\mathbb{C})$, which is automatically injective by Proposition 2.1.17. Consider the right action of $M_k(\mathbb{C})$ on $\overline{\mathbb{C}^n}$ given by pre-composing with ι. By Example 2.1.23, $\overline{\mathbb{C}^n} \cong V \otimes \overline{\mathbb{C}^k}$ as $M_k(\mathbb{C})$-modules, where V is a multiplicity space. Thus $k \mid n$ with $n = \dim(V) \cdot k$. Choosing a basis for $\overline{\mathbb{C}^n}$ coming from bases of V and $\overline{\mathbb{C}^k}$, we may identify $\iota(x) = \mathrm{id}_V \otimes x$. Identifying $\mathrm{id}_V \otimes x$ with its Kronecker product, ι is the map

$$M_k(\mathbb{C}) \ni x \mapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix} \in M_n(\mathbb{C}).$$

When ι is a $*$-homomorphism, choosing ONBs above identifies ι with the above map as $*$-homomorphisms. We conclude that any two maps $M_k(\mathbb{C}) \to M_n(\mathbb{C})$ are conjugate by an invertible elements, and any two $*$-maps are conjugate by a unitary.

When A, B are multimatrix, we will work a bit harder.

Notation 4.6.2 — We write

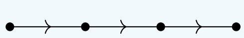
$$A = \bigoplus_{i=1}^{k} M_{m_i}(\mathbb{C}) \qquad \text{and} \qquad B = \bigoplus_{j=1}^{\ell} M_{n_j}(\mathbb{C})$$

We call $m_A = (m_1, \ldots, m_k)$ and $n_B = (n_1, \ldots, n_\ell)$ the *dimension row vectors* for A, B respectively. For $1 \leq i \leq k$, we write $p_i \in A$ for the minimal central idempotent onto the summand $M_{m_i}(\mathbb{C}) \subset A$, and for $1 \leq j \leq \ell$, we write $q_j \in B$ for the minimal central idempotent onto the summand $M_{n_j}(\mathbb{C}) \subset B$.

Definition 4.6.3 — An *oriented graph* Γ is a finite set of vertices V together with an assignment of a finite set of edges $E_{u \to v}$ to each ordered pair of vertices $(u, v) \in V^2$. The *source* of an edge $\varepsilon \in E_{u \to v}$ is u, denoted $s(\varepsilon) = u$, and the *target* is v, denoted $t(\varepsilon) = v$. A *path* on Γ is a finite sequence of edges $(\varepsilon_1, \ldots, \varepsilon_n)$ such that $s(\varepsilon_{i+1}) = t(\varepsilon_i)$ for all $i = 1, \ldots, n-1$. An oriented graph is called:

- *connected* if one can pass between any two vertices by traversing edges forwards or backwards, and
- *strongly connected* if there is always a path between any two vertices.

The *adjacency matrix* of Γ is the matrix A with rows and columns indexed by V with $A_{uv} = |E_{u \to v}|$.



connected, but not strongly connected

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

strongly connected

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

An *unoriented graph* is defined similarly, but we have a set of edges E_{uv} assigned to each *unordered* pair of vertices $\{u, v\}$. For the purposes of this book, the data of an unoriented graph is the same as the data of a graph with an involution on edges swapping source and target.

A *bipartite graph* is a graph together with a partition of the vertex set $V = V_0 \amalg V_1$ into disjoint subsets of *even* and *odd* vertices such that $E_{u\to v}$ is empty whenever the parities of u, v agree. We have the same notions of connected, strongly connected, oriented, and unoriented as before. The *bipartite adjacency matrix* of a bipartite graph Γ is the matrix Λ with rows indexed by V_0 and columns indexed by V_1 with $\Lambda_{uv} = |E_{u\to v}$. The adjacency matrix is related to the bipartite adjacency matrix by

$$A = \begin{pmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{pmatrix}.$$

Construction 4.6.4 — For multimatrix algebras $A \subset B$, consider $p_i B q_j$ as an $A - B$ bimodule, which by the folding trick (Exercise 3.1.6) can also be viewed as a right $A^{\text{op}} \otimes B$ module. Observe that the action factors through the full matrix algebra $p_i A^{\text{op}} \otimes q_j B$, so this module is isomorphic to the standard representation tensored with some multiplicity space by Example 2.1.23. Let Λ_{ij} denote the dimension of this multiplicity space.

The *inclusion matrix* of $A \subset B$ is given by $\Lambda_A^B := (\Lambda_{ij}) \in M_{k\times\ell}(\mathbb{C})$. The *Bratteli diagram* of $A \subset B$ is the unoriented bipartite graph Γ_A^B with:

- k even vertices labeled by the integers $m_1, \ldots, m_k$,
- ℓ odd vertices labeled by the integers $n_1, \ldots, n_\ell$, and
- Λ_{ij} edges from the i-th even vertex to the j-th odd vertex.

That is, Γ_A^B is the bipartite graph with bipartite adjacency matrix Λ_A^B whose even and odd vertices are labeled by the entries of the dimension row vectors of A and B respectively.

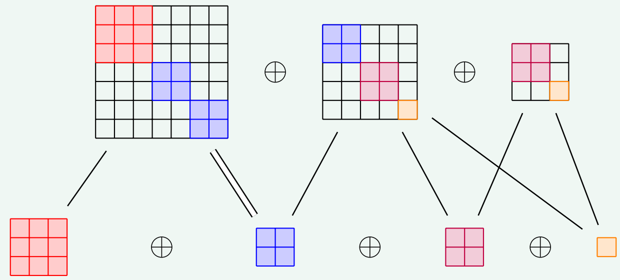
Exercise 4.6.5. Consider the inclusion map $A \hookrightarrow B$ restricted to the simple summand $M_{m_i}(\mathbb{C})$ of A. Cutting down afterward by q_j on B, we get an algebra map $M_{m_i}(\mathbb{C}) \to M_{n_j}(\mathbb{C})$.

(1) Show that the map $M_{m_i}(\mathbb{C}) \to M_{n_j}(\mathbb{C})$ is either injective or zero. Show this map is zero exactly when $p_i q_j = 0$.

(2) Now consider the map from the simple algebra $M_{a_i}(\mathbb{C})$ to the simple algebra $p_i q_j B p_i q_j$ By Example 4.6.1, up to conjugacy, this map includes some integer number of copies of $x \in M_{a_i}(\mathbb{C})$ along the diagonal. Show that this integer is Λ_{ij}.

Example 4.6.6 — We give an adjacency matrix Λ_A^B and the corresponding Bratteli diagram Γ_A^B for an inclusion $M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \subset M_7(\mathbb{C}) \oplus M_5(\mathbb{C}) \oplus M_3(\mathbb{C})$. Instead of labeling the vertices by integers $m_1, \ldots, m_4$ and $n_1, \ldots, n_3$, for the sake of pedagogy, we draw grids representing the sizes of the full matrix summands.

$$\Lambda_A^B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \rightsquigarrow \quad \Gamma_A^B =$$



Exercise 4.6.7. Suppose Γ is a graph with adjacency matrix A. Prove that $(A^k)_{uv}$ is the number of paths from u to v of length k. Deduce that Γ is strongly connected if and only if for all $u, v \in V$, there is a $k \in \mathbb{N}$ (depending on u, v) such that $(A^k)_{uv} > 0$.

Exercise 4.6.8. Suppose $A \subset B$ is a unital inclusion of multimatrix algebras.

(1) Prove that $m_A \Lambda_A^B = n_B$.

(2) Index the $Z(A)_+^\times$ torsor of faithful tracial states on $A = \bigoplus_{i=1}^k M_{m_i}(\mathbb{C})$ from Exercise 2.3.8 by *column vectors* $\lambda \in (0,1)^k$ such that $m_A \cdot \lambda = 1$. Under this normalization, what does the entry λ_i signify?

(3) Suppose λ_A and λ_B are trace column vectors for A and B respectively with strictly positive entries satisfying $m_A \lambda_A = 1 = n_B \lambda_B$. Prove that $\operatorname{tr}_B|_A = \operatorname{tr}_A$ if and only if $\Lambda_A^B \lambda_B = \lambda_A$.

Exercise 4.6.9. Show that if $A \subseteq B \subseteq C$ are all unital inclusions of multimatrix algebras, then $\Lambda_A^C = \Lambda_A^B \Lambda_B^C$. Describe Γ_A^C in terms of Γ_A^B and Γ_B^C.

Exercise 4.6.10. Let B be a multimatrix algebra. Prove that up to (unitary) conjugation in B, any unital ($*$-)subalgebra $A \subset B$ is completely determined by its Bratteli diagram.

We now show how one can view any semisimple algebra as having a system of matrix units corresponding to loops on a graph as in (TB1) for the TLJ algebras.

Trick 4.6.11 — Expanding on the identification $A \cong \operatorname{Hom}(\mathbb{C} \to A)$ as vector spaces, we can model a multimatrix algebra A as loops on the Bratteli diagram $\Gamma_{\mathbb{C}}^A$ of the inclusion $\mathbb{C} \subset A$. We label the vertex corresponding to $\mathbb{C}$ by $\star$. Loops of length 2 on $\Gamma_{\mathbb{C}}^A$ based at $\star$ with the multiplication rule

$$[\varepsilon_1 \varepsilon_2] \cdot [\varepsilon_3 \varepsilon_4] := \delta_{\varepsilon_2 = \varepsilon_3} [\varepsilon_1 \varepsilon_4]$$

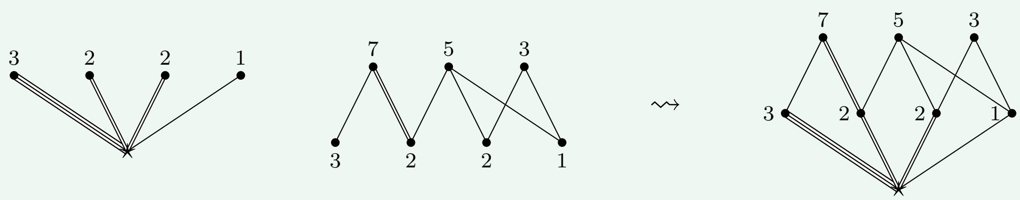
give a concrete realization of A as an algebra of loops on $\Gamma_{\mathbb{C}}^A$. When $A = \bigoplus_{i=1}^k M_{m_i}(\mathbb{C})$, loops of length 2 on $\Gamma_{\mathbb{C}}^A$ based at $\star$ passing through the m_i vertex for A gives a system of matrix units for the $M_{m_i}(\mathbb{C})$ summand of A. In the unitary setting, the adjoint

$$[\varepsilon_1\varepsilon_2]^* := [\varepsilon_2\varepsilon_1]$$

makes this a realization as a $*$-algebra.

Now append the Bratteli diagram Γ_A^B for the inclusion $A \subset B = \bigoplus_{j=1}^{\ell} M_{n_j}(\mathbb{C})$ on top of $\Gamma_{\mathbb{C}}^A$, and recall that $\Lambda_{\mathbb{C}}^B = \Lambda_{\mathbb{C}}^A \Lambda_A^B$. Observe that loops based at $\star$ passing through the n_j vertex for B give a system of matrix units for the $M_{n_j}(\mathbb{C})$ summand of B.

For the Bratteli diagram Γ_A^B in Example 4.6.6 above, appending it on top of $\Gamma_{\mathbb{C}}^A$ yields the following loop model for the inclusion $A \subset B$.

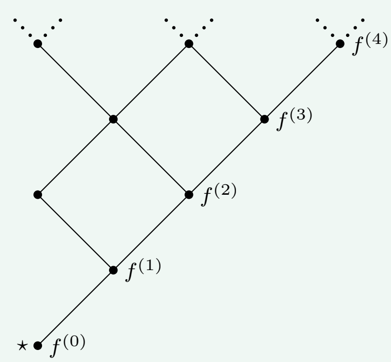


We now reprise Remark 4.4.15 to compute the Bratteli diagrams for the towers of TLJ algebras.

Example 4.6.12 (Generic d) — Suppose d is *generic*, i.e., not a root of unity, so that $[n] \neq 0$ for all $n \in \mathbb{N}$. The Bratteli diagram for the tower of algebras

$$TLJ_\bullet(d) = \Big(TLJ_0(d) \subset TLJ_1(d) \subset TLJ_2(d) \subset \cdots\Big)$$

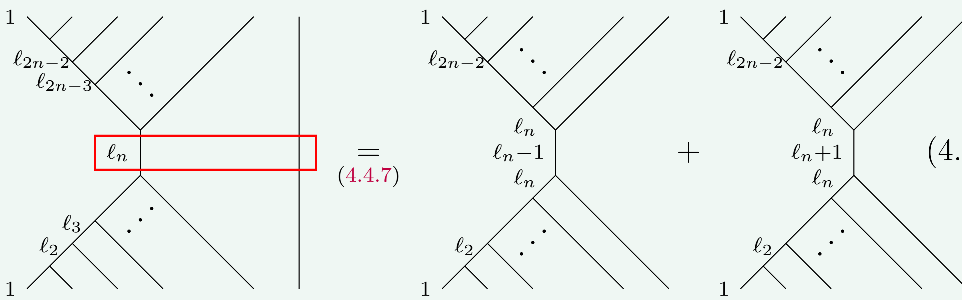
is obtained by taking the A_∞ Coxeter-Dynkin diagram A_∞ and performing the 'reflection' operation from Remark 4.4.15.



The Bratteli diagram for the inclusion $TLJ_n(d) \subset TLJ_{n+1}(d)$ is obtained from the Bratteli diagram for $TLJ_{n-1}(d) \subset TLJ_n(d)$ by reflecting vertically about the vertices corresponding to the simple summands of $TLJ_n(d)$ and adding a new vertex corresponding to $f^{(n+1)}$ connected to the old vertex corresponding to $f^{(n)}$. We see that this is correct by the following observation.

Suppose we have a tree basis diagram with waist $T_\ell \in TLJ_n(d)$ corresponding to a loop of length $2n-2$ based at $\star = 1$ on A_∞ with midpoint ℓ_n. Adding a strand to the right and applying the tree basis diagram version (4.4.7) of Wenzl's Recurrence Relation (4.3.11) produces two tree basis diagrams, one with waist $f^{(k-1)}$ and one with waist $f^{(k+1)}$ (where $f^{(-1)} = 0$ by convention).

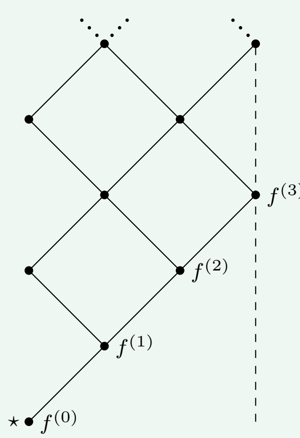
$i_n(T_\ell) =$



(4.6.13)

Observe that the two tree basis diagrams on the right hand side correspond to loops of length $2n$ on A_∞ based at $\star = 1$ with midpoints $\ell_n - 1$ and $\ell_n + 1$.

Example 4.6.14 (Quotients at roots of unity) — We now suppose $[2], \ldots, [n] \neq 0$, but $[n+1] = 0$ so that q is a $(2n+2)$-th root of unity (but not a $2k$-th root of unity for $k \leq n$. The Bratteli diagram for the tower of semisimple algebras $\mathcal{TLJ}_\bullet(d)$ is analogous to Example 4.6.12, except we have quotiented out by the relation $f^{(n)} = 0$. This has the effect of truncating the A_∞ Coxeter-Dynkin diagram to the A_n Coxeter-Dynkin diagram with vertices $0, \ldots, n-1$ corresponding to the non-zero Jones-Wenzl idempotents $f^{(0)}, \ldots, f^{(n-1)}$. Again, the Bratteli diagram is formed by reflecting the previous stage and adding a new vertex corresponding to $f^{(k)}$ if $k \leq n-1$, and we only get the reflection and no new vertex for the higher Jones-Wenzls. For example, when we set $f^{(4)} = 0$,



Again, this can be seen by setting tree basis diagrams which contain a $f^{(n)}$ equal to zero in (4.6.13).

4.7 The Frobenius-Perron theorem

We now recall the famous Frobenius-Perron Theorem.[1]

Theorem 4.7.1 (Frobenius-Perron, part 1) — Suppose $a \in M_n(\mathbb{R}_{\geq 0})$ such that for each $1 \leq i, j \leq n$, there is a $k \in \mathbb{N}$ such that $(a^k)_{ij} > 0$.

(FP1) There is a unique eigenvector $\xi \in \mathbb{R}^n_{>0}$ for a with strictly positive entries up to a multiplicative factor in $\mathbb{R}_{>0}$. The associated eigenvalue d_a is called the *Frobenius-Perron eigenvalue* of a.

(FP2) The Frobenius-Perron eigenspace $E_{d_a} = \ker(a - d_a)$ is one dimensional.

(FP3) The Frobenius-Perron eigenvalue d_a satisfies $d_a \geq |\lambda|$ for all other $\lambda \in \operatorname{spec}(a)$. Hence $d_a = \rho(a)$.

(FP4) If there is a $k \in \mathbb{N}$ such that $(a^k)_{ij} > 0$ for all i, j, then $d_a > |\lambda|$ for all other $\lambda \in \operatorname{spec}(a)$.

Proof. We adapt the proof in [EGNO15, Thm. 3.2.1]. We prove (FP1) in three steps.

(FP1a) There is an eigenvector $\xi \in \mathbb{R}^n_{\geq 0}$ for a with non-negative entries.

Proof. Consider the standard $(n-1)$-simplex

$$\Delta := \left\{ \xi \in \mathbb{R}^n_{\geq 0} \,\middle|\, \sum \xi_i = 1 \right\}.$$

Observe that $a\xi \neq 0$ for all $\xi \in \Delta$. Indeed, if $\xi \in \Delta$, some $\xi_j > 0$, and choosing $k \in \mathbb{N}$ so that $(a^k)_{ij} > 0$ implies $(a^k \xi)_i \geq (a^k)_{ij} \xi_j > 0$. The map $\Delta \to \Delta$ given by

$$\xi \longmapsto \frac{a\xi}{\|a\xi\|_1}$$

is a continuous, and thus has a fixed point. (One can prove this in many ways; some use the Brouwer Fixed Point Theorem. We provide a proof in Theorem 4.A.17 using triangulations and Sperner's Lemma in §4.A below.) This fixed point ξ is an eigenvector for a with eigenvalue $d_a := \|a\xi\| > 0$ as $a\xi = \|a\xi\|\xi$. □

(FP1b) Any eigenvector for a with non-negative entries has strictly positive entries.

[1] This theorem is more commonly called the 'Perron-Frobenius Theorem' in other branches of mathematics, but in the fusion category literature, 'Frobenius-Perron' is the preferred nomenclature.

Proof. Suppose $\eta \in \mathbb{R}^n_{\geq 0}$ is an eigenvector for a with non-negative entries with corresponding eigenvalue $\lambda \geq 0$. As some $\eta_j > 0$ and for every $1 \leq i \leq n$ there is a $k \in \mathbb{N}$ with $(a^k)_{ij} > 0$, $a\eta \neq 0$, and thus $\lambda > 0$. We now compute

$$\lambda^k \eta_i = (a^k \eta)_i = \sum_\ell (a^k)_{i\ell}\eta_\ell \geq (a^k)_{ij}\eta_j > 0.$$

As $\lambda^k > 0$, we conclude each $\eta_i > 0$. $\square$

(FP1c) There is a unique $\xi \in E_{d_a} \cap \mathbb{R}^n_{>0}$ up to scaling by $\mathbb{R}_{>0}$.

Proof. Suppose $\eta \in E_{d_a} \cap \mathbb{R}^n_{>0}$. Set $\alpha := \min \eta_i/\xi_i$, and observe $\eta - \alpha\xi \in E_{d_a}$ has non-negative entries, but at least one entry is equal to zero. By (FP1b), $\eta = \alpha\xi$.

Now suppose we have another eigenvector η for a with strictly positive entries. Then its eigenvalue λ is strictly positive. If $\lambda \leq d_a$, then set $\alpha = \min \eta_i/(2\xi_i)$, and observe $\eta - \alpha\xi \in \mathbb{R}^n_{>0}$. Note that a preserves the cone $\mathbb{R}^n_{>0}$, so $a^k(\eta - \alpha\xi) = \lambda^k\eta - \alpha d_a^k \xi \in \mathbb{R}^n_{>0}$ for all k. This is only possible if $\lambda = d_a$. The case $d_a \leq \lambda$ is similar and left to the reader. $\square$

(FP2) First, if $\eta \in E_{d_a} \cap \mathbb{R}^n$, then for a sufficiently large $r > 0$, $r\xi + \eta \in E_{d_a} \cap \mathbb{R}^n_{>0}$. By (FP1c), $r\xi + \eta = s\xi$ for some $s > 0$, and thus $\eta = (s - r)\xi \in \mathbb{R} \cdot \xi$.

Next, if $\eta \in E_{d_a}$, since a has real entries and $d_a \in \mathbb{R}$, both $\mathrm{Re}(\eta), \mathrm{Im}(\eta) \in \mathbb{R}^n$ lie in E_{d_a}, and thus $\mathrm{Re}(\eta) = r\xi$ and $\mathrm{Im}(\eta) = s\xi$ for some $r, s \in \mathbb{R}$ by the preceding paragraph. Hence $\eta = (r + is)\xi \in \mathbb{C} \cdot \xi$.

(FP3) Consider the norm on $\overline{\mathbb{C}^n}$ given by

$$\|\langle\eta|\,\|_\xi := \sum |\eta_i|\xi_i.$$

For all $\langle\eta| \in \overline{\mathbb{C}}^n$,

$$\|\langle\eta|a\|_\xi = \left|\sum_{i,j} \eta_i a_{ij}\right| \xi_j \leq \sum_{i,j} |\eta_i| a_{ij}\xi_j = d_a \sum_{i,j} |\eta_i|\xi_j = d_a \cdot \|\langle\eta|\,\|_\xi. \tag{4.7.2}$$

If $\langle\eta|$ is a left eigenvector of a associated to left eigenvalue λ, then

$$|\lambda| \cdot \|\langle\eta|\,\|_\xi = \|\langle\eta|a\|_\xi \leq d_a \cdot \|\langle\eta|\,\|_\xi,$$

and thus $|\lambda| \leq d_a$. Since a and $a^T = a^*$ have the same eigenvalues (exercise!), which is equivalent to the statement that a has the same left and right eigenvalues, we see that $d_a \geq |\lambda|$ for all $\lambda \in \mathrm{spec}(a)$.

(FP4) Now suppose for $k \in \mathbb{N}$ sufficiently large, $(a^k)_{ij} > 0$ for all i, j. If we have such a $\lambda \in \operatorname{spec}(a)$ with $|\lambda| = d_a$, then (4.7.2) above is an equality. Moreover, it remains an equality if we replace a with a^k. This is only possible if for each j,

$$\left| \sum_{i,j} \eta_i (a^k)_{ij} \right| = \sum_{i,j} |\eta_i| (a^k)_{ij},$$

which implies that $\eta_i = \alpha |\eta_i|$ for a single phase $\alpha \in U(1)$ *independent* of i. Rotating by $\overline{\alpha}$, we may thus assume that $\eta_i \geq 0$ for all i, so that $\lambda > 0$ and thus $\lambda = d_a$. □

Corollary 4.7.3 (Frobenius-Perron, part 2) — Suppose $a \in M_n(\mathbb{R}_{\geq 0})$. There is an eigenvector $\xi \in \mathbb{R}^n_{\geq 0}$ for a with eigenvalue $\rho(a)$, the spectral radius. Moreover, if there is an eigenvector $\xi \in \mathbb{R}_{>0}$ for a, then its eigenvalue is $\rho(a)$.

Proof. We again adapt the proof in [EGNO15, Thm. 3.2.1].

Consider $a_\varepsilon := a + \varepsilon \mathbf{1}$, where $\mathbf{1}$ is the matrix consisting of all ones. By the Frobenius-Perron Theorem 4.7.1, $\rho(a_\varepsilon)$ is an eigenvalue of a_ε, for which there is a $\|\cdot\|_1$-unit eigenvector ξ_ε with strictly positive entries. Since the spectral radius function $\rho : M_n(\mathbb{C}) \to [0, \infty)$ is continuous, one can produce an eigenvector $\xi \in \mathbb{R}^n_{\geq 0}$ for a with eigenvalue $\rho(a)$ by taking limits as $\varepsilon \searrow 0$. This can be proved in many ways; one basic approach is via a sequential compactness argument applied to the compact set $f^{-1}(0)$ for the continuous function

$$f : [0,1] \times \Delta \longrightarrow \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad (\varepsilon, \eta) \mapsto \|(a_\varepsilon - \rho(a_\varepsilon))\eta\|_1.$$

With more work, one can arrange that $\xi_\varepsilon \to \xi$ as $\varepsilon \searrow 0$.

For the final statement, we apply the first part to a^T to obtain a left eigenvector $\langle \eta | \in \mathbb{R}^n_{\geq 0}$ for a with eigenvalue $\rho(a^T) = \rho(a)$. If $\xi \in \mathbb{R}^n_{>0}$ so that $a\xi = \lambda \xi$, then $\langle \eta | \xi \rangle > 0$ and

$$\lambda \langle \eta | \xi \rangle = \langle \eta | a\xi \rangle = \rho(a) \langle \eta | \xi \rangle \qquad \Longrightarrow \qquad \lambda = \rho(a).$$ □

Example 4.7.4 — Expanding on Exercise 4.6.7, for the adjacency matrix A of a finite connected graph Γ, $(A^k)_{uv}$ equals the number of paths of length k on Γ from u to v. The condition that for all u, v, there is a k such that $(A^k)_{uv} > 0$ is exactly the condition that Γ is strongly connected.

Definition 4.7.5 — Suppose Γ is a finite strongly connected graph with a distinguished vertex $\star \in \Gamma$, and denote its adjacency matrix by A. For $v \in \Gamma$, the *Frobenius-Perron dimension* d_v is the entry of the Frobenius-Perron eigenvector for A normalized so that $d_\star = 1$. We define the *norm* $\|\Gamma\|$ of Γ to be the norm if its adjacency matrix A_Γ, which by Theorem 4.7.1 is the Frobenius-Perron eigenvalue of A_Γ. Writing $v \sim w$ to denote that vertices v, w in Γ are neighbors (share at least one edge), we may write the

Frobenius-Perron eigenvector criterion as

$$\|\Gamma\| \cdot d_v = \sum_{w \sim v} A_{vw} d_w \tag{4.7.6}$$

where the above sum is taken with *multiplicity*, i.e., we count w once for each edge connecting it to v.

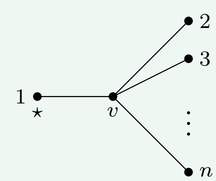
Trick 4.7.7 — While one can calculate the Frobenius-Perron dimensions of a finite connected graph using linear algebra, here are some ways to make the calculation easier in practice for small graphs. Note that we may pick *any* basepoint $\star$ and set the value of the Frobenius-Perron eigenvector to be $d_\star := 1$. Thus a wise choice of $\star$ can make the rest of the calculation trivial. In particular, we have the following two helpful tricks.

(1) We may choose $\star$ to exploit symmetries of Γ: by uniqueness of $d_\bullet$, the vector $d_\bullet$ must be invariant under any graph symmetry.

(2) If we may choose $\star$ to be univalent, then labeling its unique neighbor by v, the Frobenius-Perron criterion (4.7.6) applied to $\star$ immediately yields

$$\|\Gamma\| = \|\Gamma\| \cdot d_\star = d_v.$$

As a concrete example, consider the n-star which has n univalent vertices and one n-valent vertex.



Choosing any univalent vertex to be $\star$, we see that all univalent vertices must have Frobenius-Perron dimension equal to 1 by symmetry. Labeling the n-valent vertex by v, (4.7.6) at $\star$ implies $\|\Gamma\| = d_v$. Finally, (4.7.6) at v implies

$$n = \sum_{w \sim v} d_w \underset{(4.7.6)}{=} \|\Gamma\| \cdot d_v = d_v^2$$

so that $\|\Gamma\| = d_v = \sqrt{n}$.

Exercise 4.7.8. Suppose Γ is a connected bipartite graph with bipartite adjacency matrix Λ.

(1) Observe that $\Lambda^T\Lambda$ satisfies the hypotheses of the Frobenius-Perron Theorem 4.7.1, so there is a vector ξ with strictly positive entries such that $\Lambda^T\Lambda\xi = \|\Lambda\|^2\xi$.

(2) Setting $\eta := \Lambda\xi$, deduce that $\Lambda^T\xi = d_A^2\eta$ and $\begin{pmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \|\Lambda\|\eta \end{pmatrix} = \|\Lambda\| \begin{pmatrix} \xi \\ \|\Lambda\|\eta \end{pmatrix}$.

Example 4.7.9 — Quantum integers at $q = \exp(\pi i/n) \in Q$ give Frobenius-Perron dimensions for the A_{n-1} Coxeter-Dynkin diagram.

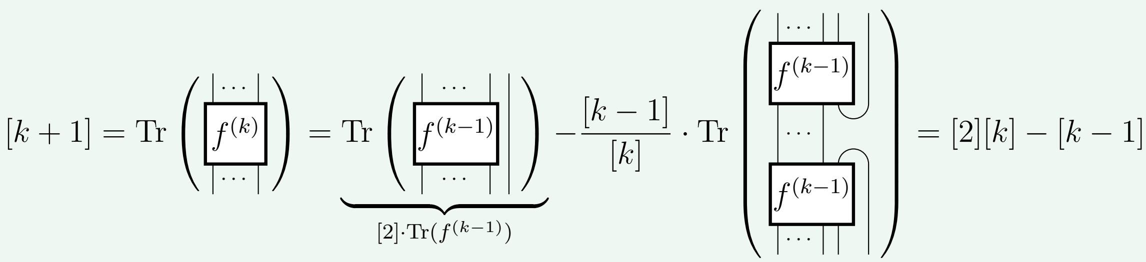
Indeed, we calculate that for each $k = 1, \ldots, n$,

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{e^{k\pi i/n} - e^{-k\pi i/n}}{e^{\pi i/n} - e^{-\pi i/n}} = \frac{\sin(k\pi/n)}{\sin(\pi/n)} \geq 0.$$

(Compare with Lemma 4.3.8.) Furthermore, $[n-k] = [k]$ so that $[n-2] = [2]$, $[n-1] = 1$, and $[n] = 0$. One now verifies the Frobenius-Perron condition by observing that at the k-th vertex, the sum of the quantum dimensions of the neighboring vertices $k-1, k+1$ equals the product of the quantum dimension at the k-th vertex by [2]:

$$[k-1] + [k+1] = [2][k].$$

Thus $d_k := [k]$ is a Frobenius-Perron eigenvector and $\|A_{n-1}\| = [2] = 2\cos(\pi/n)$. We also see this formula by taking the trace of the Wenzl Recurrence Relation (4.3.11):



Proposition 4.7.10 — If Λ is a proper strongly connected subgraph of the finite strongly connected graph Γ, then $\|\Lambda\| < \|\Gamma\|$.

Proof. Given any unit vector $\xi \in \mathbb{C}^n$ and a self-adjoint $a \in M_n(\mathbb{C})$, we always have

$$-\|a\| \leq \langle \xi | a\xi \rangle \leq \|a\|.$$

Thus the result will follow if we can exhibit a unit vector ξ such that

$$\|\Lambda\| < \langle \xi | A_\Gamma \xi \rangle.$$

Indeed, let λ be the Frobenius-Perron unit eigenvector for Λ, and let ξ be the vector obtained from λ by adding zeroes in the new entries corresponding to vertices in $\Gamma \setminus \Lambda$. By construction, ξ is a unit vector. Since $\Lambda \subset \Gamma$ is a proper subgraph and Γ is strongly connected, $(A_\Lambda)_{uv} \leq (A_\Gamma)_{uv}$ for all $u, v \in \Lambda$, and at least one of the following two situations must occur:

(1) there are vertices $u, v \in \Lambda$ with $(A_\Lambda)_{uv} < (A_\Gamma)_{uv}$, or

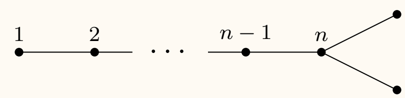
(2) there are vertices $u \in \Lambda$ and $v \in \Gamma \setminus \Lambda$ such that $(A_\Gamma)_{uv} > 0$.

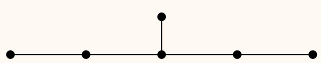
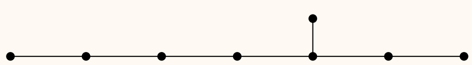
We thus conclude that

$$\|\Lambda\| = \|\|\Lambda\|\langle\lambda|\lambda\rangle = \langle\lambda|A_\Lambda\lambda\rangle = \sum_{u,v\in\Lambda} \lambda_u (A_\Lambda)_{uv}\lambda_v < \sum_{v,w\in\Gamma} \xi_u (A_\Gamma)_{uv}\xi_v = \langle\xi|A_\Gamma\xi\rangle. \qquad \square$$

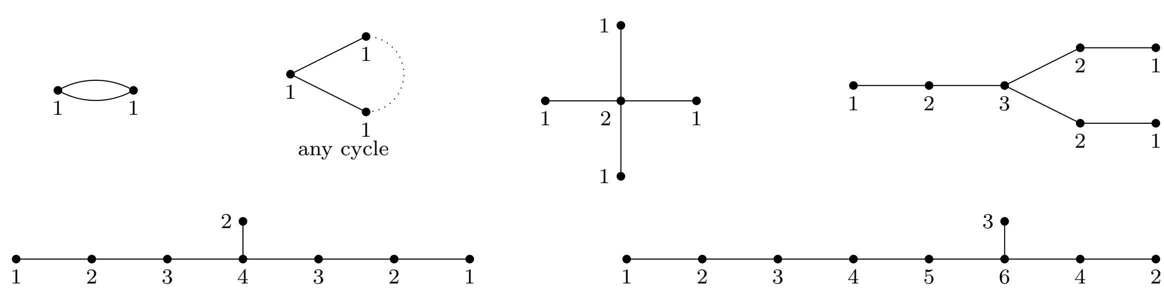
Corollary 4.7.11 ([GdlHJ89, Thm. 1.4.3]) — The connected finite bipartite graphs with Frobenius-Perron eigenvalue strictly less than 2 are:

- The type A_n Coxeter-Dynkin diagrams 

- The type D_{n+2} Coxeter-Dynkin diagrams 

- $E_6 =$ 
- $E_7 =$ 
- $E_8 =$ 

Proof. The Frobenius-Perron eigenvalues of the following graphs known as *extended/affine Coxeter-Dynkin diagrams* are equal to 2, where we have labeled the nodes below by entries of a Frobenius-Perron eigenvector (as opposed to just counting the vertices as in the statement of the corollary).



By Proposition 4.7.10, any bipartite graph with Frobenius-Perron eigenvalue less than 2 cannot contain any of these graphs as subgraphs. We leave the rest of the details to the reader. $\square$

Exercise 4.7.12. Adapt the proof of Corollary 4.7.11 to find all finite connected bipartite graphs with Frobenius-Perron eigenvalue equal to 2. Can you find all the infinite ones as well?

4.8 Braid groups, the Jones polynomial, and the Kauffman bracket

In this section, we show how the TLJ algebras give a polynomial invariant of knots and links. A *knot*, which is an embedding $S^1 \hookrightarrow \mathbb{R}^3$, is inherently a 3-dimensional object. A *link* is an embedding $\coprod_{i=1}^{n} S^1 \hookrightarrow \mathbb{R}^3$ where $n \in \mathbb{N}$, i.e., a disjoint union of knots, which can be knotted amogst themselves.

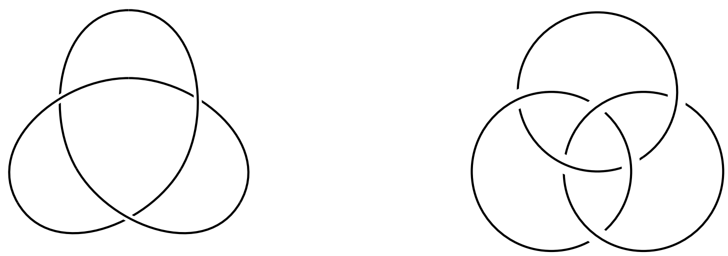
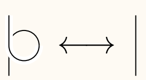
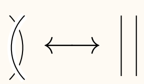


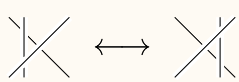
Figure 4.1: Diagram of a knot (trefoil) and of a link (borromean rings) respectively

When we work with knots on paper, a chalkboard, or a screen, we typically represent a knot via a *knot/link projection*, its image in $\mathbb{R}^2$ under a *generic regular projection*, which avoids various bad behaviors, like triple intersections and kinks. The following classical theorem of Reidemeister, whose proof we omit, characterizes when two knot projections give the same knot.

Theorem 4.8.1 ([Rei27]) — Two knot/link projections represent isotopic (equivalent) knots in $\mathbb{R}^3$ if and only if they are related by a finite number of the *Reidemeister moves*:

(R1) 

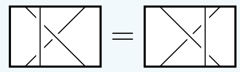
(R2) 

(R3) 

(There are several (R1), (R2), and (R3) relations; we only list one for each type above to simplify the exposition.)

Knots and links are closely related to *braids* in $\mathbb{R}^3$.

Definition 4.8.2 — The diagrammatic braid group DB_n is the group whose elements consist of string diagrams with n boundary points on the lower and upper sides of a rectangle, and the lower points are paired to the upper points by smooth strings which only intersect at a finite number of points, where we indicate which string passes over the other as in a knot/link projection. Moreover, the strings are not allowed to have any critical points. All such diagrams are considered up to isotopy and Reidemeister moves (R2) and (R3) (but not (R1)!). For example, the following elements of DB_3 are equal:



We multiply in DB_n by stacking boxes and smoothing out strings, similar to multiplication in TLJ_n, which is manifestly associative.

Exercise 4.8.3. Prove that DB_n is a group under the above multiplication. That is, find the identity element, and show every element has an inverse.

Just as the TLJ algebras afford both an algebraic and diagrammatic description, so do the braid groups.

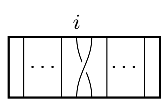
Definition 4.8.4 — The algebraic braid group AB_n is the group generated by $\beta_1, \ldots, \beta_{n-1}$ subject to the relations

(B1) $\beta_i\beta_j = \beta_j\beta_i$ for $|i-j| > 1$ and

(B2) $\beta_i\beta_{i\pm1}\beta_i = \beta_{i\pm1}\beta_i\beta_{i\pm1}$.

Exercise 4.8.5. Show that AB_2 is isomorphic to $\mathbb{Z}$, but that AB_3 contains a subgroup isomorphic to the free group $\mathbb{F}_2$.

Exercise 4.8.6. Consider the distinguished elements of DB_n given by

$b_i :=$ 

.

Prove that the elements $b_1, \ldots, b_{n-1} \in DB_n$ satisfy Relations (B1) and (B2). Deduce there is a well-defined group homomorphism $\Phi_n : AB_n \to DB_n$.

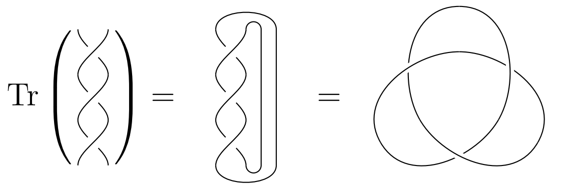
Exercise 4.8.7. Show that every element of DB_n can be written as a product of $b_1, \ldots, b_{n-1}$ from Exercise 4.8.6. Deduce that Φ_n from Exercise 4.8.6 is surjective.

We will not prove the following theorem as it would take us too far afield.

Theorem 4.8.8 ([Art25]) — The group homomorphism $\Phi_n : AB_n \to DB_n$ from Exercise 4.8.6 is an isomorphism.

Notation 4.8.9 — From this point forward, we simply write B_n to denote either AB_n or DB_n, which we identify under the group isomorphisms Φ_n.

From a braid $b \in B_n$, we obtain a link by taking its *closure/trace* $\mathrm{Tr}_n(b)$, similar to the diagrammatic trace on $TLJ_n(d)$.



Markov's Theorem (whose proof is omitted) characterizes when two braids give the same link under taking the trace.

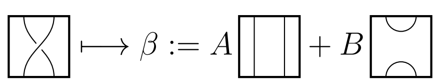
Theorem 4.8.10 (Braid closure, [Mar35]) — Every link is the closure of a braid. Moreover, two braids give the same link under closure if and only if they are related by a finite number of the following two moves:

(M1) If $b \in B_n$, we can swap $b \longleftrightarrow aba^{-1}$ for some braid $a \in B_n$.

(M2) If $b \in B_n$, we can swap $b \longleftrightarrow i_n(b)\beta_n^{\pm 1}$, the n-th generator of B_{n+1}, where we include $i_n : B_n \hookrightarrow B_{n+1}$ by adding a strand to the right.

Exercise 4.8.11. Prove that we get the same link under taking the closure of a braid under either (M1) or (M2).

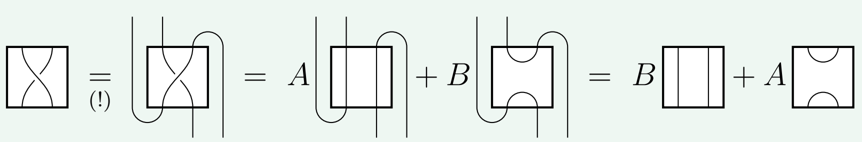
Now looking at the striking resemblance between the braid relations (B1),(B2) and two of the TLJ relations (J2),(J3) respectively, it is natural to wonder if we can find a family of maps $B_n \to TLJ_n(d)$ which is compatible with the inclusions $B_n \hookrightarrow B_{n+1}$ and $TLJ_n(d) \hookrightarrow TLJ_{n+1}(d)$ by adding a string to the right or left. Any such family of maps would be completely determined by



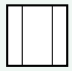
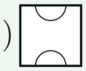
(4.8.12)

for some $A, B \in \mathbb{C}$. Observe that (B1) is always satisfied. One must check that β is invertible together with (B2). To check the first, we can use the following trick to guess and check the answer.

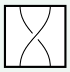
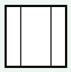
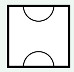
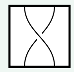
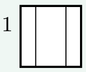
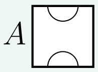
Trick 4.8.13 — While we cannot bend strings in B_n, we can bend strings in $TLJ_n(d)$. Rotating β by one strand should give us the inverse of β if it exists:



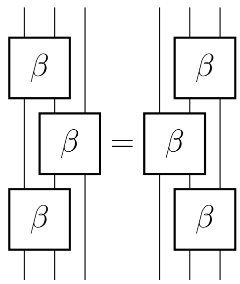
Using this guess for β^{-1}, we multiply it by β to get

$\beta\beta^{-1} = AB$  $+ (A^2 + B^2 + dAB)$  $= \beta^{-1}\beta\,.$

To set this equal to id_2, we require $B = A^{-1}$ and $d = -A^2 - A^{-2}$ so that

 $= \beta = A$  $+ A^{-1}$   $= \beta^{-1} = A^{-1}$  $+ A$  $.$ (4.8.14)

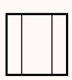
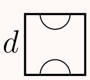
Exercise 4.8.15. Defining β, β^{-1} as in (4.8.14), prove that relation (B2) is satisfied, i.e.,



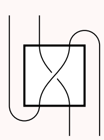
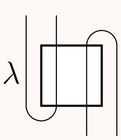
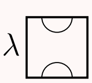
Warning 4.8.16 — Recall from Lemma 4.3.4 that $[n]$ for $TLJ_n(d)$ only depended on d and not q. However, when $d = -A^2 - A^{-2}$, defining β, β^{-1} as in (4.8.14) not only depends on q, but also a choice of square root of q! We deduce there are exactly 4 choices for β for a fixed d: first choose q, for which there are 2 choices, and then choose a square root of q, for which there are 2 choices, so that

$$A = \pm i q^{1/2} \qquad \text{or} \qquad A = \pm i q^{-1/2}.$$

However, when $d = \pm 1$ (so $d = d^{-1}$), these four choices in the above warning are actually only two choices! Observe that the two diagrams on the right hand side of (4.8.12) are proportional in the semisimple quotient $\mathcal{TLJ}_2 \cong \mathbb{C}$. In this case,

 $= d$  and $\beta = \lambda\,\mathrm{id}_2$

for some $\lambda \in \mathbb{C}$. The rotation argument above implies that

$\lambda^{-1}\,\mathrm{id}_2 = \beta^{-1} =$  $= \lambda$  $= \lambda$  $= d\lambda\,\mathrm{id}_2\,.$

We conclude that $\lambda^2 = d$, so there are exactly 2 choices: $\beta = \pm\,\mathrm{id}_2$ if $d = 1$ and $\beta = \pm i\,\mathrm{id}_2$ if $d = -1$.

Proposition 4.8.17 — When $TLJ_n(d)$ is a unitary algebra (see (TB6) and ($\mathcal{TB}$6)), β is unitary if and only if q is a root of unity.

Proof. By Jones' Modulus Restriction Theorem 4.3.17 and Exercise 4.3.18, it suffices to prove q is unimodular. By (4.8.14), β is unitary if and only if $\overline{A} = A^{-1}$, whence the result. $\square$

The above discussion proves the following theorem.

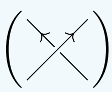
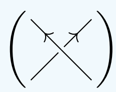
Theorem 4.8.18 — The map $\Psi_n : B_n \to TLJ_n(d)$ given by

$$\beta_i \mapsto A\,\mathrm{id}_n + A^{-1}E_i \qquad\qquad \beta_i^{-1} \mapsto A^{-1}\,\mathrm{id}_n + AE_i$$

where $d = -A^2 - A^{-2}$ preserves (B1) and (B2) and thus gives a well-defined group homomorphism. Moreover, the family of maps $\{\Psi_n\}$ is compatible with the inclusions $B_n \hookrightarrow B_{n+1}$ and $TLJ_n(d) \hookrightarrow TLJ_{n+1}(d)$ under adding strands to the right and left.

We now go through Jones' original construction of his famous polynomial invariant of knots and links using Markov traces on TLJ algebras [Jon85].[2] We then connect it to the more common skein theoretic definition via the Kauffman bracket [Kau87].

Definition 4.8.19 — Let $\vec{\ell}$ be an *oriented* link. For each crossing in a projection of $\vec{\ell}$, we define the *sign* of the crossing as follows:

sign  $:= +1$ sign  $:= -1$

We define the *writhe factor* $\mathrm{wr}(\vec{\ell})$ to be the number of crossings, counted with their signs.

Exercise 4.8.20. For a braid $b \in B_n$, define the *exponent sum* $\exp(b)$ as the sum of the exponents in any expression of b as a word in $\beta_1, \ldots, \beta_{n-1}$.

(1) Show that (B1) and (B2) preserve the exponent sum. Deduce $\exp(b)$ is well-defined.

(2) Prove that $\exp(b)$ is exactly the writhe factor of $\mathrm{Tr}(\vec{b})$, where $\vec{b}$ is the *oriented* braid obtained from b by orienting all strands from bottom to top.

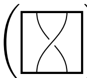
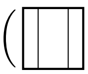
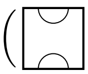
Definition 4.8.21 — Suppose $\vec{\ell}$ is an oriented link. Write $\vec{\ell} = \mathrm{Tr}(\vec{b})$ for some braid $b \in B_n$. The *Jones polynomial* is

$$V_{\vec{\ell}}(A) := d^{-1}(-A^3)^{-\exp(b)} \cdot \mathrm{Tr}_{TLJ_n(d)}(\Psi(b)) \tag{4.8.22}$$

[2]The name 'Markov' appears twice in the development of the Jones polynomial. Markov traces are named after Markov processes, due to A.A. Markov (1856-1922). Markov's Braid Closure Theorem 4.8.10 is due to A.A. Markov (1903-1979). They are father and son.

where $d = -A^2 - A^{-2}$, $\exp(b)$ is the exponent sum, and $\Psi_n : B_n \to TLJ_n(d)$ is the group homomorphism from Theorem 4.8.18.

Remark 4.8.23. The formula for $V_{\vec{\ell}}$ may appear cryptic to the reader at this point. e normalize $V_{\vec{\ell}}(A)$ by d^{-1} so that $V_{\vec{\ell}}(\mathsf{unknot}) = 1$. The factor $(-A^3)^{-\exp(b)}$ is included to remedy the fact that $d^{-1} \cdot \mathrm{Tr}_{TLJ_n(d)}$ is not invariant under (R1) as

$d^{-1} \cdot \mathrm{Tr}\Big(\Psi\Big($  $\Big)\Big) = Ad^{-1} \cdot \mathrm{Tr}\Big($  $\Big) + A^{-1}d^{-1} \cdot \mathrm{Tr}\Big($  $\Big)$

$$= A(-A^2 - A^{-2}) + A^{-1} = -A^3.$$

Theorem 4.8.24 — The formula (4.8.22) for $V_{\vec{\ell}}$ is well-defined, i.e., it does not depend on the choice of b.

Proof. By Theorem 4.8.10, it suffices to show that (4.8.22) is invariant under the Markov moves (M1) and (M2).

(M1): This is immediate from $\exp(a^{-1}) = -\exp(a)$ for all $a \in B_n$, together with the facts that Ψ is a homomorphism and Tr is a trace:

$$\mathrm{Tr}(\Psi(aba^{-1})) = \mathrm{Tr}(\Psi(a)\Psi(b)\Psi(a)^{-1}) = \mathrm{Tr}(\Psi(a)^{-1}\Psi(a)\Psi(b)) = \mathrm{Tr}(\Psi(b)).$$

(M2): We prove that $B_n \ni b \leftrightarrow b\beta_n \in B_{n+1}$ does not change (4.8.22), and the proof for $b \leftrightarrow b\beta_n^{-1}$ is similar. Note that $\exp(b\beta_n) = 1 + \exp(b)$. Expanding $\Psi(\beta_n) = A\,\mathrm{id}_n + A^{-1}E_n$, we have

$$\begin{aligned}(-A^3)^{-\exp(b\beta_n)} \cdot \mathrm{Tr}_{n+1}(\Psi(b\beta_n)) &= (-A^3)^{-1-\exp(b)} \cdot (A\,\mathrm{Tr}_{n+1}(\Psi(b)) + A^{-1}\,\mathrm{Tr}_n(\Psi(b)E_n)) \\ &= (-A^3)^{-1-\exp(b)} \cdot (Ad + A^{-1}) \cdot \mathrm{Tr}_n(\Psi(b)) \\ &= (-A^3)^{-1-\exp(b)} \cdot (-A^3) \cdot \mathrm{Tr}_n(\Psi(b)) \\ &= (-A^3)^{-\exp(b)} \cdot \mathrm{Tr}_n(\Psi(b)).\end{aligned}$$

This completes the proof. □

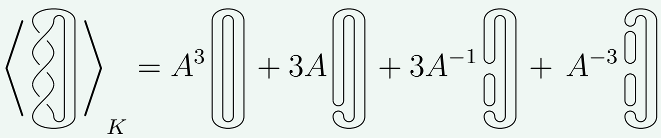
Kauffman gave another construction of an invariant which is essentially the Jones polynomial (minus the write factor) using *skein theory*, similar to how we analyzed the semisimple quotients of the TLJ algebras at roots of unity. The basic idea is that we replace crossings in a link locally using (4.8.14), leaving a disjoint union of closed loops, which is a multiplicative factor of a power of d.

Definition 4.8.25 — Given a link ℓ, we define an element $\langle \ell \rangle_K \in \mathbb{C}(A)$ (Laurent polynomials in A, A^{-1}) called the *Kauffman bracket* of ℓ by replacing the crossings by $\beta^{\pm 1}$ as in (4.8.14).[a] By (4.8.14) and Exercise 4.8.15, we see that $\langle \ell \rangle_K$ is *invariant* under applying (R2) and (R3) to ℓ anywhere locally. Thus the Kauffman bracket is almost an invariant of knots and links, modulo (R1).

[a]This differs from Kauffman's original definition of the bracket polynomial by a normalization.

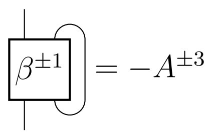
Kauffman normalized so that the uknot has bracket equal to 1, whereas we normalize so that the unknot has bracket equal to d.

Example 4.8.26 — We calculate the Kauffman bracket of a trefoil knot as follows:



$$= -A^{-9} + A^{-1} + A^3 + A^7.$$

This proves a trefoil is not isotopic to its mirror image ($\beta \leftrightarrow \beta^{-1}$, $A \leftrightarrow A^{-1}$).

Exercise 4.8.27. Show that 

. Deduce $\langle \ell \rangle_K$ is not invariant under (R1).

Exercise 4.8.28. Let $\vec{\ell}$ be an oriented link and let ℓ be the link obtained from forgetting the orientation. Show that

$$V_{\vec{\ell}}(A) := d^{-1}(-A)^{-3\,\mathrm{wr}(\vec{\ell})} \cdot \langle \ell \rangle_K \tag{4.8.29}$$

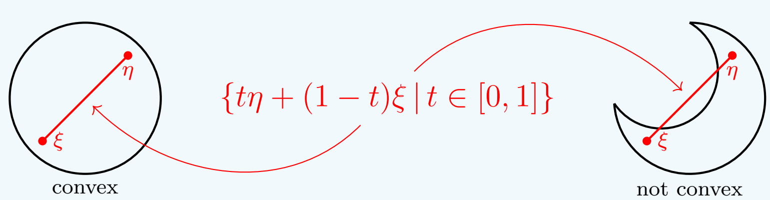
is invariant under (R1), (R2), and (R3). Then verify (4.8.29) agrees with (4.8.22).

Remark 4.8.30. The discovery of the Jones polynomial led to the profusion of quantum invariants of knots, links, and 3-manifolds from quantum groups and braided tensor categories. We will study the latter in Part[[II]] §[[?]]. Its discovery via operator algebras earned Vaughan Jones the Fields Medal in 1990. He gave a plenary lecture at the 1990 International Congress of Mathematicians in a New Zealand All Blacks rugby jersey.

4.A Convexity, simplices, and a fixed point theorem

In this section, V is a vector space over $\mathbb{R}$ or $\mathbb{C}$.

Definition 4.A.1 — A subset $K \subset V$ is called *convex* if for any $\eta, \xi \in K$, $t\eta + (1-t)\xi \in K$ for all $t \in [0, 1]$.

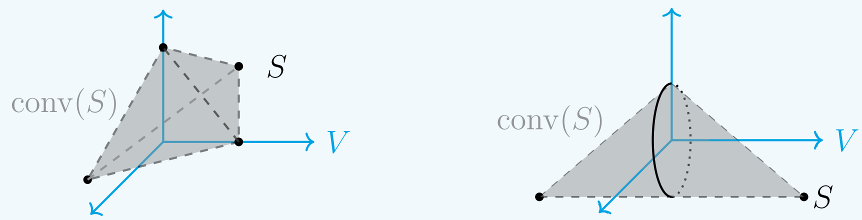


An *extreme point* of a convex set K is a point $\xi \in K$ such that

$$\xi = t\eta + (1-t)\zeta \text{ for some } t \in [0,1] \qquad \Longleftrightarrow \qquad \xi = \eta = \zeta.$$

We denote the set of extreme points of K by $\operatorname{ext}(K)$. Given a subset $S \subset V$, the *convex hull* of S is

$$\operatorname{conv}(S) := \left\{ \sum_{i=1}^{n} t_i s_i \,\middle|\, n \in \mathbb{N},\ \text{each } t_i \in [0,1] \text{ with } \sum t_i = 1,\ \text{and each } s_i \in S \right\}.$$



Exercise 4.A.2. Prove that $\operatorname{conv}(S)$ is convex.

Exercise 4.A.3. Find a compact convex subset $K \subset \mathbb{R}^3$ such that $\operatorname{ext}(K)$ is not closed.

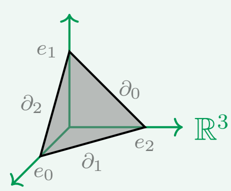
Exercise 4.A.4. Prove that if V, W are vector spaces, $T : V \to W$ is linear, and $K \subset V$ is convex, then $TK \subset W$ is convex. Deduce that if T is injective, then T maps extreme points to extreme points.

Example 4.A.5 — The *standard n-simplex* is

$$\Delta^n := \left\{ \xi = (\xi_0, \ldots, \xi_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} \xi_i = 1 \text{ and } \xi_i \geq 0 \text{ for all } i \right\}.$$

For $0 \leq i \leq n$, the *i-th face* of Δ^n is $\partial_i \Delta^n := \{\xi = (\xi_0, \ldots, \xi_n) \in \Delta^n \,|\, \xi_i = 0\}$.

Δ^2 :



Exercise 4.A.6. Consider the map $d_i : \Delta^{n-1} \hookrightarrow \Delta^n$ which inserts 0 as the i-th coordinate, which is an isometry onto $\partial_i \Delta^n$. Prove that $d_j d_i = d_i d_{j-1} : \Delta^{n-1} \hookrightarrow \Delta^{n+1}$ for all $0 \leq i < j \leq n+1$.

There are several reasonable notions of an n-simplex in a space, depending on the structure of the space.

Definition 4.A.7 — An *n-simplex* in a *topological space* X is the image of Δ^n under a topological embedding $\Delta^n \hookrightarrow X$, i.e. $\mathrm{im}(\Delta^n) \cong \Delta^n$ are homeomorphic as spaces.

An *n-simplex* in a *vector space* V over $\mathbb{R}$ or $\mathbb{C}$ is an image of Δ^n under an injective linear map $\mathbb{R}^n \hookrightarrow V$.

Remark 4.A.8. Since linear maps from $\mathbb{R}^{n+1}$ are continuous with respect to any norm on V, n-simplices in V are also n-simplices with respect to any norm topology on V. However, since the map is linear, an n-simplex in V is automatically convex by Exercise 4.A.4.

Definition 4.A.9 — A *triangulation* T of a topological space X is a finite collection T_n of n-simplices contained in X for every $n \geq 0$ such that:

- (cover) Every $x \in X$ is in an n-simplex $S \in T_n$ for some $n \geq 0$.
- (closed) If $S \in T_n$ is an n-simplex, then its faces $\partial_i S$ are distinct simplices in T_{n-1}.
- (disjoint) For distinct $S_1, S_2 \in T_n$, their interiors are disjoint, and their intersection is a lower dimensional face of both S_1 and S_2. In particular, $S_1 \cap S_2 \in T_k$ for some $k < n$.

Exercise 4.A.10. Show that finiteness of T_0 implies that $T_n = \emptyset$ for n sufficiently large.

Exercise 4.A.11. Suppose X has a triangulation T. Prove that for each $x \in X$, either $x \in T_0$ or there exists a unique n-simplex $S \in T_n$ whose interior contains x.

Exercise 4.A.12. In this exercise, we show that a topological space X with a triangulation T is the *topological realization* of a *simplicial complex*. We write

$$\mathcal{T}_n \coloneqq \{\mathrm{ext}(S) \mid S \in T_n\} \qquad \text{and} \qquad \mathcal{T} = \bigcup_n \mathcal{T}_n,$$

and note that $\mathcal{T}_0 \coloneqq \{\{x\} \mid x \in T_0\}$.

(1) Prove that $S \mapsto \mathrm{ext}(S)$ is a bijection $T_n \to \mathcal{T}_n$.

(2) Prove that for all $S_1, S_2 \in T$, $\mathrm{ext}(S_1) \cap \mathrm{ext}(S_2) = \mathrm{ext}(S_1 \cap S_2)$ and $S_1 \subseteq S_2$ if and only if $\mathrm{ext}(S_1) \subset \mathrm{ext}(S_2)$.

(3) Show that X is the union of the convex hulls of the elements of $\mathcal{T}$. We may thus reconstruct X by inductively gluing copies of Δ^n for each $\mathrm{ext}(S) \in \mathcal{T}_n$ along the $(n-1)$-dimensional boundary of S.

Example 4.A.13 — A triangulation T of $[0, 1]$ consists of:

(0) a finite collection of points

$$T_0 = \{0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = 1\},$$

(1) the 1-simplices are uniquely determined by T_0 and must be

$$T_1 = \{[t_i, t_{i+1}]\}_{i=0}^{k};$$

(n) there are no higher n-simplices i.e. $T_n = \emptyset$ for $n \geq 2$.

Observe that we may identify $[0,1]$ with the standard 1-simplex Δ^1 under the bijective linear map

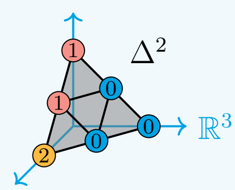
$$[0,1] \ni t \longmapsto (t, (1-t)) \in \Delta^1.$$

Exercise 4.A.14. Show that if T is a triangulation of Δ^n, then $T_k = \emptyset$ for all $k > n$.

Definition 4.A.15 — Let T be a triangulation of the standard 1-simplex Δ^1. A *Sperner coloring* of Δ^1 is an assignment of the *colors* $0, 1$ to the vertices of T such that the extreme points of Δ^1 are assigned distinct colors.

Now consider a triangulation T of the standard n-simplex Δ^n. We define a *Sperner coloring* of (Δ^n, T) is an assignment of the *colors* $0, \ldots, n$ to the vertices of T satisfying the following co-inductive requirement:

- the extreme points of Δ^n are assigned distinct colors,
- The coloring restricted to the triangulation of every face $\partial_i \Delta^n$ of Δ^n is a Sperner coloring of that face, omitting the color assigned to the extreme point $e_i \in \Delta^n$.



Lemma 4.A.16 (Sperner) — Suppose T is a triangulation of Δ^n equipped with a Sperner coloring. There is an odd number of n-simplices in T whose extreme points have distinct colors.

Proof. We proceed by induction on n. If $n = 1$, since the color changes from 0 to 1, there are an odd number of subintervals with distinctly colored vertices.

Suppose that the result holds for any triangulation of Δ^{n-1}. Form a graph whose vertices are the n-simplices in T together with one vertex corresponding to the exterior of Δ^n such that vertices u, v are connected by an edge if and only if the edge passes through a single $(n-1)$-simplex in T whose coloring contains $1, \ldots, n$, but not 0. We have the following immediate observations.

- The vertices internal to Δ^n have valence at most 2 in Γ.

- The vertices internal to Δ^n with valence 1 have distinctly colored extreme points.
- Any connected component of Γ whose vertices are all internal to Δ^n has an even number of valence 1 vertices.
- The exterior vertex only connects to vertices inside Δ^n through $(n-1)$-simplices contained in the face $\partial_0\Delta^n$. By the inductive hypothesis, there are an odd number of $(n-1)$-simplices in $\partial_0\Delta^n$ whose coloring contains $1,\ldots,n$, but not 0, so the exterior vertex has odd valence.
- Since connected graphs always have an even number of vertices with odd valence, the connected component of Γ containing the external vertex contains an odd number of internal vertices with odd valence.

We conclude that there are an odd number of n-simplices in T whose extreme points have distinct colors. □

Theorem 4.A.17 — Any continuous map $f : \Delta^n \to \Delta^n$ has a fixed point.

Proof. For every $\xi \in \Delta^n$, since

$$\sum f(\xi)_i = 1 = \sum \xi_i, \tag{4.A.18}$$

there is a minimal index $0 \le j \le n$ such that $f(\xi)_j \le \xi_j \ne 0$; call this index $j(\xi)$.

Claim 1: For any triangulation T of Δ^n, the map $\xi \mapsto j(\xi)$ is a Sperner coloring of (Δ^n, T).

Proof. The extreme points $\{e_i\}_{i=0}^n$ of Δ^n have exactly one non-zero entry, so $j(e_i) = i$, which are all distinct. If $\xi \in \partial_i\Delta^n$, then since $\xi_i = 0$, we have $j(\xi) \ne i$, so $j|_{\partial_i\Delta^n}$ is a Sperner coloring by a coinductive argument. □

Claim 2: For any $\varepsilon > 0$, there is a $\xi \in \Delta^n$ such that $f(\xi_i) \le \xi_i + \varepsilon$ for all $i = 0,\ldots,n$.

Proof. Let $\varepsilon > 0$. Since Δ^n is compact, f is uniformly continuous with respect to $\|\cdot\|_1$ on Δ^n. Thus, there is a $0 < \delta < \varepsilon/2$ such that if we choose T_δ such that every n-simplex $S \in T$ has diameter less than δ, then the diameter of $f(S)$ is less than $\varepsilon/2$. By Claim 1 and Sperner's Lemma 4.A.16, there is a simplex S in T_δ with distinctly colored vertices $\eta^0,\ldots,\eta^n$ where η^i has color i. For any $\xi \in S$, since $\|\xi - \eta^i\|_1 < \delta < \varepsilon/2$ and $\|f(\xi) - f(\eta^i)\|_1 < \varepsilon/2$ for all $i = 0,\ldots,n$,

$$f(\xi)_i \le f(\eta^i)_i + \frac{\varepsilon}{2} \le (\eta^i)_i + \frac{\varepsilon}{2} \le \xi_i + \varepsilon \qquad \forall\, i = 0,\ldots,n.$$ □

Now setting $\varepsilon = 1/k$, for each $k \in \mathbb{N}$, there is a $\xi_k \in \Delta^n$ such that

$$f(\xi_k)_i \le (\xi_k)_i + \frac{1}{k} \qquad \forall\, i = 0,\ldots,n.$$

Since Δ^n is compact, there is a convergent subsequence (ξ_{k_j}) which converges to some $\xi \in \Delta^n$. Since $\xi_{k_j} \to \xi$ coordinatewise, we conclude that $f(\xi_i) \le \xi_i$ for all $i = 0,\ldots,n$. By (4.A.18), we must have $f(\xi)_i = \xi_i$ for all i. □