Chapter 7

Semisimple categories

Category theory is an elegant and powerful language for studying various branches of mathematics. Often proofs can be formalized to apply in many different situations, and many definitions apply to many different mathematical structures simultaneously. Deep connections between different areas of mathematics can be seen from the existence of functors (maps) between categories.

7.1 Basic category theory

Definition 7.1.1 — A (locally small) *category* has a collection of objects, and between any two objects, a set of morphisms. We write $c \in \mathcal{C}$ to denote an object of a category $\mathcal{C}$, and we write $\mathcal{C}(a \to b)$ or $\mathrm{Hom}(a \to b)$ for the set of morphisms from a to b, and we write $\mathrm{End}(a) := \mathrm{Hom}(a \to a)$. There is a composition law for morphisms: if $f \in \mathcal{C}(a \to b)$ and $g \in \mathcal{C}(b \to c)$, there is a morphism $g \circ f \in \mathcal{C}(a \to c)$. This composition must be associative, and every object $c \in \mathcal{C}$ has an identity morphism $\mathrm{id}_c \in \mathcal{C}(c \to c)$.

An *isomorphism* is an invertible morphism. We say two objects a, b are *isomorphic* if there is an isomorphism $a \to b$. We always assume that $\mathcal{C}$ is *essentially small*, i.e., the isomorphism classes of our category form a set, unless stated otherwise.

A category is called a *groupoid* when every morphism is an isomorphism.

Example 7.1.2 — Typical examples of categories one works with are:

- Set, the category of sets and functions (which is not essentially small),
- Top, the category of topological spaces and continuous maps (which is not essentially small),
- Vec, the category of finite dimensional complex vector spaces and linear maps,
- Hilb, the category of finite dimensional Hilbert spaces and linear maps,

- $\mathsf{Rep}(G)$, the category of finite dimensional complex representations of a finite group G. In more detail, objects are pairs (V, π) where $V \in \mathsf{Vec}$ and $\pi : G \to \operatorname{End}(V)$ is a homomorphism. The morphisms are G-equivariant maps, i.e., $T : (V, \pi) \to (W, \rho)$ is a linear map $T : V \to W$ such that $T\pi_g = \rho_g T$ for all $g \in G$.

- $\mathsf{Rep}(A)$, the category of all finite dimensional complex representations of a finite dimensional unital algebra A.

- $\mathsf{Mod}(A)$, the category of all finite dimensional right A-modules of a finite dimensional unital algebra A. (Observe $\mathsf{Mod}(A) \cong \mathsf{Rep}(A^{\mathrm{op}})$, where A^{op} denotes the opposite algebra with the opposite multiplication $a^{\mathrm{op}} \cdot b^{\mathrm{op}} := (ba)^{\mathrm{op}}$.)

Example 7.1.3 — Let M be a monoid. The *delooping* $\mathrm{B}M$ is the category with one object $\star$ and $\operatorname{End}(\star) := M$, where the composition law is multiplication in M and $\mathrm{id}_\star$ is the unit of M. In this sense, one should think of a category as a monoid with more than one object.

Example 7.1.4 (Opposite category) **—** Given a category $\mathcal{C}$, its *opposite* is the category $\mathcal{C}^{\mathrm{op}}$ with the same objects as $\mathcal{C}$, but $\mathcal{C}^{\mathrm{op}}(a \to b) := \mathcal{C}(b \to a)$.

Example 7.1.5 — (Small) Categories and functors form a category (which is not locally small) $\mathsf{Cat}_{\tau\leq 1}$. The reason for this subscript will become clear in Part[[III]]§[[??]].

Definition 7.1.6 — A map of categories is called a *functor*. A functor $F : \mathcal{A} \to \mathcal{B}$ assigns an object $F(a) \in \mathcal{B}$ to each $a \in \mathcal{A}$ and a morphism $F(f) : F(a) \to F(b)$ to each $f \in \mathcal{A}(a \to b)$. We require that F preserves identities and the composition law:

- $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ for all $a \in \mathcal{A}$, and

- $F(f \circ_{\mathcal{A}} g) = F(f) \circ_{\mathcal{B}} F(g)$ for all composable morphisms f, g in $\mathcal{A}$.

Example 7.1.7 — A common type of functor is a *forgetful functor* which forgets extra structure. For example, there are organic forgetful functors $\mathsf{Top} \to \mathsf{Set}$, $\mathsf{Vec} \to \mathsf{Set}$, and $\mathsf{Hilb} \to \mathsf{Vec}$.

Exercise 7.1.8. For a set S, define the *power set*

$$\mathcal{P}(S) := \{A \,|\, A \subseteq S\}\,.$$

Given a function $f\colon S \to T$, for $A \subset S$ and $B \subset T$, we define the *direct image* and *preimage* under f respectively by

$$f(A) := \{f(a) \,|\, a \in A\} \qquad \text{and} \qquad f^{-1}(B) := \{s \in S \,|\, \exists b \in B \text{ such that } f(s) = b\}\,.$$

- Verify that $S \mapsto \mathcal{P}(S)$ and $f : S \to T$ maps to the function $A \mapsto f(A)$ defines a functor $\mathsf{Set} \to \mathsf{Set}$.
- Verify that $S \mapsto \mathcal{P}(S)$ and $f : S \to T$ maps to the function $B \mapsto f^{-1}(B)$ defines a functor on $\mathsf{Set} \to \mathsf{Set}^{\mathrm{op}}$.

Exercise 7.1.9. We can also define the *excluded image* of $A \subset S$ under $f : S \to T$ as

$$f_{\forall}(A) := \{f(a) \mid s \in S \text{ with } f(s) = f(a) \text{ implies } s \in A\} \subseteq f(A).$$

One can repackage this definition as to be more similar to the image as follows:

$$\begin{aligned} f_{\exists}(A) &:= \left\{t \in T \,\middle|\, \exists s \in f^{-1}(t) : s \in A\right\} = f(A) \\ f_{\forall}(A) &:= \left\{t \in T \,\middle|\, \forall s \in f^{-1}(t) : s \in A\right\} \end{aligned}$$

Show that the excluded image also defines a functor on Set.

Definition 7.1.10 — We call a category $\mathcal{C}$ *pointed* if it comes equipped with a choice of distinguished object $c \in \mathcal{C}$. A functor between pointed categories $(\mathcal{C}, c) \to (\mathcal{D}, d)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(c) = d$. We denote the category of pointed (small) categories by $\mathsf{Cat}^*_{\tau\leq 1}$.

Example 7.1.11 — Consider the category Mon of monoids and monoid homomorphisms. Delooping gives a functor $\mathrm{B} : \mathsf{Mon} \to \mathsf{Cat}^*_{\tau\leq 1}$, where a monoid M is mapped to $(\mathrm{B}M, \star)$.

Now for $(\mathcal{C}, c) \in \mathsf{Cat}^*_{\tau\leq 1}$, we can define the *loops* of $\mathcal{C}$ at c by $\Omega(\mathcal{C}, c) := \mathrm{End}_{\mathcal{C}}(c)$, which gives us a monoid. One can check that $\Omega : \mathsf{Cat}^*_{\tau\leq 1} \to \mathsf{Mon}$ is a functor such that $\Omega \circ \mathrm{B} = \mathrm{id}_{\mathsf{Mon}}$.

Example 7.1.12 — Suppose A, B are semisimple algebras and $\mathsf{Mod}(A)$ and $\mathsf{Mod}(B)$ denote their categories of right modules. An $A - B$ bimodule ${}_AM_B$ gives a functor $\mathsf{Mod}(A) \to \mathsf{Mod}(B)$ by $- \otimes_A M_B$. On morphisms, an A-intertwiner $f \colon N_A \to N'_A$ gets mapped to the B-intertwiner

$$N \otimes_A M_B \xrightarrow{f \otimes \mathrm{id}_M} N' \otimes_A M_B$$

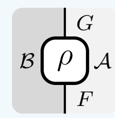
from Definition 3.1.12.

Definition 7.1.13 — The functors from $\mathcal{A} \to \mathcal{B}$ themselves form another category. Given two functors $F, G : \mathcal{A} \to \mathcal{B}$, a *natural transformation* $\rho : F \Rightarrow G$ is an assignment of a map $\rho_a \in \mathcal{B}(F(a) \to G(a))$ for every $a \in \mathcal{A}$ such that the following diagram commutes for every $f \in \mathcal{A}(a \to b)$.

$$\begin{tikzcd} F(a) \arrow[r, "F(f)"] \arrow[d, "\rho_a"'] & F(b) \arrow[d, "\rho_b"] \\ G(a) \arrow[r, "G(f)"'] & G(b) \end{tikzcd}$$

Typically, natural transformations $\rho : F \Rightarrow G$ are depicted by a *2-cell*

$$\begin{tikzcd}[column sep=large] \mathcal{A} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow \rho"] & \mathcal{B} \end{tikzcd}$$



The diagram on the left is called a *pasting diagram*, and the diagram on the right is called a *string diagram*. String diagrams, which are formally dual to pasting diagrams, give a 2D graphical calculus for Cat, the collection of all categories, functors, and natural transformations, similar to the graphical calculus for bimodules over an algebra from Chapter §3.7.

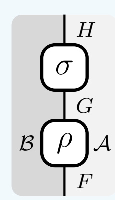
We warn the reader that the string diagram on the right transposes $\mathcal{A}$ and $\mathcal{B}$, for reasons which will become clear after Example 7.1.19 below. In order for pasting diagrams to match string diagrams on the nose, we should point our arrows leftwards so that composing arrows corresponds to composing maps.

Functors $\mathcal{A} \to \mathcal{B}$ and their natural transformations form a category, denoted

$$\mathsf{Fun}(\mathcal{A} \to \mathcal{B}).$$

Given three functors $F, G, H : \mathcal{A} \to \mathcal{B}$, composition of natural transformations $\rho : F \Rightarrow G$ and $\sigma : G \Rightarrow H$ is given by $(\sigma \cdot \rho)_a := \sigma_a \circ_{\mathcal{B}} \rho_a$. It is straightforward to show $\sigma \cdot \rho$ satisfies the above commutative square. Typically, the composite $\sigma \cdot \rho$ is denoted by vertically stacking 2-cells, which can be represented by either pasting or string diagrams.

$$\begin{tikzcd}[column sep=large, row sep=small] \mathcal{A} \arrow[r, bend left=50, "H"] \arrow[r, "G" description] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, shift left=2.5ex, "\Uparrow\sigma"] \arrow[r, phantom, shift right=2.5ex, "\Uparrow\rho"] & \mathcal{B} \end{tikzcd}$$

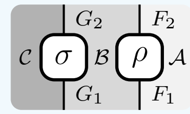


There is a second composition operation for natural transformations. Given functors $F_1, F_2 : \mathcal{A} \to \mathcal{B}$ and $G_1, G_2 : \mathcal{B} \to \mathcal{C}$ and natural transformations $\rho : F_1 \Rightarrow F_2$ and $\sigma : G_1 \Rightarrow G_2$, there is another composite $\sigma \circ \rho : G_1 \circ F_1 \Rightarrow G_2 \circ F_2$ defined by

$$(\sigma \circ \rho)_a := \sigma_{F_2(a)} \circ_{\mathcal{C}} G_1(\rho_a)$$

One verifies $\sigma \circ \rho$ is indeed a natural transformation. Typically, the composite $\sigma \circ \rho$ is denoted by horizontally concatenating 2-cells.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "F_1"'] \arrow[r, phantom, "\Uparrow \rho"] & \mathcal{B} \arrow[r, bend left, "G_2"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Uparrow \sigma"] & \mathcal{C} \end{tikzcd}$$

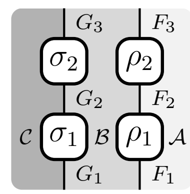


Exercise 7.1.14. Suppose we have functors $F_1, F_2, F_3 : \mathcal{A} \to \mathcal{B}$ and $G_1, G_2, G_3 : \mathcal{B} \to \mathcal{C}$ and natural transformations $\rho_1 : F_1 \Rightarrow F_2$, $\rho_2 : F_2 \Rightarrow F_3$, $\sigma_1 : G_1 \Rightarrow G_2$, and $\sigma_2 : G_2 \Rightarrow G_3$. Prove the *interchange law* for natural transformations

$$(\sigma_2 \cdot \sigma_1) \circ (\rho_2 \cdot \rho_1) = (\sigma_2 \circ \rho_2) \cdot (\sigma_1 \circ \rho_1).$$

We depict this common natural transformation $G_1 \circ F_1 \Rightarrow G_3 \circ F_3$ as the following composite 2-cell.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=40, "F_3"] \arrow[r, "F_2" description] \arrow[r, bend right=40, "F_1"'] \arrow[r, phantom, shift left=2.5ex, "\Uparrow \rho_2"] \arrow[r, phantom, shift right=2.5ex, "\Uparrow \rho_1"] & \mathcal{B} \arrow[r, bend left=40, "G_3"] \arrow[r, "G_2" description] \arrow[r, bend right=40, "G_1"'] \arrow[r, phantom, shift left=2.5ex, "\Uparrow \sigma_2"] \arrow[r, phantom, shift right=2.5ex, "\Uparrow \sigma_1"] & \mathcal{C} \end{tikzcd}$$



Definition 7.1.15 — An *equivalence* of categories $\mathcal{C} \to \mathcal{D}$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$, a functor $G : \mathcal{D} \to \mathcal{C}$, and natural isomorphisms $\epsilon\colon F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ and $\eta\colon \mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} G \circ F$.

Just as we may canonically identify a set S with maps into S from a singleton set, we may identify a category with functors from a category with one object and one morphism.

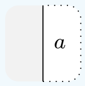
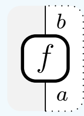
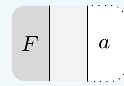
Example 7.1.16 — Let $\star$ denote the singleton category with one object $\star$ and one morphism id_*.[a] There is a canonical equivalence $\mathcal{C} \cong \mathsf{Fun}(\star \to \mathcal{C})$ given by $a \mapsto (\star \mapsto a)$ and $(f : a \to b) \mapsto (f_\star : (\star \mapsto a) \Rightarrow (\star \mapsto b))$.

[a]Later, when considering linear categories, we replace $\star$ with $\mathrm{B}\mathbb{C}$, the category with one object $\star$ and $\mathrm{End}(\star) = \mathbb{C}\,\mathrm{id}_\star$.

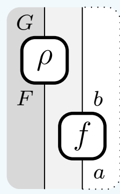
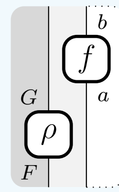
Example 7.1.17 — Building on Example 7.2.2, a Morita equivalence between semisimple algebras A, B gives an equivalence of categories $\mathsf{Mod}(A) \cong \mathsf{Mod}(B)$. We will see later in Exercise 7.7.13 that any equivalence of linear categories $\mathsf{Mod}(A) \cong \mathsf{Mod}(B)$ arises from a Morita equivalence.

Exercise 7.1.18. Show that there is an equivalence of categories $\mathsf{Fun}(\mathrm{B}A \to \mathsf{Vec}) \cong \mathsf{Rep}(A)$ where A is a finite dimensional algebra and $\mathrm{B}G$ is its delooping from Example 7.1.3.

Notation 7.1.19 — We represent $\star$ by the empty shaded region, and by Example 7.1.16, we may represent objects $a \in \mathcal{C}$ by strings and morphisms by coupons with a $\mathcal{C}$-shading on the left and $\star$-shading on the right (recall we compose right-to-left in Cat). We then apply the functor $F : \mathcal{C} \to \mathcal{D}$ by adding an F strand to the left of the a-strand.

 $\leftrightarrow a \in \mathcal{C}$  $\leftrightarrow (f : a \to b)$  $\leftrightarrow F(a) \in \mathcal{D}$

Natural transformations $\rho : F \Rightarrow G$ are then represented by coupons between the F, G strings, and naturality is represented by the exchange relation:

 $= \rho_b \circ F(f) = G(f) \circ \rho_a =$  .

Definition 7.1.20 — A functor $F : \mathcal{C} \to \mathcal{D}$ is called:

- *faithful* if for every $a, b \in \mathcal{C}$, the map $\mathcal{C}(a \to b) \to \mathcal{D}(F(a) \to F(b))$ is injective,
- *full* if for every $a, b \in \mathcal{C}$, the map $\mathcal{C}(a \to b) \to \mathcal{D}(F(a) \to F(b))$ is surjective,
- *essentially surjective* if every object $d \in \mathcal{D}$ is isomorphic to an object of the form $F(c)$ for some $c \in \mathcal{C}$.

Theorem 7.1.21 (Whitehead Theorem) — A functor $F \colon \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective if and only if it forms part of an equivalence of categories.

Proof. Suppose $F \colon \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective. Since F is essentially surjective, we may use the axiom of choice to pick $Gd \in \mathcal{C}$ together with an isomorphism $\epsilon_d \in \mathcal{D}(FGd \to d)$. Since F is fully faithful, for each $g \in \mathcal{D}(d_1 \to d_2)$ there exists a unique $Gg \in \mathcal{C}(Gd_1 \to Gd_2)$ such that

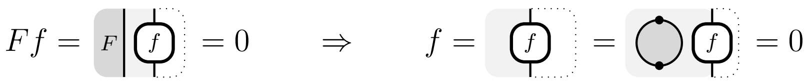
$$FGd_1 \xrightarrow{FGg} FGd_2 = FGd_1 \xrightarrow{\epsilon_{d_1}} d_1 \xrightarrow{g} d_2 \xrightarrow{\epsilon_{d_2}^{-1}} FGd_2.$$

It is easy to verify that $G \colon \mathcal{D} \to \mathcal{C}$ is a functor and $\epsilon \colon F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism. Moreover, since F is fully faithful, for each $c \in \mathcal{C}$, there exists a unique isomorphism $\eta_c \in \mathcal{C}(c \to GFc)$ such that $F\eta_c = \epsilon_{Fc}^{-1}$. One checks η is indeed natural in $c \in \mathcal{C}$.

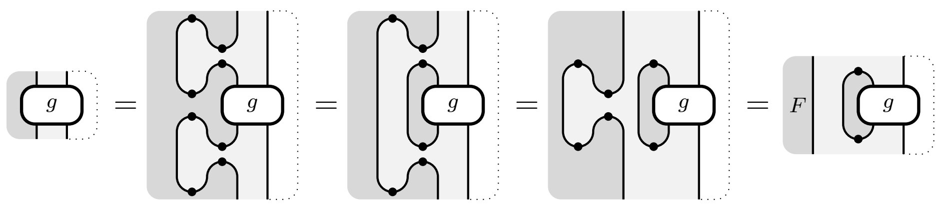
Conversely, suppose that a functor $F \colon \mathcal{C} \to \mathcal{D}$ admits a functor $G \colon \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $\eta \colon \mathrm{id}_C \Rightarrow G \circ F$ and $\epsilon \colon F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$. We represent η, ϵ by cups and caps with nodes on them (so that we do not confuse them with evaluations and coevaluations in the next section!), where we omit strings for the identity functors for convenience.

These maps satisfy the following relations:

It is clear that F is essentially surjective as each $c \in \mathcal{C}$ is isomorphic to FGc via ϵ_c. Moreover, F is faithful since



To see that F is full, consider $g \in \mathcal{C}(Fc \to Fc')$ and observe



□

Exercise 7.1.22. Verify G and ϵ, η constructed in the proof of Theorem 7.1.21 are indeed functorial and natural respectively.

Corollary 7.1.23 — The forgetful functor $\mathsf{Hilb} \to \mathsf{Vec}$ is an equivalence.

Remark 7.1.24. The Whitehead Theorem in topology states that given a continuous map $f\colon X \to Y$ between (nice enough) spaces, then f is a *homotopy equivalence* (there is a $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, where $\simeq$ means *homotopic*) if and only if the induced map $f_*\colon \pi_0(X) \to \pi_0(Y)$ is a bijection and the

$$f_*\colon \pi_k(X, x) \to \pi_k(Y, f(x)) \qquad x \in X,\ k \geq 1,$$

are isomorphisms on all homotopy groups.

Given a topological space X, there is a corresponding category $\Pi_1(X)$ whose objects are points $x \in X$ and morphisms $p\colon x \to y$ are given by (homotopy equivalence classes of) continuous paths $p\colon [0,1] \to X$ with $p(0) = x$ and $p(1) = y$. Notice $\Pi_1(X)$ is actually a groupoid, that is, each morphism admits an inverse as we can reverse the orientation of each path. Moreover, a continuous map $f\colon X \to Y$ induces a functor $f_*\colon \Pi_1(X) \to \Pi_1(Y)$, whereas homotopies between maps induce natural transformations.

The homotopy hypothesis [BS10, Hyp. 1] states that this construction provides an equivalence[1] between groupoids and homotopy 1-types (spaces with $\pi_k(X,x)$ trivial for $k > 1$).

$$\left\{\begin{matrix}\text{groupoids}\\ \text{functors}\\ \text{natural transformations}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{homotopy 1-types}\\ \text{continuous maps}\\ \text{homotopies}\end{matrix}\right\}$$

From this perspective, notice that $\pi_0(X)$ is the set of equivalence classes of objects in $\Pi_1(X)$ and $\pi_1(X,x) = \mathrm{End}_{\Pi_1(X)}(x)$. One then verifies that:

- $f_* \colon \pi_0(X) \to \pi_0(Y)$ is bijective if and only if $f_* \colon \Pi_1(X) \to \Pi_1(Y)$ is essentially surjective;
- $f_* \colon \pi_1(X) \to \pi_1(Y)$ is bijective if and only if $f_* \colon \Pi_1(X) \to \Pi_1(Y)$ is fully faithful;
- $f \colon X \to Y$ is a homotopy equivalence between homotopy 1-types if and only if $f_* \colon \Pi_1(X) \to \Pi_1(Y)$ is an equivalence of categories.

Thus, Theorem 7.1.21 indeed recovers the classical Whitehead Theorem for homotopy 1-types.

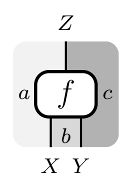
As an immediate corollary of this discussion, a homotopy 1-type X is *contractible* (homotopy equivalent to a point) if and only if there is a unique isomorphism between any two objects in $\Pi_1(X)$. This motivates the following definition.

> **Definition 7.1.25 —** A groupoid is called *contractible* if there is a unique map between any two objects.

7.2 Adjoint functors and graphical calculus

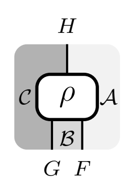
In §3.7, we used string diagrams to represent intertwiners between bimodules over algebras.

$$f : {}_aX \otimes_b Y_c \Rightarrow {}_aZ_c \quad\rightsquigarrow\quad \begin{tikzcd} a \arrow[dr, "X"'] \arrow[rr, bend left, "Z"] & {} \arrow[d, phantom, "\Uparrow f"] & c \\ & b \arrow[ur, "Y"'] & \end{tikzcd} \quad\rightsquigarrow$$



We saw above that we may represent functors and natural transformations by both pasting diagrams and string diagrams, but we swapped source and target horizontally when depicting composition of functors as opposed to the tensor product of bimodules.

$$\rho : GF \Rightarrow H \quad\rightsquigarrow\quad \begin{tikzcd} \mathcal{A} \arrow[dr, "F"'] \arrow[rr, bend left, "H"] & {} \arrow[d, phantom, "\Uparrow \rho"] & \mathcal{C} \\ & \mathcal{B} \arrow[ur, "G"'] & \end{tikzcd} \quad\rightsquigarrow$$



[1]In Part [[III]] we will see Π_1 is, more formally, an equivalence of *2-categories*.

The relative tensor product and composition of functors are both represented by horizontal concatenation. The composition of intertwiners and the composition of natural transformations is represented by vertical stacking.

These two graphical calculi are in fact instances of the same graphical calculus for *2-categories*, which we will see later in Part[[III]]. The only difference is the order of composition.

Just as bends in string diagrams were given meaning as evaluation and coevaluation maps for bimodules

$\mathrm{coev}_X =$ 

$\mathrm{ev}_X =$ 

,

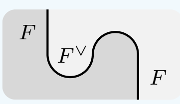
we may also discuss bends in string diagrams for functors.

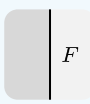
Definition 7.2.1 — Suppose $\mathcal{C}, \mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$. A *dual* for F is a functor $F^\vee : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms

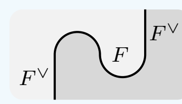
$\mathrm{coev}_F =$ 

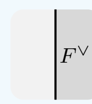
$\mathrm{ev}_F =$ 

which satisfy the zig-zag/snake equations.



$=$ 

$= \mathrm{id}_F$ and 

$=$ 

$= \mathrm{id}_{F^\vee}$.

Note that the duality conventions for bimodules and functors is swapped, since the order of tensor product/composition is swapped.

Example 7.2.2 — Suppose A, B are semisimple algebras and $\mathsf{Mod}(A)$ and $\mathsf{Mod}(B)$ denote their categories of (finite dimensional) right modules. An $A - B$ bimodule ${}_AM_B$ gives a functor $\mathsf{Mod}(A) \to \mathsf{Mod}(B)$ by $- \otimes_A M_B$. Its dual is the functor $- \otimes_B M_A^\vee$: $\mathsf{Mod}(B) \to \mathsf{Mod}(A)$.

Example 7.2.3 — Suppose $A \subset B$ is a unital inclusion of semisimple algebras. We have *induction* and *restriction* functors

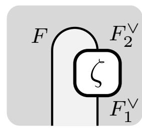
$$- \otimes_A B_B : \mathsf{Mod}(A) \longrightarrow \mathsf{Mod}(B)$$
$$- \otimes_B B_A : \mathsf{Mod}(B) \longrightarrow \mathsf{Mod}(A).$$

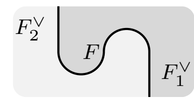
Since $({}_AB_B)^\vee = {}_BB_A$ (exercise!), these two functors are dual to one another.

Proposition 7.2.4 — Any two duals for F are canonically isomorphic. That is, if $F_1^\vee$ and $F_2^\vee$ are both duals for F, there is a unique natural isomorphism $\zeta : F_1^\vee \Rightarrow F_2^\vee$ that is compatible with the evaluations/coevaluations, i.e.,

$$\mathrm{ev}_1 = \mathrm{ev}_2 \cdot (\mathrm{id}_F \circ \zeta) \qquad \text{and} \qquad (\zeta \circ \mathrm{id}_F) \cdot \mathrm{coev}_1 = \mathrm{coev}_2$$

Proof. If ζ is compatible with the evaluations, then

$=$


$\iff$ $\zeta =$


.

Checking this formula works is a straightforward exercise. □

Remark 7.2.5. Suppose $F_1^\vee, F_2^\vee, F_3^\vee$ are all duals of F. Write $\zeta_{ij} : F_j^\vee \Rightarrow F_i^\vee$ for the unique natural isomorphism afforded by Proposition 7.2.4. Since the composite $\zeta_{32} \cdot \zeta_{21}$ is compatible with the evaluations (exercise!) it must be equal to ζ_{31}. In particular, the space of duals of F forms a contractible groupoid. Thus the existence of a dual should be considered a *property* and not extra *structure* cf. Ethos 1.5.2.

Exercise 7.2.6. Suppose $F^\vee$ is dual to F. Prove that F is fully faithful if and only if coev_F is invertible if and only if $F^\vee$ is essentially surjective. Which conditions are equivalent to invertibility of ev_F?

These bends in string diagrams for functors are related to the notion of *adjoint functors.*

Definition 7.2.7 — Suppose $\mathcal{C}, \mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors. We say that F is *left adjoint* to G, equivalently G is *right adjoint* to F, denoted $F \dashv G$, if there is a family of isomorphisms

$$\mathcal{D}(F(c) \to d) \cong \mathcal{C}(c \to G(d)). \tag{7.2.8}$$

which is *natural* in $c \in \mathcal{C}$ and $d \in \mathcal{D}$. The word *natural* here means that for every $f : c_1 \to c_2$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(F(c_2) \to d) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "- \circ F(f)"] & \mathcal{C}(c_2 \to G(d)) \arrow[d, "- \circ f"] \\
\mathcal{D}(F(c_1) \to d) \arrow[r, leftrightarrow, "\cong"] & \mathcal{C}(c_1 \to G(d))
\end{tikzcd}$$

and for every $g : d_1 \to d_2$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(F(c) \to d_1) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "g \circ -"] & \mathcal{C}(c \to G(d_1)) \arrow[d, "G(g) \circ -"] \\
\mathcal{D}(F(c) \to d_2) \arrow[r, leftrightarrow, "\cong"] & \mathcal{C}(c \to G(d_2))
\end{tikzcd}$$

Observe here that precomposition by f maps $\mathcal{C}(c_2 \to G(d)) \to \mathcal{C}(c_1 \to G(d))$, whereas postcomosition by g maps $\mathcal{D}(F(c) \to d_1) \to \mathcal{D}(F(c) \to d_2)$.

We will show in Proposition 7.2.15 below that duals and adjoints for functors are identical notions.

Example 7.2.9 — There are many free/forget adjoint functor pairs across mathematics, where the free functor is left adjoint to the forgetful functor. Examples include:

- Forget : $\mathsf{Vec} \to \mathsf{Set}$ which forgets the vector space structure and Free : $\mathsf{Set} \to \mathsf{Vec}$ defined by $S \mapsto \mathbb{C}[S]$
- Forget : $\mathsf{Rep}(G) \to \mathsf{Vec}$ which forgets the group action and Free : $\mathsf{Vec} \to \mathsf{Rep}(G)$ defined by $V \mapsto \mathbb{C}[G] \otimes V$
- Forget : $\mathsf{Mod}(A) \to \mathsf{Vec}$ has left adjoint given by $V \mapsto V \otimes A$.
- Forget : $\mathsf{Top} \to \mathsf{Set}$ which forgets the topology and Free : $\mathsf{Set} \to \mathsf{Top}$ which endows a set with the discrete topology, i.e., every subset is open.

Exercise 7.2.10. There is another functor $\mathsf{Set} \to \mathsf{Top}$ which endows a set S with the trivial topology, i.e., only $\emptyset$ and S are open. Show that this functor is right adjoint to the forgetful functor $\mathsf{Top} \to \mathsf{Set}$.

Exercise 7.2.11. Recall the direct image, preimage, and excluded image power set functors from Exercises 7.1.8 and 7.1.9. Show that the direct image functor is the left adjoint to the preimage functor, and the excluded image functor is right adjoint to the preimage functor.

Remark 7.2.12. Preimage preserves unions and intersections, but direct image only preserves unions and excluded image only preserves intersections. There is a categorical explanation for this fact (which is beyond the scope of this book). Left adjoints always preserve *colimits* in categories, whereas right adjoints always preserve *limits* in categories.

We refer the reader to [**?**, Thm. 4.6.2.] for further discussions on limits and colimits.

Definition 7.2.13 — Suppose $\mathcal{C}, \mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors with $F \dashv G$. We say $f : F(c) \to d$ and $g : c \to G(d)$ are *mates* if they map to each other under the natural isomorphism (7.2.8). The *unit* of the adjunction is the natural transformation $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow GF$ given by

$$\eta_c \coloneqq \mathrm{mate}(\mathrm{id}_{F(c)}) \in \mathcal{C}(c \to GF(c)) \cong \mathcal{D}(F(c) \to F(c)),$$

and the *counit* of the adjunction is the natural transformation $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$ given by

$$\epsilon_d \coloneqq \mathrm{mate}(\mathrm{id}_{G(d)}) \in \mathcal{D}(FG(d) \to d) \cong \mathcal{C}(G(d) \to G(d)).$$

Lemma 7.2.14 — The operations of taking mate are natural with respect to pre-composition and post-composition by another morphism:

(mate1) $\operatorname{mate}(f_2 \circ f_1) = G(f_2) \circ \operatorname{mate}(f_1)$ for all $f_1 : F(c) \to d_1$ and $f_2 : d_1 \to d_2$.

(mate2) $\operatorname{mate}(g_2 \circ g_1) = \operatorname{mate}(g_2) \circ F(g_1)$ for all $g_1 : c_1 \to c_2$ and $g_2 : c_2 \to G(d)$.

Proof. We prove the first, and the second is similar. By naturality of the adjunction isomorphism (7.2.8), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{D}(F(c) \to d_1) \arrow[r, "\cong"] \arrow[d, "f_2 \circ -"] & \mathcal{C}(c \to G(d_1)) \arrow[d, "G(f_2)\circ -"] \\
\mathcal{D}(F(c) \to d_2) \arrow[r, "\cong"] & \mathcal{C}(c \to G(d_2)).
\end{tikzcd}$$

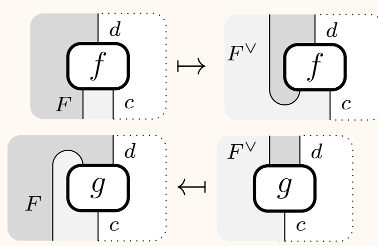
Going down and then left gives $\operatorname{mate}(f_2 \circ f_1)$, and going right and then down gives $G(f_2) \circ \operatorname{mate}(f_1)$. □

The next proposition shows that adjoints are the same thing as duals for functors.

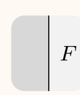
Proposition 7.2.15 — Suppose $\mathcal{C}, \mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors.

1. If there exist natural transformations $\operatorname{ev}_F : F \circ G \Rightarrow \operatorname{id}_{\mathcal{D}}$ and $\operatorname{coev}_F : \operatorname{id}_{\mathcal{C}} \Rightarrow G \circ F$ satisfying the snake equations (so $G \cong F^\vee$), then $F \dashv G$ with adjunction natural isomorphism (7.2.8) given by the *Frobenius reciprocity* isomorphisms:

$$\mathcal{D}(F(c) \to d) \cong \mathcal{C}(c \to F^\vee(d))$$

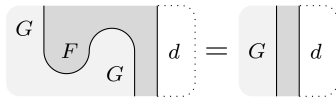


2. If $F \dashv G$, then the unit and counit satisfy the snake equations and thus can be represented by

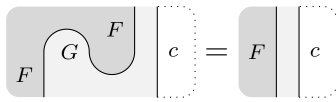
$\eta =$  and $\epsilon =$  where  $= F$,  $= G$.

Proof. Statement (1) is immediate from naturality of ev_F and coev_F. To prove statement

(2), we must prove that



$$\iff G(\epsilon_d) \circ \eta_{G(d)} = \mathrm{id}_{G(d)} \qquad \forall\, d \in \mathcal{D}$$



$$\iff \epsilon_{F(c)} \circ F(\eta_c) = \mathrm{id}_{F(c)} \qquad \forall\, c \in \mathcal{C}.$$

The first is (mate1) with $f_1 = \mathrm{id}_{FG(d)}$ and $f_2 = \epsilon_d$, and the second is (mate2) with $g_1 = \eta_c$ and $g_2 = \mathrm{id}_{GF(c)}$. □

Exercise 7.2.16. Suppose $F : \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective. In this exercise, we will prove that F participates in an *adjoint equivalence*, i.e., there is a $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\epsilon : F \circ G \Rightarrow \mathrm{id}_{b}$ which exhibit G as a dual of F.

(1) Prove that the G, η, ϵ constructed in the proof of the Whitehead Theorem 7.1.21 satifies the zig-zag equation for id_F.

(2) Deduce that the zig-zag for $F^\vee$ is then an invertible idempotent, and must therefore be $\mathrm{id}_{F^\vee}$.

7.3 Linear categories and linear functors

We saw in the Quantum Information Chapter 5 that superposition and entanglement were expressed in terms of taking linear combinations in a vector space. When working with higher quantum structures, we work with *linear categories* in which we may take linear combinations of morphisms.

Definition 7.3.1 — We call a category $\mathcal{C}$ *linear* if it is enriched in finite dimensional complex vector spaces. This means that $\mathcal{C}(a \to b)$ is a finite dimensional complex vector space for each $a, b \in \mathcal{C}$, and pre- and post-composition by any composable morphism in $\mathcal{C}$ is a linear operation. In other words, the composition operation

$$- \circ_{\mathcal{C}} - : \mathcal{C}(b \to c) \otimes \mathcal{C}(a \to b) \longrightarrow \mathcal{C}(a \to c) \qquad \text{given by} \qquad g \otimes f \mapsto g \circ f$$

is a linear map.

A functor $F : \mathcal{C} \to \mathcal{D}$ between linear categories is called *linear* if it is linear on hom spaces, i.e., for all $f, g \in \mathcal{C}(a \to b)$ and $\lambda \in \mathbb{C}$, $F(\lambda f + g) = \lambda F(f) + F(g)$.

Two linear categories $\mathcal{C}, \mathcal{D}$ are *equivalent* if there is an equivalence consisting of linear functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.

Remark 7.3.2. Just as we canonically identified $V = \mathrm{Hom}(\mathbb{C} \to V)$ for a vector space V (column vectors are maps $\mathbb{C} \to \mathbb{C}^n$), we may canonically identify a linear category $\mathcal{C} = \mathsf{Fun}(\mathrm{B}\mathbb{C} \to \mathcal{C})$.

Example 7.3.3 — Let A be a unital finite dimensional complex algebra. The category $\mathrm{B}A$, called the *delooping* of A, has one object $\star$ and $\mathrm{End}(\star) := A$. In this sense, one should think of a linear category as an algebra with more than one object.

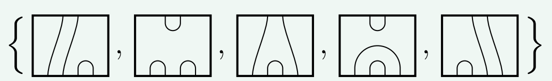
Example 7.3.4 — Both Vec and Hilb are linear categories. Moreover, the forgetful functor $\mathsf{Hilb} \to \mathsf{Vec}$ is an equivalence of linear categories.

Example 7.3.5 — Let S be a set. The category $\mathsf{Vec}(S)$ has objects finite dimensional S-graded complex vector spaces $V = \bigoplus_{s \in S} V_s$ and grading-preserving linear maps, i.e., if $T : V \to W$, then $T(V_s) \subset W_s$.

Example 7.3.6 — Let G be a finite group. The category $\mathsf{Rep}(G)$ is linear.

Example 7.3.7 — When $\mathcal{C}, \mathcal{D}$ are linear categories, $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ is also a linear category, where for $\eta, \theta : F \Rightarrow G$, we define $(\eta + \theta)_c := \eta_c + \theta_c$. Note, however, that the spaces $\mathrm{Hom}(F \Rightarrow G)$ of natural transformations may be infinite dimensional.

Example 7.3.8 — Let $d \in \mathbb{C}$. The category $\mathsf{TLJ}(d)$ has objects $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathsf{TLJ}(m \to n)$ consists of complex linear combinations of Kauffman diagrams with m boundary points on the bottom and n boundary points on the top. For example, the basis for $\mathsf{TLJ}(d)(4 \to 2)$ is given by



Composition is given by the usual stacking of diagrams and bubble popping relation from Definition 4.1.4 in Chapter 4. Observe that $\mathsf{TLJ}(\delta)(m, n) = 0$ if $m + n$ is odd.

Definition 7.3.9 — Given a linear category $\mathcal{C}$ and objects $a, b \in \mathcal{C}$, the *linking algebra*

$$L(a,b) := \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(b \to a) \\ \mathcal{C}(a \to b) & \mathcal{C}(b \to b) \end{pmatrix}$$

has multiplication given by matrix multiplication and composition in $\mathcal{C}$:

$$\begin{pmatrix} f_{aa} & f_{ab} \\ f_{ba} & f_{bb} \end{pmatrix} \circ \begin{pmatrix} g_{aa} & g_{ab} \\ g_{ba} & g_{bb} \end{pmatrix} := \begin{pmatrix} f_{aa}g_{aa} + f_{ab}g_{ba} & f_{aa}g_{ab} + f_{ab}g_{bb} \\ f_{ba}g_{aa} + f_{bb}g_{ba} & f_{ba}g_{ab} + f_{bb}g_{bb} \end{pmatrix}.$$

Here, we write $f_{ab} \in \mathcal{C}(b \to a)$, similar to how $e_{ij} \in M_n(\mathbb{C})$ maps e_j to e_i. More generally, given $a_1, \ldots, a_n \in \mathcal{C}$, we can define the n-fold linking algebra $L(a_1, \ldots, a_n)$.

Definition 7.3.10 — A linear category is called *pre-semisimple* if for every $a_1, \ldots, a_n \in \mathcal{C}$, the linking algebra $L(a_1, \ldots, a_n)$ is a finite dimensional semisimple complex algebra.

We call a pre-semisimple category *finite* if there is a global bound on the dimensions of the centers of all linking algebras. That is, there is a $K > 0$ such that $\dim(Z(L(a_1, \ldots, a_n))) < K$ for all $a_1, \ldots, a_n \in \mathcal{C}$.

Example 7.3.11 — If S a set, $\mathsf{Vec}(S)$ is pre-semisimple.

Exercise 7.3.12. Prove that $\mathsf{Vec}(S)$ is finite if and only if S is finite.

Example 7.3.13 — Given a finite dimensional complex semisimple algebra A, its delooping $\mathrm{B}A$ is finite pre-semisimple.

7.4 Direct sums in linear categories

In this section, unless stated otherwise, $\mathcal{C}$ denotes a linear category.

Definition 7.4.1 — Given objects $a, b \in \mathcal{C}$, an object $a \oplus b \in \mathcal{C}$ equipped with morphisms $\iota_a : a \to a \oplus b$, $\iota_b : b \to a \oplus b$, $\pi_a : a \oplus b \to a$, and $\pi_b : a \oplus b \to b$ is called the *direct sum* of a and b if

($\oplus$1) $\pi_a \circ \iota_a = \mathrm{id}_a$ and $\pi_b \circ \iota_b = \mathrm{id}_b$, and

($\oplus$2) $\iota_a \circ \pi_a + \iota_b \circ \pi_b = \mathrm{id}_{a \oplus b}$

The direct sum is canonical in the sense that the space of direct sums is contractible, as was the case for Hilb.

Observe there is a canonical isomorphism of algebras

$$\mathrm{End}_{\mathcal{C}}(a \oplus b) \cong L(a, b) = \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(b \to a) \\ \mathcal{C}(a \to b) & \mathcal{C}(b \to b) \end{pmatrix}. \tag{7.4.2}$$

This map and its inverse are given by

$$f \mapsto \begin{pmatrix} \pi_a \circ f \circ \iota_a & \pi_a \circ f \circ \iota_b \\ \pi_b \circ f \circ \iota_a & \pi_b \circ f \circ \iota_b \end{pmatrix}$$

$$\sum_{i,j \in \{a,b\}} \iota_i \circ g_{ij} \circ \pi_j \leftarrow\!\shortmid \begin{pmatrix} g_{aa} & g_{ab} \\ g_{ba} & g_{bb} \end{pmatrix}$$

We say $\mathcal{C}$ *admits direct sums* if $a \oplus b$ exists for all $a, b \in \mathcal{C}$, and there is a zero object $0 \in \mathcal{C}$ which is simultaneously *initial* and *terminal* in $\mathcal{C}$. This means for every $c \in \mathcal{C}$, there are unique morphisms $0 \to c$ and $c \to 0$ respectively.

Exercise 7.4.3. Show that any two initial objects in $\mathcal{C}$ are uniquely isomorphic. Then do the same for two terminal objects.

Exercise 7.4.4. Suppose that $(a \oplus b, \iota_a, \iota_b, \pi_a, \pi_b)$ is the direct sum of a and b. Show that $(a \oplus b, \iota_a, \iota_b)$ is the *coproduct* of a and b, meaning that for any $c \in \mathcal{C}$ and morphisms $f : a \to c$ and $g : b \to c$, there is a unique map $a \oplus b \to c$ such that the following diagram commutes:

$$\begin{tikzcd}
a \arrow[r, "\iota_a"] \arrow[dr, "f"'] & a \oplus b \arrow[d, dashed, "\exists!"] & b \arrow[l, "\iota_b"'] \arrow[dl, "g"] \\
& c &
\end{tikzcd}$$

Then prove $(a \oplus b, \pi_a, \pi_b)$ is the *product* of a and b, meaning that for any $c \in \mathcal{C}$ and morphisms $f : c \to a$ and $g : c \to b$, there is a unique map $c \to a \oplus b$ such that the following diagram commutes:

$$\begin{tikzcd}
& c \arrow[dl, "f"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "g"] & \\
a & a \oplus b \arrow[l, "\pi_a"] \arrow[r, "\pi_b"'] & b
\end{tikzcd}$$

Exercise 7.4.5. Prove that the space of direct sums of a and b is contractible as in §1.5 for Hilbert spaces. That is, if $(a \oplus' b, \iota'_a, \iota'_b, \pi'_a, \pi'_b)$ is another direct sum of a and b, then there is a unique isomorphism $a \oplus b \to a \oplus' b$ which is compatible with the ι, π.

Exercise 7.4.6. Show that direct sums are preserved by all linear functors between linear categories. That is, if $F : \mathcal{C} \to \mathcal{D}$ is a linear functor and $(a \oplus b, \iota_a, \iota_b, \pi_a, \pi_b)$ witnesses the direct sum of $a, b \in \mathcal{C}$, then $(F(a \oplus b), F(\iota_a), F(\iota_b), F(\pi_a), F(\pi_b))$ witnesses the direct sum of $F(a), F(b) \in \mathcal{D}$, i.e., there is a canonical isomorphism $F(a \oplus b) \cong F(a) \oplus F(b)$.

Exercise 7.4.7. Suppose $\mathcal{C}, \mathcal{D}$ are linear categories such that $\mathcal{D}$ admits direct sums. Show that $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ admits direct sums.

Construction 7.4.8 — Given a linear category $\mathcal{C}$, the *additive envelope* of $\mathcal{C}$ is the linear category $\mathcal{C}^{\oplus}$ whose objects are *formal tuples* $(a_i)_{i=1}^n$ for $a_1, \ldots, a_n \in \mathcal{C}$, $n \geq 0$, and whose morphism sets are given by matrices of operators:

$$\mathcal{C}^{\oplus}\left((b_j)_{j=1}^n \to (a_i)_{i=1}^m\right) := \{(x_{ij}) \,|\, x_{ij} \in \mathcal{C}(b_j \to a_i)\} \tag{7.4.9}$$

where composition is given by $(x_{ij}) \circ (y_{jk}) := (\sum_j x_{ij} \circ y_{jk})$. By convention, 0 is the only morphism to or from the empty tuple.

Observe that $c \mapsto (c)$ for $c \in \mathcal{C}$ and $x \mapsto (x)$ for $x \in \mathcal{C}(a \to b)$ is a fully faithful linear functor $\mathcal{C} \hookrightarrow \mathcal{C}^{\oplus}$.

Lemma 7.4.10 — $\mathcal{C}^{\oplus}$ admits finite direct sums.

Proof. Consider objects $c_1 = (a_j)_{j=1}^m$ and $c_2 = (a_j)_{j=m+1}^n$ in $\mathcal{C}^{\oplus}$. We define their direct sum $c_1 \oplus c_2$ as the tuple $(a_j)_{j=1}^n$, and we define $\iota_1 : (a_j)_{j=1}^m \to (a_j)_{j=1}^n$ and $\iota_2 : (a_j)_{j=m+1}^n \to (a_j)_{j=1}^n$ by

$$(\iota_1)_{ij} = \begin{cases} \delta_{i=j} \operatorname{id}_{a_i} & \text{if } i, j \leq m \\ 0 & i > m \end{cases} \qquad (\iota_2)_{ij} = \begin{cases} \delta_{i=j} \operatorname{id}_{a_i} & \text{if } i, j > m \\ 0 & i \leq m \end{cases}$$

and $\pi_1 : (a_j)_{j=1}^n \to (a_j)_{j=1}^m$ and $\pi_2 : (a_j)_{j=1}^n \to (a_j)_{j=m+1}^n$ by

$$(\pi_1)_{ij} = \begin{cases} \delta_{i=j} \operatorname{id}_{a_i} & \text{if } i, j \leq m \\ 0 & j > m \end{cases} \qquad (\pi_2)_{ij} = \begin{cases} \delta_{i=j} \operatorname{id}_{a_i} & \text{if } i, j > m \\ 0 & j \leq m. \end{cases}$$

The rest of the verification is left to the reader. $\square$

The additive envelope $\mathcal{C}^{\oplus}$ with the canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\oplus}$ satisfies the following universal property.

Proposition 7.4.11 — For every linear category $\mathcal{D}$ which admits finite direct sums, pre-composition with the canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\oplus}$ gives an equivalence

$$\iota^* : \mathsf{Fun}(\mathcal{C}^{\oplus} \to \mathcal{D}) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathcal{D}).$$

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a linear functor. We define a linear functor $F^{\oplus} : \mathcal{C}^{\oplus} \to \mathcal{D}$ by setting $F^{\oplus}((a_j)_{j=1}^n) := \bigoplus_{j=1}^n F(a_j)$ and $F^{\oplus}(x_{ij}) = \sum_{i,j} \iota_i \circ F(x_{ij}) \circ \pi_j$ where $(\iota_j, \pi_j)_{j=1}^n$ witness the direct sum of the $F(a_j)$ in $\mathcal{D}$. Clearly F equals the composite of the canonical inclusion followed by $F^{\oplus}$. Hence ι^* is essentially surjective.

Suppose now that $\alpha : F \Rightarrow G$ is a natural transformation between functors $\mathcal{C} \to \mathcal{D}$. Since $(a_j)_{j=1}^n \in \mathcal{C}^{\oplus}$ is the direct sum of the $(a_j) \in \mathcal{C}^{\oplus}$, we get a natural transformation $\alpha^{\oplus} : F^{\oplus} \Rightarrow G^{\oplus}$ by defining

$$\alpha^{\oplus}_{(a_j)} := \sum_j \iota_j^G \circ \alpha_{a_j} \circ \pi_j^F : \bigoplus_{j=1}^n F(a_j) \to \bigoplus_{j=1}^n G(a_j). \tag{7.4.12}$$

Since, $\alpha^{\oplus}_{(a)} = \alpha_a : F(a) \to G(a)$ for all $a \in \mathcal{C}$, $\iota^*(\alpha^{\oplus}) = \alpha$, so ι^* is full.

If $\beta : F^{\oplus} \Rightarrow G^{\oplus}$ with $\iota^*(\beta) = 0$, then $\beta_{(a)} : F(a) \to G(a)$ is zero for all $a \in \mathcal{C}$. By naturality, (7.4.12) still holds replacing $\alpha^{\oplus}$ with β and α_{a_j} with $\beta_{(a_j)}$. Thus $\beta = 0$ and ι^* is faithful. $\square$

Remark 7.4.13. Unpacking the universal property in Proposition 7.4.11 above, essential surjectivity of ι^* means that for any $F : \mathcal{C} \to \mathcal{D}$, there is a linear functor $F^\oplus : \mathcal{C}^\oplus \to \mathcal{D}$ and a natural isomorphism $\eta : F \Rightarrow F^\oplus \circ \iota$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\iota"] \arrow[dr, "F"'] & \mathcal{C}^\oplus \arrow[d, "F^\oplus"] \arrow[dl, phantom, "\Nearrow\eta"] \\ & \mathcal{D}. \end{tikzcd} \tag{7.4.14}$$

That ι^* is fully faithful means that whenever $(F_i^\oplus, \eta_i)$ for $i = 1, 2$ both satisfy (7.4.14) above, there is a unique natural transformation $\alpha : F_1^\oplus \Rightarrow F_2^\oplus$ such that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\iota"] \arrow[dr, "F"'] & \mathcal{C}^\oplus \arrow[d, "F_2^\oplus"] \arrow[dl, phantom, "\eta_2\Nearrow"] \\ & \mathcal{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[r, "\iota"] \arrow[dr, "F"'] & \mathcal{C}^\oplus \arrow[d, "F_1^\oplus"'] \arrow[d, bend left=80, "F_2^\oplus"] \arrow[dl, phantom, "\eta_1\Nearrow"] \arrow[d, phantom, shift left=4, "\overset{\alpha}{\Longrightarrow}"] \\ & \mathcal{D} \end{tikzcd} .$$

Corollary 7.4.15 — If $\mathcal{C}$ admits all finite direct sums, then $\mathcal{C}$ is equivalent to $\mathcal{C}^\oplus$.

Proof. This is a formal consequence of the universal property. First, since

$$\iota^* : \mathsf{Fun}(\mathcal{C}^\oplus \to \mathcal{C}) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathcal{C})$$

is an equivalence, there is a functor $\mathrm{id}_\mathcal{C}^\oplus : \mathcal{C}^\oplus \to \mathcal{C}$ such that $\mathrm{id}_\mathcal{C}^\oplus \circ \iota \cong \mathrm{id}_\mathcal{C}$. Next, since

$$\iota^* : \mathsf{Fun}(\mathcal{C}^\oplus \to \mathcal{C}^\oplus) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathcal{C}^\oplus)$$

is an equivalence and both $\mathrm{id}_{\mathcal{C}^\oplus}$ and $\iota \circ \mathrm{id}_\mathcal{C}^\oplus$ are objects on the left hand side, we see that

$$\iota^*(\iota \circ \mathrm{id}_\mathcal{C}^\oplus) = \iota \circ \mathrm{id}_\mathcal{C}^\oplus \circ \iota \cong \iota \circ \mathrm{id}_\mathcal{C} = \iota = \mathrm{id}_{\mathcal{C}^\oplus} \circ \iota = \iota^*(\mathrm{id}_{\mathcal{C}^\oplus}).$$

Since ι^* is an equivalence, we must have $\mathrm{id}_{\mathcal{C}^\oplus} \cong \iota \circ \mathrm{id}_\mathcal{C}^\oplus$. □

Remark 7.4.16. We will see in the next chapter that using Proposition 7.4.11, we can extend $(-)^\oplus$ to a 2-functor from the 2-category of linear categories to the 2-category of linear categories which admit direct sums. Indeed, given a linear functor $F : \mathcal{C} \to \mathcal{D}$, we can post-compose with the inclusion $\mathcal{D} \hookrightarrow \mathcal{D}^\oplus$ and apply Proposition 7.4.11 to obtain a functor $F^\oplus : \mathcal{C}^\oplus \to \mathcal{D}^\oplus$.

We leave it to the reader to show that $\mathrm{id}_\mathcal{C}^\oplus \cong \mathrm{id}_{\mathcal{C}^\oplus}$ and $F^\oplus \circ G^\oplus \cong (F \circ G)^\oplus$ for composable F, G.

Exercise 7.4.17. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a linear functor and $\mathcal{D}$ admits direct sums. Prove that if F is fully faithful, then so is $F^\oplus : \mathcal{C}^\oplus \to \mathcal{D}$.

7.5 Multiplicity spaces

Let $\mathcal{C}$ be a linear category which admits direct sums and let $c \in \mathcal{C}$. We now define a functor $-\otimes c : \mathsf{Vec} \to \mathcal{C}$, which we can think of as a categorification of the scalar product of $\mathbb{C}$ on Vec. Really, the object $V \otimes c$ is a fancy version of the direct sum $\bigoplus_{j=1}^{\dim(V)} c$ which allows us to define a canonical morphism $f \otimes \mathrm{id}_c : V \otimes c \to W \otimes c$ for every linear map $f : V \to W$.

Construction 7.5.1 — First, define $\mathbb{C} \otimes c := c$. For a (finite dimensional) vector space $V \in \mathsf{Vec}$, choose an ordered basis $\{v_1, \ldots, v_n\}$, which we may identify as a collection of maps $\mathbb{C} \to V$ given by $1 \mapsto v_i$. Define $V \otimes c := \bigoplus_{j=1}^n c$ with maps ι_j^V, π_j^V for $j = 1, \ldots, n$.

If $W \in \mathsf{Vec}$ with an ordered basis $\{w_1, \ldots, w_m\}$ and $f : V \to W$, recall from §1.1 that $[f] := [f]_{\{v_j\}}^{\{w_i\}} \in M_{m\times n}(\mathbb{C})$ is the matrix such that the following diagram commutes:

$$\begin{tikzcd}
(V, \{v_j\}_{j=1}^n) \arrow[r, "f"] \arrow[d, "{[\,\cdot\,]}"] & (W, \{w_i\}_{i=1}^m) \arrow[d, "{[\,\cdot\,]}"] \\
\mathbb{C}^n \arrow[r, "{[f]}"] & \mathbb{C}^m
\end{tikzcd}$$

Letting $W \otimes c = \bigoplus_{i=1}^m c$ with maps ι_i^W, π_i^W for $i = 1, \ldots, m$, we define the map $f \otimes \mathrm{id}_c : V \otimes c \to W \otimes c$ as $\sum_{i=1}^m \sum_{j=1}^n \iota_i^W([f]_{ij}\, \mathrm{id}_{\mathcal{C}})\pi_j^V$.

Exercise 7.5.2. Prove that $(f \otimes \mathrm{id}_c) \circ (g \otimes \mathrm{id}_c) = (f \circ g) \otimes \mathrm{id}_c$ whenever f, g are composable.

Since it is clear that $\mathrm{id}_V \otimes \mathrm{id}_c = \mathrm{id}_{V\otimes c}$ by construction, we can conclude that $-\otimes c : \mathsf{Vec} \to \mathcal{C}$ is a functor. However, we relied on choosing ordered bases for each vector space $V \in \mathsf{Vec}$. If we had chosen different bases, say $\{v_1', \ldots, v_n'\}$ for V and $\{w_1', \ldots, w_m'\}$ for W giving a different coordinate map $[f]' \in M_{m\times n}(\mathbb{C})$, there are unique invertible matrices β_V, β_W making the following diagram commute.

$$\begin{tikzcd}
 & (V, \{v_j'\}_{j=1}^n) \arrow[rr, "f"] \arrow[dd, "{[\,\cdot\,]'}"'] & & (W, \{w_i'\}_{i=1}^m) \arrow[dd, "{[\,\cdot\,]'}"] \\
(V, \{v_j\}_{j=1}^n) \arrow[ur, "\mathrm{id}_V"] \arrow[rr, "f"] \arrow[dd, "{[\,\cdot\,]}"'] & & (W, \{w_i\}_{i=1}^m) \arrow[ur, "\mathrm{id}_W"'] \arrow[dd, "{[\,\cdot\,]}"'] & \\
 & \mathbb{C}^n \arrow[rr, "{[f]'}"] & & \mathbb{C}^m \\
\mathbb{C}^n \arrow[ur, "\beta_V"] \arrow[rr, "{[f]}"] & & \mathbb{C}^m \arrow[ur, "\beta_W"'] &
\end{tikzcd}$$

Hence $\beta = \{\beta_V\}_{V\in\mathsf{Vec}}$ gives a canonical natural isomorphism $-\otimes c \Rightarrow -\otimes' c$. Moreover, these canonical natural isomorphisms compose properly, meaning that there is a contractible choice of functor $-\otimes c : \mathsf{Vec} \to \mathcal{C}$.

Definition 7.5.3 — Given $V \in \mathsf{Vec}$ and $c \in \mathcal{C}$, we call V the *multiplicity space* for the object $V \otimes c \in \mathcal{C}$.

Notation 7.5.4 — Given $a, b \in \mathcal{C}$ and $g : a \to b$, we can *amplify* $g : a \to b$ to a map $\bigoplus_{j=1}^n g : \bigoplus_{j=1}^n a \to \bigoplus_{j=1}^n b$ given by $\sum_{j=1}^n \iota_j^b \circ g \circ \pi_j^a$. This means for every vector space V and choice of ordered basis, we get a canonical map $\mathrm{id}_V \otimes g : V \otimes a \to V \otimes b$.

Exercise 7.5.5. Show that $\mathrm{id}_V \otimes g$ is natural in V, i.e., for all linear maps $f : V \to W$, the following diagram commutes.

$$\begin{tikzcd}
V \otimes a \arrow[r, "f \otimes \mathrm{id}_a"] \arrow[d, "\mathrm{id}_V \otimes g"'] & W \otimes a \arrow[d, "\mathrm{id}_W \otimes g"] \\
V \otimes b \arrow[r, "f \otimes \mathrm{id}_b"'] & W \otimes b
\end{tikzcd}$$

Deduce that defining $f \otimes g$ by either of the above composites is well-defined, and that the maps $\mathrm{id}_V \otimes g : V \otimes a \to V \otimes b$ compile into a natural transformation $- \otimes g : - \otimes a \Rightarrow - \otimes b$.

Exercise 7.5.6. Show that since $\bigoplus_{j=1}^n h \circ \bigoplus_{j=1}^n g = \bigoplus_{j=1}^n h \circ g$ for composable $g : a \to b$ and $h : b \to c$, the composite

$$(- \otimes a) \xRightarrow{- \otimes g} (- \otimes b) \xRightarrow{- \otimes h} (- \otimes c)$$

is equal to $- \otimes (h \circ g)$.

We summarize the above discussion into the following corollary below, in which $\mathsf{Vec} \times \mathcal{C}$ is the category whose objects are pairs (V, c) with $V \in \mathsf{Vec}$ and $c \in \mathcal{C}$ and whose morphisms are pairs of morphisms (f, g) with f in Vec and g in $\mathcal{C}$.

Corollary 7.5.7 — $- \otimes - : \mathsf{Vec} \times \mathcal{C} \to \mathcal{C}$ is a functor.

Proof. We combine Exercises 7.5.2, 7.5.5, and 7.5.6. First, $\mathrm{id}_V \otimes \mathrm{id}_c = \mathrm{id}_{V \otimes c}$ as $- \otimes c$ is a functor. If $f_1 : U \to V$ and $f_2 : V \to W$, and $g_1 : a \to b$ and $g_2 : b \to c$, then

$$\begin{aligned}
(f_2 \otimes g_2) \circ (f_1 \otimes g_1) &= (f_2 \otimes \mathrm{id}_c) \circ (\mathrm{id}_V \circ g_2) \circ (f_1 \otimes \mathrm{id}_b) \circ (\mathrm{id}_U \otimes g_1) && \text{(Ex. 7.5.5)} \\
&= (f_2 \otimes \mathrm{id}_c) \circ (f_1 \otimes \mathrm{id}_c) \circ (\mathrm{id}_U \circ g_2) \circ (\mathrm{id}_U \otimes g_1) && \text{(Ex. 7.5.5)} \\
&= ((f_2 \circ f_1) \otimes \mathrm{id}_c) \circ (\mathrm{id}_U \otimes (g_2 \circ g_1)) && \text{(Ex. 7.5.2 and 7.5.6)} \\
&= (f_2 \circ f_1) \otimes (g_2 \circ g_1). && \text{(Ex. 7.5.5)}
\end{aligned}$$

□

This corollary is the categorification of a scalar product on a vector space to a Vec-product on linear categories.

Now as linear functors preserve direct sums, each linear functor $F : \mathcal{C} \to \mathcal{D}$ comes equipped with a canonical isomorphism

$$\mu_{V,c} : F(V \otimes c) \longrightarrow V \otimes F(c) \tag{7.5.8}$$

which is natural in both $V \in \mathsf{Vec}$ and $c \in \mathcal{C}$. That is, for every $f : V \to W$ and $g : a \to b$, the following diagrams commute:

$$
\begin{tikzcd}
F(V \otimes c) \arrow[r, "F(f \otimes \mathrm{id}_c)"] \arrow[d, "\mu_{V,c}"] & F(W \otimes c) \arrow[d, "\mu_{W,c}"] \\
V \otimes F(c) \arrow[r, "f \otimes \mathrm{id}_{F(c)}"] & W \otimes F(c)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(V \otimes a) \arrow[r, "F(\mathrm{id}_V \otimes f)"] \arrow[d, "\mu_{V,a}"] & F(V \otimes b) \arrow[d, "\mu_{V,b}"] \\
V \otimes F(a) \arrow[r, "\mathrm{id}_V \otimes F(f)"] & V \otimes F(b)
\end{tikzcd}
$$

7.6 Idempotent completion

In this section $\mathcal{C}$ is merely a category which is not assumed to be linear.

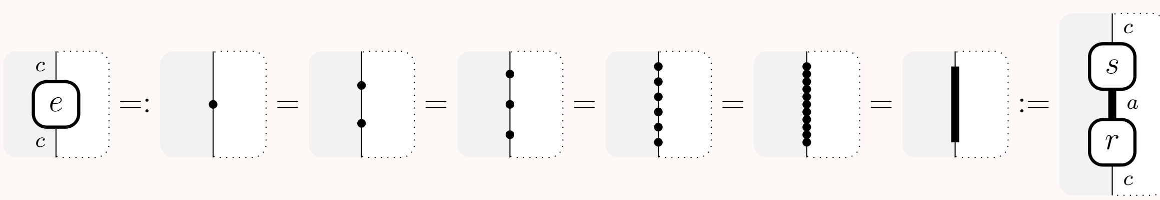
Definition 7.6.1 — An *idempotent* in $\mathcal{C}$ is a pair (c, e) where $c \in \mathcal{C}$ and $e \in \mathcal{C}(c \to c)$ such that $e \circ e = e$. A *splitting* for an idempotent (c, e) is an triple (a, r, s) where $a \in \mathcal{C}$, $r \in \mathcal{C}(c \to a)$ called a *retract*, and $s \in \mathcal{C}(a \to c)$ such that $s \circ r = e$ and $r \circ s = \mathrm{id}_a$. A linear category $\mathcal{C}$ is called *idempotent complete* if every idempotent admits a splitting.

Example 7.6.2 — Set and Vec are both idempotent complete.

Exercise 7.6.3. Show that the category $\mathsf{Vec}_{\text{even}}$ of even dimensional vector spaces is *not* idempotent complete.

Example 7.6.4 — If A is an algebra, then $\mathrm{B}A$ is idempotent complete if and only if A has no non-trivial idempotents.

Ethos 7.6.5 (Ethos 1.6.6, reprise) — In Proposition 1.4.25, we saw that projections in $B(H)$ correspond to subspaces of H. Similarly, one should think of idempotents (c, e) as *putative subobjects* of c, which may or may not really exist in $\mathcal{C}$. A splitting (a, r, s) of (c, e) is a *witness* to the existence of such a subobject, namely $a \in \mathcal{C}$. We may also think of this splitting as densely proliferating the idemptotent along the c-strand:



One may thus think of idempotent completeness of a category as including all of its subobjects. We will see that we may formally add all such subobjects, leading to the idempotent completion of a category.

Exercise 7.6.6. Suppose (a, r, s) is a splitting of (c, e). Show that given $f \in \mathcal{C}(c \to d)$ such that $f = f \circ e$, there exists a unique $f|_a \in \mathcal{C}(a \to d)$ such that $f|_a \circ r = f$. Then show that given $f \in \mathcal{C}(d \to c)$ such that $f = e \circ f$, there exists a unique $f|^a \in \mathcal{C}(d \to a)$ such that $s \circ f|^a = f$.

Exercise 7.6.7. Suppose $(a, r_a, s_a), (b, r_b, s_b)$ are two splittings of (c, e). Show that there is a unique isomorphism $f : a \to b$ which is compatible with (r_a, s_a) and (r_b, s_b), i.e., the space of splittings is contractible.

Exercise 7.6.8. Suppose $\mathcal{C}, \mathcal{D}$ are categories. Show that the property that the idempotent (c, e) admits a splitting is preserved by all functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. That is, if (a, r, s) splits (c, e), prove that $(F(a), F(r), F(s))$ splits $(F(c), F(e))$.

Exercise 7.6.9. Suppose $\mathcal{C}, \mathcal{D}$ are categories such that $\mathcal{D}$ is idempotent complete. Show that $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ is idempotent complete.

Definition 7.6.10 — Suppose $\mathcal{C}, \mathcal{D}$ are categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *dominant* if for every $d \in \mathcal{D}$, there is a $c \in \mathcal{C}$, a retract $r : F(c) \to d$, and a splitting $s : d \to F(c)$ such that $r \circ s = \mathrm{id}_d$.

Proposition 7.6.11 — Suppose $\mathcal{C}, \mathcal{D}$ are categories with $\mathcal{C}$ idempotent complete. A fully faithful functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is dominant.

Proof. It suffices to show dominant implies essentially surjective. Suppose F is dominant and $d \in \mathcal{D}$. Let $c \in \mathcal{C}$ and $s : d \to F(c)$ and $r : F(c) \to d$ such that $r \circ s = \mathrm{id}_d$. Consider the idempotent $s \circ r \in \mathrm{End}_{\mathcal{D}}(F(c))$. Since F is full, there is an $e \in \mathrm{End}_{\mathcal{C}}(c)$ such that $F(e) = s \circ r$, and since F is faithful, e is also an idempotent. Since $\mathcal{C}$ is idempotent complete, there is a splitting (a, r', s') of (c, e). We claim that $F(a) \cong d$. Indeed, since $s' \circ r' = e$ and $r' \circ s' = \mathrm{id}_a$, we see that the maps

$$F(a) \xrightarrow{F(s')} F(c) \xrightarrow{r} d \qquad \text{and} \qquad d \xrightarrow{s} F(c) \xrightarrow{F(r')} F(a)$$

are mutually inverse. □

Construction 7.6.12 — The *idempotent/Karoubi completion* $\mathcal{C}^{\ominus}$ is the category whose objects are pairs (c, e) where $c \in \mathcal{C}$ and $e \in \mathcal{C}(c \to c)$ is an idempotent. The morphism spaces are given by

$$\mathcal{C}((a, e) \to (b, f)) := \{x \in \mathcal{C}(a \to b) \,|\, x = f \circ x \circ e\}.$$

Observe that $\mathcal{C}((a, e) \to (b, f)) \subseteq \mathcal{C}(a \to b)$ is a linear subspace, and if $x \in \mathcal{C}((a, e) \to (b, f))$, then $x = x \circ e = f \circ x$. Composition of morphisms is exactly composition in $\mathcal{C}$, i.e., if $x \in \mathcal{C}((a,e) \to (b,f))$ and $y \in \mathcal{C}((b,f) \to (c,g))$, then $y \circ x \in \mathcal{C}((a,e) \to (c,g))$.

There is a faithful inclusion functor $\mathcal{C} \hookrightarrow \mathcal{C}^{\ominus}$ given by $c \mapsto (c, \mathrm{id}_c)$.

Remark 7.6.13. The notation $\mathcal{C}^{\ominus}$ makes more sense for linear categories. Indeed, suppose $\mathcal{C}$ is a linear category and $c \in \mathcal{C}$. If (c,e) is an idempotent, then so is $(c, \mathrm{id}_c - e)$. Moreover, if (a, r_a, s_a) is a splitting for (c,e) and (b, r_b, s_b) is a splitting for $(c, \mathrm{id}_c - e)$, then $c = a \oplus b$ with the maps $\iota_a = s_a$, $\iota_b = s_b$, $\pi_a = r_a$, and $\pi_b = r_b$. One may thus think of the splitting of an idempotent (c,e) as $c \ominus b$ for some $b \in \mathcal{C}$.

Proposition 7.6.14 — $\mathcal{C}^{\ominus}$ is idempotent complete.

Proof. If $f : (c,e) \to (c,e)$ is an idempotent, then $f : c \to c$ satisfies $fe = ef = f$ and $f^2 = f$. Thus (c,f) is another idempotent. We claim that $r = f : (c,e) \to (c,f)$ and $s = f : (c,f) \to (c,e)$ splits $((c,e), f)$ as an idempotent. Indeed, $rs = f^2 = f = \mathrm{id}_{(c,f)}$ and $sr = f^2 = f \in \mathrm{End}(c,e)$. □

The pair $\mathcal{C}^{\ominus}$ and the inclusion $\iota : \mathcal{C} \hookrightarrow \mathcal{C}^{\ominus}$ satisfy the following universal property. We leave it to the reader to unpack this as in Remark 7.4.13.

Proposition 7.6.15 — For every idempotent complete category $\mathcal{D}$, pre-composition with the canonical inclusion $\iota : \mathcal{C} \hookrightarrow \mathcal{C}^{\ominus}$ gives an equivalence

$$\iota^* : \mathsf{Fun}(\mathcal{C}^{\ominus} \to \mathcal{D}) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathcal{D}).$$

Proof. We define $F^{\ominus}(c,e) = d$ where (d,r,s) is any splitting of the idempotent $F(e) : F(c) \to F(c)$, where if $e = \mathrm{id}_c$, we of course pick the trivial splitting $d = F(c), r = s = \mathrm{id}_c$. For $x : (c_1, e_1) \to (c_2, e_2)$ and splittings (d_j, r_j, s_j) of $F(e_j) : F(c_j) \to F(c_j)$ for $j = 1, 2$, $x^{\ominus} := r_2 F(x) s_1 : d_1 \to d_2$. One checks that $F^{\ominus}$ is a well-defined functor such that $F^{\ominus} \circ \iota = F$.

Suppose now $\alpha : F \Rightarrow G$ between functors $\mathcal{C} \to \mathcal{D}$. Suppose (c,e) is an idempotent, and suppose (d_F, r_F, s_F) and (d_G, r_G, s_G) are our chosen splittings of $F(e)$ and $G(e)$ respectively. Since $(c,e) \in \mathcal{C}^{\ominus}$ splits $e : (c,e) \to (c,e)$ as an idempotent, we get a natural transformation $\alpha^{\ominus} : F^{\ominus} \Rightarrow G^{\ominus}$ by defining

$$\alpha^{\ominus}_{(c,e)} := r_G \circ \alpha_c \circ s_F : d_F \to d_G. \tag{7.6.16}$$

Since $\alpha^{\ominus}_{(c,\mathrm{id}_c)} = \alpha_c$, we have that $\iota^*(\alpha^{\ominus}) = \alpha$, so ι^* is full.

If $\beta^1, \beta^2 : F^{\ominus} \Rightarrow G^{\ominus}$ with $\iota^*(\beta^1) = \iota^*(\beta^2)$, then $\beta^1_{(c,\mathrm{id}_c)} = \beta^2_{(c,\mathrm{id}_c)} : F(c) \to G(c)$ for all $c \in \mathcal{C}$. By naturality, (7.6.16) still holds replacing $\alpha^{\ominus}$ with β^i and α_c with $\beta^i_{(c,\mathrm{id}_c)}$ for $i = 1, 2$. Thus $\beta^1 = \beta^2$ and ι^* is faithful. □

We omit the proof of the next corollary, which is similar to proof of Corollary 7.4.15 via the universal property.

Exercise 7.6.17. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a linear functor and $\mathcal{D}$ is idempotent complete. Prove that if F is fully faithful, then so is $F^{\ominus} : \mathcal{C}^{\ominus} \to \mathcal{D}$.

Corollary 7.6.18 — If $\mathcal{C}$ is idempotent complete, then $\mathcal{C}$ is equivalent to $\mathcal{C}^{\ominus}$.

Similar to Remark 7.4.16, we will see Proposition 7.6.15 above can be used to extend $(-)^{\ominus}$ to a 2-functor.

Proposition 7.6.19 — If $\mathcal{C}$ is a linear category that admits direct sums, then $\mathcal{C}^{\ominus}$ is also.

Proof. The relation $x = f \circ x \circ e$ is preserved under linear combinations, so $\mathcal{C}^{\ominus}$ is linear. Suppose $c_1 = (a, e)$ and $c_2 = (b, f)$ are objects in $\mathcal{C}^{\ominus}$. Using the matrix notation (7.4.2) for morphisms between direct sums, it is easily verified that

$$c_1 \oplus c_2 = \left(a \oplus b, \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right)$$

is an idempotent, and the maps

$$\iota_1 := \begin{pmatrix} e \\ 0 \end{pmatrix} : c_1 \to c_1 \oplus c_2 \qquad \iota_2 := \begin{pmatrix} 0 \\ f \end{pmatrix} : c_2 \oplus c_2 \to c_1$$

$$\pi_1 := \begin{pmatrix} e & 0 \end{pmatrix} : c_1 \oplus c_2 \to c_2 \qquad \pi_2 := \begin{pmatrix} 0 & f \end{pmatrix} : c_1 \to c_1 \oplus c_2$$

witness $c_1 \oplus c_2$ as the direct sum. □

7.7 Cauchy complete linear categories

Definition 7.7.1 — A linear category $\mathcal{C}$ is called *Cauchy complete* if it admits all finite direct sums and it is idempotent complete.

Example 7.7.2 — When S is a set, $\mathsf{Vec}(S)$ is Cauchy complete. If $V = \bigoplus_{s \in S} V_s$ and $W = \bigoplus_{s \in S} W_s$, then $V \oplus W = \bigoplus_{s \in S} V_s \oplus W_s$. If $e : V \to V$ is am idempotent, then $e = (e_s)_{s \in S}$ and each $e_s : V_s \to V_s$ is an idempotent. Split each e_s and then take the direct sum to obtain a splitting for e.

Construction 7.7.3 — The *Cauchy completion* of a linear category $\mathcal{C}$ is $\mathcal{C}^{\cent} := (\mathcal{C}^{\oplus})^{\ominus}$. Observe that $c \mapsto (c, \mathrm{id}_c)$ gives a faithful linear functor $\mathcal{C} \hookrightarrow \mathcal{C}^{\cent}$.

Corollary 7.7.4 — $\mathcal{C}^{\cent}$ is Cauchy complete.

Proof. We know $\mathcal{C}^{\cent}$ is idempotent complete by Proposition 7.6.14. By Proposition 7.6.19, $\mathcal{C}^{\cent}$ also admits finite direct sums. □

Example 7.7.5 — Suppose A is a finite dimensional complex semisimple algebra. Since every finite dimensional A-module is projective by Theorem 2.2.20, we have a canonical equivalence $(\mathrm{B}A)^{¢} \cong \mathsf{Mod}(A)$.

The following theorem is immediate.

Theorem 7.7.6 — $(\mathrm{B}\mathbb{C})^{¢} \cong \mathsf{Vec}$.

Exercise 7.7.7. Find an example of a linear category $\mathcal{C}$ such that $\mathcal{C}^{\ominus\oplus}$ is not equivalent to $\mathcal{C}^{\oplus\ominus}$.
Hint: Try $\mathrm{B}A$ *for an algebra* A *that is not finitely semisimple. In particular, consider* A *without non-trivial idempotents with projective modules which are not free, e.g.,* $C(S^2)$*, the continuous functions* $S^2 \to \mathbb{C}$.

"Hot Take" 7.7.8 — A way to remember that $\mathcal{C}^{¢} = \mathcal{C}^{\oplus\ominus}$ and not $\mathcal{C}^{\ominus\oplus}$ is that we cannot take away before we add. There is no such thing as a 'negative' category (yet).

Proposition 7.7.9 — For every Cauchy complete linear category $\mathcal{D}$, pre-composition with the canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{¢}$ gives an equivalence

$$\mathsf{Fun}(\mathcal{C}^{¢} \to \mathcal{D}) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathcal{D}).$$

Proof. Define $F^{¢} := (F^{\oplus})^{\ominus}$. We omit the rest of the proof, which is similar to the proofs of Propositions 7.4.11 and 7.6.15. □

Corollary 7.7.10 — If $\mathcal{C}$ is Cauchy complete, then $\mathcal{C}$ is equivalent to $\mathcal{C}^{¢}$.

Example 7.7.11 — Given finite dimensional semisimple algebras A, B, recall that we have a functor $\mathsf{Bim}(A,B) \to \mathsf{Fun}(\mathsf{Mod}(A) \to \mathsf{Mod}(B))$ given by ${}_AM_B \mapsto - \otimes_A M_B$. We can use Example 7.7.5 and Proposition 7.7.9 to prove that this functor is an equivalence of categories. To construct an inverse functor, we observe that an $A-B$ bimodule ${}_AM_B$ is a right B-module M_B equipped with a B-linear left A-action $A \to \mathrm{End}(M_B)$. Thus we may identify

$$\mathsf{Bim}(A,B) = \mathsf{Fun}(\mathrm{B}A \to \mathsf{Mod}(B)).$$

Precomposing with the functor $\mathrm{B}A \to \mathsf{Mod}(A)$ via the map $\star \mapsto A_A$ gives an equivalence

$$\mathsf{Fun}(\mathsf{Mod}(A) \to \mathsf{Mod}(B)) \xrightarrow{\cong} \mathsf{Fun}(\mathrm{B}A \to \mathsf{Mod}(B)) = \mathsf{Bim}(A,B)$$

determined by $F \mapsto F(A_A) \in \mathsf{Bim}(A,B)$. Indeed, since the linear functors $- \otimes_A F(A_A)$

and F agree on A_A, they agree on all of $\mathsf{Mod}(A) = (\{A_A\})^{¢}$.

Exercise 7.7.12. Prove that the equivalences

$$\mathsf{Fun}(\mathsf{Mod}(A) \to \mathsf{Mod}(B)) \cong \mathsf{Bim}(A, B)$$

are compatible with composition. That is, the composite of $- \otimes_A M_B : \mathsf{Mod}(A) \to \mathsf{Mod}(B)$ and $- \otimes_B N_C : \mathsf{Mod}(B) \to \mathsf{Mod}(C)$ is naturally isomorphic to $- \otimes_A (M \otimes_B N)_C$.

Exercise 7.7.13. Deduce that any equivalence of categories $\mathsf{Mod}(A) \cong \mathsf{Mod}(B)$ arises from a Morita equivalence.

Similar to Remark 7.4.16, we will see Proposition 7.7.9 above can be used to extend $(-)^{¢}$ to a functor.

Exercise 7.7.14. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a linear functor and $\mathcal{D}$ is Cauchy complete. Prove that if F is fully faithful, then so is $F^{¢} : \mathcal{C}^{¢} \to \mathcal{D}$.

Proposition 7.7.15 — When $\mathcal{C}$ is linear and $\mathcal{D}$ is Cauchy complete, $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ is Cauchy complete. In particular, direct sums of linear functors $F, G : \mathcal{C} \to \mathcal{D}$ are given on objects by $(F \oplus G)(c) := F(c) \oplus G(c)$ and similarly for morphisms.

Proof. Verifying our formula for $F \oplus G$ is a linear functor is a straightforward computation.

Suppose now $\pi : F \Rightarrow F$ is an idempotent. Then $\pi_c : F(c) \to F(c)$ is an idempotent in $\mathcal{D}$ for all $c \in \mathcal{C}$. Choose any splitting (a_c, r_c, s_c) for $(F(c), \pi_c)$ in $\mathcal{D}$, and define $A : \mathcal{C} \to \mathcal{D}$ by $A(c) := a_c$ and $A(f : c \to d) := r_d \circ F(f) \circ s_c : a_c \to a_d$. Since $r_c \circ s_c = \mathrm{id}_c$ for all $c \in \mathcal{C}$, A is a functor. Observe that $r = (r_c)_{c \in \mathcal{C}}$ is a natural transformation $F \Rightarrow A$, and similarly and $s = (s_c)_{c \in \mathcal{C}} : A \Rightarrow F$ is a natural transformation such that $r \circ s = \mathrm{id}_A$. Since $s_c \circ r_c = \pi_c$ for all $c \in \mathcal{C}$, we see that $s \circ r = \pi : F \Rightarrow F$ as desired. □

7.8 Direct sum and Deligne product of linear categories

We now define the direct sum of two linear categories.

Definition 7.8.1 — Given two linear categories $\mathcal{C}, \mathcal{D}$, we define the category $\mathcal{C} \boxplus \mathcal{D}$ whose objects are formal direct sums $c \boxplus d$ with $c \in \mathcal{C}$ and $d \in \mathcal{D}$ and whose morphisms are given by

$$\mathrm{Hom}(c_1 \boxplus d_1 \to c_2 \boxplus d_2) := \mathcal{C}(c_1 \to c_2) \oplus \mathcal{D}(d_1 \to d_2)$$

where the direct sum on the right hand side is the direct sum of vector spaces.

Example 7.8.2 — For algebras A, B, every representation of $A \oplus B$ decomposes canonically as a direct sum of representations, giving a canonical equivalence $\mathsf{Mod}(A \oplus B) \cong \mathsf{Mod}(A) \boxplus \mathsf{Mod}(B)$. In particular, if A is a finite dimensional complex semisimple algebra with n simple summands, then $\mathsf{Mod}(A) \cong \mathsf{Vec}^{\boxplus n}$.

Example 7.8.3 — When S is finite, $\mathsf{Vec}(S) \cong \mathsf{Vec}^{\boxplus |S|}$.

Exercise 7.8.4. Show that if $\mathcal{C}, \mathcal{D}$ admit direct sums, are idempotent complete, or are Cauchy complete, then the same applies to $\mathcal{C} \boxplus \mathcal{D}$ respectively.

Exercise 7.8.5. In this exercise, we assume all categories are linear and Cauchy complete, and all functors are linear.

1. Given $\mathcal{C}, \mathcal{D}$, construct functors $I_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C} \boxplus \mathcal{D}$, $I_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{C} \boxplus \mathcal{D}$, $P_{\mathcal{C}} : \mathcal{C} \boxplus \mathcal{D} \to \mathcal{C}$, and $P_{\mathcal{D}} : \mathcal{C} \boxplus \mathcal{D} \to \mathcal{D}$ such that

$$P_{\mathcal{C}} \circ I_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}} \qquad P_{\mathcal{C}} \circ I_{\mathcal{D}} = 0$$
$$P_{\mathcal{D}} \circ I_{\mathcal{C}} = 0 \qquad P_{\mathcal{D}} \circ I_{\mathcal{D}} = \mathrm{id}_{\mathcal{D}}$$

and $I_{\mathcal{C}}(c) \oplus I_{\mathcal{D}}(d) = c \boxplus d$ for $c \in \mathcal{C}$, $d \in \mathcal{D}$.

2. Given $F_1 \colon \mathcal{C}_1 \to \mathcal{D}_1$ and $F_2 \colon \mathcal{C}_2 \to \mathcal{D}_2$, show there is a functor $F = F_1 \boxplus F_2 \colon \mathcal{C}_1 \boxplus \mathcal{C}_2 \to \mathcal{D}_1 \boxplus \mathcal{D}_2$ such that

$$P_{\mathcal{D}_1} \circ (F_1 \boxplus F_2) \circ I_{\mathcal{C}_1} = F_1 \qquad P_{\mathcal{D}_1} \circ (F_1 \boxplus F_2) \circ I_{\mathcal{C}_2} = 0$$
$$P_{\mathcal{D}_2} \circ (F_1 \boxplus F_2) \circ I_{\mathcal{C}_1} = 0 \qquad P_{\mathcal{D}_2} \circ (F_1 \boxplus F_2) \circ I_{\mathcal{C}_2} = F_2$$

3. Deduce that in part (1), $I_{\mathcal{C}} \circ P_{\mathcal{C}} \oplus I_{\mathcal{D}} \circ P_{\mathcal{D}} = \mathrm{id}_{\mathcal{C} \boxplus \mathcal{D}}$.

4. Suppose we also have a category $\mathcal{E}$ and functors $J_{\mathcal{C}} : \mathcal{C} \to \mathcal{E}$, $J_{\mathcal{D}} : \mathcal{D} \to \mathcal{E}$, $Q_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$, $Q_{\mathcal{D}} : \mathcal{E} \to \mathcal{D}$ satisfying $Q_{\mathcal{C}} \circ J_{\mathcal{C}} \cong \mathrm{id}_{\mathcal{C}}$, $Q_{\mathcal{D}} \circ J_{\mathcal{D}} \cong \mathrm{id}_{\mathcal{D}}$, and $J_{\mathcal{C}} \circ Q_{\mathcal{C}} \oplus J_{\mathcal{D}} \circ Q_{\mathcal{D}} \cong \mathrm{id}_{\mathcal{E}}$. Show that $G := J_{\mathcal{C}} \circ P_{\mathcal{C}} \oplus J_{\mathcal{D}} \circ P_{\mathcal{D}} \colon \mathcal{C} \boxplus \mathcal{D} \to \mathcal{E}$ satisfies

$$\begin{aligned} G \circ I_{\mathcal{C}} &\cong J_{\mathcal{C}} & Q_{\mathcal{C}} \circ G &\cong P_{\mathcal{C}} \\ G \circ I_{\mathcal{D}} &\cong J_{\mathcal{D}} & Q_{\mathcal{D}} \circ G &\cong P_{\mathcal{D}} \end{aligned} \tag{7.8.6}$$

(In fact, $G \circ I_{\mathcal{C}} = J_{\mathcal{C}}$ and $G \circ I_{\mathcal{D}} = J_{\mathcal{D}}$, but we leave the weaker version above for the next part.)

5. Given any other functor $H \colon \mathcal{C} \boxplus \mathcal{D} \to \mathcal{E}$ satisfying (7.8.6), show there is a unique natural isomorphism $G \cong H$ compatible with (7.8.6). Deduce that the space of direct sums of $\mathcal{C}$ and $\mathcal{D}$ is contractible.

Definition 7.8.7 — Given two linear categories $\mathcal{C}, \mathcal{D}$, their *Deligne product* is the linear category $\mathcal{C} \boxtimes \mathcal{D}$ whose objects are formal tensors $c \boxtimes d$ with $c \in \mathcal{C}$ and $d \in \mathcal{D}$ and whose morphisms are given by

$$\mathrm{Hom}(c_1 \boxtimes d_1 \to c_2 \boxtimes d_2) \coloneqq \mathcal{C}(c_1 \to c_2) \otimes \mathcal{D}(d_1 \to d_2)$$

where the tensor product on the right hand side is the tensor product of vector spaces.

Now when $\mathcal{C}, \mathcal{D}$ admit direct sums, are idempotent complete, or are Cauchy complete, it may not be the case that $\mathcal{C} \boxtimes \mathcal{D}$ is as well. We thus *redefine* the Deligne product for these categories as follows: When $\mathcal{C}, \mathcal{D}$:

- admit direct sums, we define $\mathcal{C} \boxtimes_{\oplus} \mathcal{D} \coloneqq (\mathcal{C} \boxtimes \mathcal{D})^{\oplus}$.
- are idempotent complete, we define $\mathcal{C} \boxtimes_{\ominus} \mathcal{D} \coloneqq (\mathcal{C} \boxtimes \mathcal{D})^{\ominus}$.
- are Cauchy complete, we define $\mathcal{C} \boxtimes_{¢} \mathcal{D} \coloneqq (\mathcal{C} \boxtimes \mathcal{D})^{¢}$.

We will usually omit the decoration from the notation on the left hand side and simply use $\boxtimes$ for this redefined Deligne product when working with such linear categories.

Exercise 7.8.8. Given two categories $\mathcal{C}$ and $\mathcal{D}$, one can similarly form their cartesian product $\mathcal{C} \times \mathcal{D}$ whose objects are tuples (c, d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$ and morphisms

$$\mathrm{Hom}((c_1, d_1) \to (c_2, d_2)) \coloneqq \mathcal{C}(c_1 \to c_2) \times \mathcal{D}(d_1 \to d_2).$$

For linear categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$, we say a functor $F \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is bilinear if it is linear in each entry.

1. Verify that $\boxtimes : (c, d) \mapsto c \boxtimes d$ is a bilinear functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$.
2. Given a bilinear functor $F \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, construct a linear functor $\widetilde{F} \colon \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{E}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{C} \boxtimes \mathcal{D} \arrow[dr, dashed, "\widetilde{F}"] & \\ \mathcal{C} \times \mathcal{D} \arrow[u, "\boxtimes"] \arrow[r, "F"] & \mathcal{E} \end{tikzcd}$$

3. Given a bilinear functors $F, G \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ and a natural transformation $\eta \colon F \Rightarrow G$, show there is a unique natural transformation $\widetilde{\eta} \colon \widetilde{F} \Rightarrow \widetilde{G}$ such that the following equality holds:

$$\begin{tikzcd} \mathcal{C} \boxtimes \mathcal{D} \arrow[dr, dashed, "\widetilde{G}"] & \\ \mathcal{C} \times \mathcal{D} \arrow[u, "\boxtimes"] \arrow[r, "G"] \arrow[r, bend right=50, "F"', ""{name=B}] & \mathcal{E} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \boxtimes \mathcal{D} \arrow[dr, dashed, "\widetilde{F}"'] \arrow[dr, dashed, bend left=50, "\widetilde{G}"] \arrow[dr, phantom, bend left=20, "\widetilde{\eta}\Nearrow"] & \\ \mathcal{C} \times \mathcal{D} \arrow[u, "\boxtimes"] \arrow[r, "F"'] & \mathcal{E} \end{tikzcd}$$

Exercise 7.8.9. For linear categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$, show that

$$\begin{aligned} \mathsf{Fun}(\mathcal{C} \boxtimes \mathcal{D} \to \mathcal{E}) &\xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathsf{Fun}(\mathcal{D} \to \mathcal{E})). \\ F &\mapsto (c \mapsto F(c \boxtimes -)) \end{aligned}$$

is an equivalence of categories. Deduce this equivalence holds when the relevant categories admit direct sums, are idempotent complete, or Cauchy complete and interpreting $\boxtimes$ accordingly.

7.9 Representable functors and the Yoneda embedding

For this section, $\mathcal{C}$ is a linear category. Every $c \in \mathcal{C}$ gives two *representable* functors:

$$\mathcal{C}(c \to -) : \mathcal{C} \to \mathsf{Vec} \qquad \text{and} \qquad \mathcal{C}(- \to c) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}.$$

We will focus on functors of the second type in this section, and the functors of the first type can be treated similarly.

Lemma 7.9.1 (Yoneda) — Let $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}$ be a linear functor. For each $c \in \mathcal{C}$, the map

$$\begin{aligned} よ(a, F) : \mathrm{Hom}(\mathcal{C}(- \to a) \Rightarrow F) &\longrightarrow F(a) \\ \rho &\longmapsto \rho_a(\mathrm{id}_a) \end{aligned}$$

is an isomorphism which is natural in both c and F when both are considered as bilinear functors $\mathcal{C}^{\mathrm{op}} \times \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}) \to \mathsf{Vec}$.

Proof. Observe that $\rho_a : \mathcal{C}(a \to a) \to F(a)$ is a linear map which can be evaluated at id_a, so the above map is well-defined. Since ρ is natural, we see that for every other $b \in \mathcal{C}$ and every $f \in \mathcal{C}(b \to a)$, the following square commutes.

$$\begin{tikzcd} \mathcal{C}(a \to a) \arrow[r, "- \circ_{\mathcal{C}} f"] \arrow[d, "\rho_a"] & \mathcal{C}(b \to a) \arrow[d, "\rho_b"] \\ F(a) \arrow[r, "F(f)"] & F(b) \end{tikzcd}$$

This means that $F(f) \circ \rho_a = \rho_b \circ (- \circ f)$. Evaluating at id_a, we see $\rho_b(f) = F(f)(\rho_a(\mathrm{id}_a))$, so ρ_b is completely determined by $\rho_a(\mathrm{id}_a)$. Conversely, given an element $x \in F(a)$, we can define a natural transformation by $\rho_b(f) := F(f)(x)$, establishing the isomorphism.

To see naturality in a, for all $f \in \mathcal{C}(b \to a)$, we must show that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{C}(- \to a) \Rightarrow F) \arrow[r, "- \circ_{\mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})} (f \circ_{\mathcal{C}} -)"] \arrow[d, "よ(a,F)"] & \mathrm{Hom}(\mathcal{C}(- \to b) \Rightarrow F) \arrow[d, "よ(b,F)"] \\ F(a) \arrow[r, "F(f)"] & F(b) \end{tikzcd}$$

Given $\rho : \mathcal{C}(- \to a) \Rightarrow F$, the map right and then down is

$$(\rho \cdot (f \circ_{\mathcal{C}} -))_b(\mathrm{id}_b) = \rho_b((f \circ_{\mathcal{C}} -)_b(\mathrm{id}_b)) = \rho_b(f \circ \mathrm{id}_b) = \rho_b(f),$$

which is exactly the map down and then right by the first part of the argument.

To see naturality in F, we must show the following diagram commutes for all $\sigma : F \Rightarrow G$.

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{C}(- \to a) \Rightarrow F) \arrow[r, "\sigma \circ_{\mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})} -"] \arrow[d, "よ(a,F)"] & \mathrm{Hom}(\mathcal{C}(- \to a) \Rightarrow G) \arrow[d, "よ(a,G)"] \\
F(a) \arrow[r, "\sigma_a"] & G(a)
\end{tikzcd}$$

Given $\rho : \mathcal{C}(- \to a) \Rightarrow F$, the map right and then down is

$$(\sigma \cdot \rho)_a(\mathrm{id}_a) = \sigma_a(\rho_a(\mathrm{id}_a)),$$

which is exactly the map down and then right. $\square$

Exercise 7.9.2. Write down the statement and the proof of the Yoneda Lemma 7.9.1 on your own without any references. Repeat this as many times as necessary until you truly understand it.

Remark 7.9.3. For notational simplicity let us denote $\mathcal{C}^* := \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})$ and consider the functor $\mathrm{ev} \colon \mathcal{C} \to \mathcal{C}^{**}$ given by $c \mapsto (F \mapsto F(c))$. The natural isomorphism $よ(a, F)$ from the Yoneda Lemma 7.9.1 can be repackaged as natural isomorphisms of the following two forms:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "よ"] \arrow[dr, "\mathrm{ev}"'] & \mathcal{C}^* \arrow[d, "よ"] \arrow[dl, phantom, "\Leftarrow"] \\
& \mathcal{C}^{**}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C}^* \arrow[r, "よ"] \arrow[dr, "\mathrm{id}"'] & \mathcal{C}^{**} \arrow[d, "よ^*"] \arrow[dl, phantom, "\Leftarrow"] \\
& \mathcal{C}^*
\end{tikzcd}$$

Corollary 7.9.4 — The *Yoneda embedding* functor

$$\begin{aligned} よ : \mathcal{C} &\hookrightarrow \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}) \\ c &\mapsto \mathcal{C}(- \to c) \end{aligned}$$

is fully faithful.

Proof. The Yoneda Lemma 7.9.1 states that $\mathrm{Hom}(\mathcal{C}(- \to a) \Rightarrow \mathcal{C}(- \to b))$ is canonically isomorphic to $\mathcal{C}(a \to b)$. $\square$

Exercise 7.9.5. Let S be a finite set. Show that the Yoneda embedding $よ : \mathsf{Vec}(S) \hookrightarrow \mathsf{Fun}(\mathsf{Vec}(S)^{\mathrm{op}} \to \mathsf{Vec})$ is an equivalence of linear categories.

Definition 7.9.6 — A functor $F : \mathcal{C}^{\text{op}} \to \mathsf{Vec}$ is called *representable* if there is an $a \in \mathcal{C}$ and a natural isomorphism $\alpha : F \Rightarrow \mathcal{C}(- \to a)$. We call (a, α) a representing pair for F.

Remark 7.9.7. Representing pairs for F form a contractible space (when they exist). Indeed, given (a, α) and (b, β) for F, we get canonical natural isomorphisms

$$\mathcal{C}(- \Rightarrow a) \overset{\alpha^{-1}}{\Longrightarrow} F \overset{\beta}{\Longrightarrow} \mathcal{C}(- \to b) \qquad \text{and} \qquad \mathcal{C}(- \Rightarrow b) \overset{\beta^{-1}}{\Longrightarrow} F \overset{\alpha}{\Longrightarrow} \mathcal{C}(- \to a).$$

By the Yoneda embedding, these natural isomorphisms must come from $\mathcal{C}(a \to b)$, i.e., there is an isomorphism $f : a \to b$ such that $\beta \circ \alpha^{-1}$ is postcomposition with f. Plugging a into the left hand side above, we see that $f \in \mathcal{C}(a \to b)$ is exactly equal to $(\beta_a \circ \alpha_a^{-1})(\mathrm{id}_a)$.

We conclude the object representing a representable functor is uniquely determined (up to a contractible space).

7.10 Semisimplicity and 2-vector spaces

In this section, $\mathcal{C}$ is a Cauchy complete linear category.

Definition 7.10.1 — An object $c \in \mathcal{C}$ is called *simple* if $\mathrm{End}_{\mathcal{C}}(c) = \mathbb{C}\,\mathrm{id}_c$. Two simple objects $a, b \in \mathcal{C}$ are called *distinct* if $\mathcal{C}(a \to b) = (0)$ and $\mathcal{C}(b \to a) = (0)$.

A Cauchy complete linear category $\mathcal{C}$ is called *semisimple* ([**?**, Adapted from Def. 2.10], see also [**?**]) if there is a set $\mathrm{Irr}(\mathcal{C})$ of pairwise distinct simple objects such that for any $a, b \in \mathcal{C}$, the composition map

$$\bigoplus_{s \in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \to s) \otimes_{\mathbb{C}} \mathcal{C}(s \to b) \longrightarrow \mathcal{C}(a \to b) \tag{7.10.2}$$

is an isomorphism. (The direct sum in (7.10.2) is the direct sum in Vec.)

If $\mathrm{Irr}(\mathcal{C})$ is finite, then $\mathcal{C}$ is called *finite semisimple* or a *2-vector space*.

Example 7.10.3 — For a set S, the category $\mathcal{C} = \mathsf{Vec}(S)$ is semisimple with $\mathrm{Irr}(\mathcal{C}) = \{\mathbb{C}_s \,|\, s \in S\}$. In particular, $\mathsf{Rep}(A)$ is semisimple when A is a finite dimensional complex semisimple algebra by Example 7.8.2.

Remark 7.10.4. Simplicity of objects is defined differently in an abelian category, where kernels and cokernels are already assumed to exist. We include a short section on abelian categories at the end of this chapter.

Lemma 7.10.5 (Schur) — Suppose $\mathcal{C}$ is semisimple. Any two simple objects are either isomorphic or distinct.

Proof. It suffices to prove that if $a \in \mathcal{C}$ is simple, then there is exactly one $c \in \operatorname{Irr}(\mathcal{C})$ isomorphic to a, and a is distinct from every other $d \in \operatorname{Irr}(\mathcal{C})$. Since composition gives an isomorphism

$$\bigoplus_{c \in \operatorname{Irr}(\mathcal{C})} \mathcal{C}(a \to c) \otimes_{\mathbb{C}} \mathcal{C}(c \to a) \longrightarrow \mathcal{C}(a \to a) \cong \mathbb{C},$$

we can conclude:

- there is exactly one $c \in \operatorname{Irr}(\mathcal{C})$ such that both $\mathcal{C}(a \to c)$ and $\mathcal{C}(c \to a)$ are nonzero,
- $\mathcal{C}(a \to c) \cong \mathbb{C}$ and $\mathcal{C}(c \to a) \cong \mathbb{C}$, and
- if $f : a \to c$ and $g : c \to a$ are non-zero, then $g \circ f \neq 0$.

Choose $f : a \to c$ and $g : c \to a$ such that $g \circ f = \mathrm{id}_a$. Let $\lambda \in \mathbb{C}$ such that $f \circ g = \lambda \, \mathrm{id}_c$. Precompose both sides with g to obtain $g = \mathrm{id}_a \circ g = g \circ f \circ g = \lambda g$, so $\lambda = 1$ and $a \cong c$.

It remains to prove that for all other $d \in \operatorname{Irr}(\mathcal{C})$ with $d \neq c$, $\mathcal{C}(a \to d) = 0$ and $\mathcal{C}(d \to a) = 0$. For all $h \in \mathcal{C}(a \to d)$, observe that

$$h = h \circ \mathrm{id}_a = h \circ (g \circ f) = \underbrace{(h \circ g)}_{\in \mathcal{C}(c \to d) = 0} \circ f = 0.$$

Similarly, $\mathcal{C}(d \to a) = 0$. $\square$

Facts 7.10.6. We gather a list of elementary properties about a semisimple category $\mathcal{C}$.

(EP1) Let $c \in \mathcal{C}$. By (7.10.2), for each $s \in \operatorname{Irr}(\mathcal{C})$ there are finite sets $\{\lambda_i^s\}_{i=1}^{n_s} \subset \mathcal{C}(s \to c)$ and $\{\rho_i^s\}_{i=1}^{n_s} \subset \mathcal{C}(c \to s)$ such that $\mathrm{id}_c = \sum_{s \in \operatorname{Irr}(\mathcal{C})} \sum_{i=1}^{n_s} \lambda_i^s \circ \rho_i^s$.

(EP2) Since the simples in $\operatorname{Irr}(\mathcal{C})$ are pairwise distinct, the morphisms $e_s := \sum_{i=1}^{n_s} \lambda_i^s \circ \rho_i^s$ for $s \in \operatorname{Irr}(\mathcal{C})$ satisfy $e_s \circ e_t = 0$ when $s \neq t$. Indeed,

$$e_s \circ e_t = \left(\sum_{i=1}^{n_s} \lambda_i^s \circ \rho_i^s \right) \circ \left(\sum_{j=1}^{n_t} \lambda_j^t \circ \rho_j^t \right) = \sum_{i=1}^{n_s} \sum_{j=1}^{n_t} \lambda_i^s \circ \underbrace{\rho_i^s \circ \lambda_j^t}_{\in \mathcal{C}(t \to s) = 0} \circ \rho_j^t = 0.$$

(EP3) Since the left hand side of (7.10.2) is a direct sum and

$$\sum_s e_s = \mathrm{id}_c = \mathrm{id}_c \circ \mathrm{id}_c = \left(\sum_s e_s \right) \circ \left(\sum_t e_t \right) = \sum_{s,t} e_s \circ e_t = \sum_s e_s \circ e_s,$$

each e_s is an idempotent. It is straightforward to show that the idempotents e_s are independent of the choice of $\{\lambda_i^s\}_{i=1}^{n_s} \subset \mathcal{C}(s \to c)$ and $\{\rho_i^s\}_{i=1}^{n_s} \subset \mathcal{C}(c \to s)$. Splitting e_s gives an object in $\mathcal{C}$ called the *s-isotypic component* of c.

(EP4) If $f \in \mathcal{C}(c \to s)$ with $s \in \operatorname{Irr}(\mathcal{C})$, then

$$f = f \circ \mathrm{id}_c = \left(\sum_{t \in \operatorname{Irr}(\mathcal{C})} \sum_{i=1}^{n_t} \lambda_i^t \circ \rho_i^t\right) = \sum_{t \in \operatorname{Irr}(\mathcal{C})} \sum_{i=1}^{n_t} \underbrace{f \circ \lambda_i^t}_{\in \mathcal{C}(t \to s)} \circ \rho_i^t = f \circ e_s,$$

and similarly, if $g \in \mathcal{C}(s \to c)$, then $g = e_s \circ g$.

(EP5) If $f \in \mathcal{C}(c \to c)$, then $e_s \circ f = f \circ e_s$, so each e_s is a central idempotent. Indeed, expanding $f = \sum_{t \in \operatorname{Irr}(\mathcal{C})} \sum_{i=1}^{m_t} g_i^t \circ h_i^t$ with $\{g_i^s\}_{i=1}^{n_t} \subset \mathcal{C}(t \to c)$ and $\{h_i^s\}_{i=1}^{n_t} \subset \mathcal{C}(c \to t)$, we have

$$f \circ e_s = \sum_{t \in \operatorname{Irr}(\mathcal{C})} \sum_{i=1}^{m_t} g_i^t \circ h_i^t \circ e_s = \sum_{t \in \operatorname{Irr}(\mathcal{C})} \sum_{i=1}^{m_t} g_i^t \circ h_i^t \circ e_t \circ e_s = \sum_{i=1}^{m_s} g_i^s \circ h_i^s,$$

and similarly for $e_s \circ f$. We call $e_s \circ f = f \circ e_s$ the *s-isotypic component of f*.

(EP6) Suppose $c, s \in \mathcal{C}$ with $s \in \operatorname{Irr}(\mathcal{C})$ simple. For every non-zero $g \in \mathcal{C}(s \to c)$, there is an $f \in \mathcal{C}(c \to s)$ such that $f \circ g = \mathrm{id}_s$, and for every non-zero $f \in \mathcal{C}(c \to s)$, there is an $g \in \mathcal{C}(s \to c)$ such that $f \circ g = \mathrm{id}_s$, Indeed, since $f \circ e_s = f$ for all $f \in \mathcal{C}(c \to s)$ and $e_s \circ g = g$ for all $g \in \mathcal{C}(s \to c)$, expanding $e_s = \sum_{i=1}^{n_s} \lambda_i^s \circ \rho_i^s$ with $\{\rho_i^s\}_{i=1}^{n_t} \subset \mathcal{C}(c \to s)$ and $\{\lambda_i^s\}_{i=1}^{n_s} \subset \mathcal{C}(s \to c)$ yields this result.

This elementary property has the following two immediate consequences.

(EP7) For any non-zero $f \in \mathcal{C}(s \to b)$ and $g \in \mathcal{C}(a \to s)$, $f \circ g \neq 0$.

(EP8) $\mathcal{C}(c \to s) \neq 0$ if and only if $\mathcal{C}(s \to c) \neq 0$.

(EP9) For each $c \in \mathcal{C}$, (7.10.2) and the last elementary property implies that $\mathcal{C}(s \to c) \neq 0$ for only finitely many $s \in \mathcal{C}$. Hence the object $\bigoplus_{s \in \operatorname{Irr}(\mathcal{C})} \mathcal{C}(s \to c) \otimes s$ is well-defined.

Lemma 7.10.7 — Suppose $\mathcal{C}$ is semisimple and $c \in \mathcal{C}$. There is a canonical isomorphism $\upsilon_c : c \cong \bigoplus_{s \in \operatorname{Irr}(\mathcal{C})} \mathcal{C}(s \to c) \otimes s$. Moreover, for all $f \in \mathcal{C}(a \to b)$, the following diagram commutes.

$$\begin{tikzcd}
a \arrow[r, "\upsilon_a"] \arrow[d, "f"] & \bigoplus_{s \in \operatorname{Irr}(\mathcal{C})} \mathcal{C}(s \to a) \otimes s \arrow[d, "(f \circ -) \otimes \mathrm{id}_s"] \\
b \arrow[r, "\upsilon_b"] & \bigoplus_{s \in \operatorname{Irr}(\mathcal{C})} \mathcal{C}(s \to b) \otimes s
\end{tikzcd} \tag{7.10.8}$$

Proof. Since the composition map is natural (7.10.2) is natural in a, we get the following isomorphism between representable functors:

$$\begin{aligned}
\mathcal{C}\left(a \to \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to c)\otimes s\right) &\cong \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}\left(a \to \mathcal{C}(s\to c)\otimes s\right) && \text{(Ex. 7.4.6)}\\
&\cong \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(a\to s)\otimes \mathcal{C}(s\to c) && (7.5.8)\\
&\cong \mathcal{C}(a\to c) && (7.10.2).
\end{aligned}$$

Now the Yoneda Lemma 7.9.1 and Remark 7.9.7 gives the desired isomorphism υ_c. The final claim follows either from naturality of the Yoneda isomorphism or from naturality of the composition map (7.10.2) in b. □

Corollary 7.10.9 — A semisimple category $\mathcal{C}$ with distinguished set of simples $S = \mathrm{Irr}(\mathcal{C})$ is canonically equivalent to $\mathsf{Vec}(S)$ via the functor $\mathsf{Vec}(S) \to \mathcal{C}$ given by $\bigoplus_{s\in S} V_s \mapsto \bigoplus_{s\in S} V_s \otimes s$.

Proof. Essential surjectivity follows the canonical isomorphism $\upsilon_c : c \cong \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to c)\otimes s$. Fully faithful follows from (7.10.8). □

When $\mathcal{C}$ is finite semisimple, the above corollary says that every 2-vector space has a basis, in that every object can be written as a linear Vec-combination of simples, and morphisms just move around the multiplicity spaces. The next proposition shows that functors out of 2-vector spaces are completely determined by where they send a basis.

Proposition 7.10.10 — If $\mathcal{C}$ is semisimple and $\mathcal{D}$ is linear, then every linear functor $F : \mathcal{C} \to \mathcal{D}$ is completely determined up to unique natural isomorphism by where it sends simple objects via the commutative diagram

$$\begin{tikzcd}
F(a) \arrow[r, "F(\upsilon_a)"] \arrow[d, "F(f)"] & \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s\to a)\otimes F(s) \arrow[d, "(f\circ -)\otimes \mathrm{id}_{F(s)}"] \\
F(b) \arrow[r, "F(\upsilon_b)"] & \bigoplus_{s\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s\to b)\otimes F(s)
\end{tikzcd} \qquad \forall f : a \to b. \tag{7.10.11}$$

Moreover, every natural transformation $\rho : F \Rightarrow G$ is completely determined by $\{\rho_s\}_{s\in \mathrm{Irr}(\mathcal{C})}$. In other words, we have an equivalence of categories

$$\mathsf{Fun}(\mathcal{C} \to \mathcal{D}) \cong \mathsf{Fun}(\mathcal{C}_0 \to \mathcal{D})$$

where $\mathcal{C}_0$ is the full subcategory of $\mathcal{C}$ whose objects are $\mathrm{Irr}(\mathcal{C})$.

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ is linear. Then for each $c \in \mathcal{C}$, combining that linear functors preserve direct sums, the canonical isomorphism and Lemma 7.10.7, we have a canonical isomorphism

$$\begin{aligned} F(c) &\xrightarrow{F(v_c)} F\left(\bigoplus_{s\in\operatorname{Irr}(\mathcal{C})} \mathcal{C}(s\to c)\otimes s\right) && \text{(Lem. 7.10.7)}\\ &\cong \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})} F(\mathcal{C}(s\to c)\otimes s) && \text{(Ex. 7.4.6)}\\ &\xrightarrow{\bigoplus_s \mu_{\mathcal{C}(s\to c),s}} \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})} \mathcal{C}(s\to c)\otimes F(s) && (7.5.8). \end{aligned}$$

By Lemma 7.10.7, for every morphism $f\in\mathcal{C}(a\to b)$, (7.10.11) commutes. We conclude that F is completely determined by where it sends simples.

Now suppose $\rho : F \Rightarrow G$. Using the canonical isomorphism $v_c : c \to \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})}\mathcal{C}(s\to c)\otimes s$ and natruality of ρ, the following diagram commutes.

$$\begin{tikzcd} F(c) \arrow[r, "F(v_c)"] \arrow[d, "\rho_c"] & F\left(\bigoplus_{s\in\operatorname{Irr}(\mathcal{C})}\mathcal{C}(s\to c)\otimes s\right) \arrow[r, "\cong"] \arrow[d, "\rho_{\bigoplus_{s\in\operatorname{Irr}(\mathcal{C})}\mathcal{C}(s\to c)\otimes s}"] & \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})}\mathcal{C}(s\to c)\otimes F(s) \arrow[d, "\bigoplus_{s\in\operatorname{Irr}(\mathcal{C})}\mathrm{id}_{\mathcal{C}(s\to c)}\otimes\rho_s"] \\ G(c) \arrow[r, "G(v_c)"] & G\left(\bigoplus_{s\in\operatorname{Irr}(\mathcal{C})}\mathcal{C}(s\to c)\otimes s\right) \arrow[r, "\cong"] & \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})}\mathcal{C}(s\to c)\otimes G(s) \end{tikzcd}$$

Thus ρ_c is completely determined by the ρ_s. □

Remark 7.10.12. By Corollary 7.10.9, when $\mathcal{C}$ is semisimple, $\mathcal{C}=\mathcal{C}_0^{\natural}$ where $\mathcal{C}_0$ is the full subcategory of $\mathcal{C}$ whose objects are $\operatorname{Irr}(\mathcal{C})$. When $\mathcal{D}$ admits direct sums, we get a quicker proof of the above corollary by the universal property from Proposition 7.4.11 applied to the functor $F_0 = F|_{\mathcal{C}_0} : \mathcal{C}_0 \to \mathcal{D}$, as $F\cong F_0^{\oplus}$.

Corollary 7.10.13 — When $\mathcal{C}$ is finite semisimple, the Yoneda embedding よ $: \mathcal{C} \hookrightarrow \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}\to\mathsf{Vec})$ is an equivalence.

Proof. We already know よ is fully faithful. Suppose $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}$ is a linear functor. By Proposition 7.10.10,

$$F \cong \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})}\mathcal{C}(-\to s)\otimes F(s) \cong \mathcal{C}(-\to \bigoplus_{s\in\operatorname{Irr}(\mathcal{C})} F(s)\otimes s),$$

which is clearly a representable functor. Thus よ is essentially surjective. □

We summarize the results above in the following theorem.

Theorem 7.10.14 (Fundamental Theorem of semisimple categories) — Suppose $\mathcal{C}$ is a Cauchy complete linear category. The following conditions are equivalent.

(SS1) $\mathcal{C}$ is semisimple.

(SS2) $\mathcal{C} \cong \mathsf{Vec}(S)$ for some set S (cf. Example 7.3.5).

(SS3) Every $c \in \mathcal{C}$ is isomorphic to a direct sum of simples $c \cong \bigoplus_{i=1}^n s_i^{\oplus m_i}$, where the $s_i \in \mathcal{C}$ are mutually distinct simples.

(SS4) For every object $c \in \mathcal{C}$, the endomorphism algebra $\mathcal{C}(c \to c)$ is a finite dimensional complex semisimple algebra, i.e., a multimatrix algebra.

Proof.

(SS1)⇒(SS2): This is exactly Corollary 7.10.9.

(SS2)⇒(SS4): Every $V = \bigoplus_{s\in S} V_s \in \mathsf{Vec}(S)$ is isomorphic to $\bigoplus_{s\in S} \mathbb{C}_s^{\oplus \dim(V_s)}$.

(SS3)⇒(SS4): Observe that

$$\operatorname{End}_{\mathcal{C}}\left(\bigoplus_{i=1}^n s_i^{\oplus m_i}\right) \cong \bigoplus_{i=1}^n \operatorname{End}(s_i^{\oplus m_i}) \cong \bigoplus_{i=1}^n M_{m_i}(\mathbb{C})$$

which is semisimple.

(SS4)⇒(SS1): We claim that if every endomorphism algebra is semisimple, then the result of Schur's Lemma 7.10.5 holds. That is, simples are either pairwise isomorphic or distinct. Indeed, if s, t are simples, then semisimplicity of

$$\operatorname{End}_{\mathcal{C}}(s \oplus t) \cong \begin{pmatrix} \mathcal{C}(s \to t) & \mathcal{C}(t \to s) \\ \mathcal{C}(s \to t) & \mathcal{C}(t \to t) \end{pmatrix} \cong \begin{pmatrix} \mathbb{C} & \mathcal{C}(t \to s) \\ \mathcal{C}(s \to t) & \mathbb{C} \end{pmatrix}$$

implies $\operatorname{End}_{\mathcal{C}}(s \oplus t)$ is either $M_2(\mathbb{C})$ or $\mathbb{C} \oplus \mathbb{C}$.

Now let $\operatorname{Irr}(\mathcal{C})$ be a set of representatives for the simple objects in $\mathcal{C}$ under the equivalence relation of isomorphism. Observe that the elements of $\operatorname{Irr}(\mathcal{C})$ are pairwise distinct. We first show we can split $\mathcal{C}(c \to c)$ over simples. For $c \in \mathcal{C}$, we have $\operatorname{End}_{\mathcal{C}}(c) = \bigoplus_{k=1}^n M_{m_k}(\mathbb{C})$. For each $k = 1, \ldots, n$, let $(e_{ij}^k)_{i,j=1}^{m_j}$ be a system of matrix units. For each k, let (a_k, r_k, s_k) be a splitting of (c, e_{11}^k). Since $\operatorname{End}((c, e_{11}^k)) = e_{11}^k M_{m_k}(\mathbb{C}) e_{11}^k = \mathbb{C}e_{11}^k$, we see that a_k is simple, so without loss of generality, we may assume $a_k \in \operatorname{Irr}(\mathcal{C})$.

We claim that the composition map

$$\bigoplus_{k=1}^n \mathcal{C}(c \to a_k) \otimes \mathcal{C}(a_k \to c) \longrightarrow \mathcal{C}(c \to c)$$

is an isomorphism. Surjectivity follows by observing that

$$e_{ij}^k = e_{i1}^k e_{1j}^k = e_{i1}^k e_{1j}^k = e_{i1}^k s_k r_k e_{1j}^k,$$

and injectivity follows from observing that $\{r_k e^k_{1j}\}_{j=1}^{m_k} \subset \mathcal{C}(c \to a_k)$ and $\{e^k_{i1} s_k\}_{i=1}^{m_k} \subset \mathcal{C}(a_k \to c)$ are bases. We prove the first set is a basis and the second is similar. If $\sum_{j=1}^{m_k} \lambda_j r_k e^k_{1j} = 0$, then applying e^k_{ii} on the right shows each $\lambda_i = 0$. If $f : c \to a_k$ is an arbitrary morphism, then since $\mathrm{id}_{a_k} = r_k s_k = r_k e^k_{11} s_k$ and $\mathrm{id}_c = \sum_{\ell=1}^{n} \sum_{j=1}^{m_\ell} e^\ell_{jj}$, we have

$$f = \mathrm{id}_{a_k} \circ f \circ \mathrm{id}_c = r_k e^k_{11} s_k f \sum_{\ell=1}^{n} \sum_{j=1}^{m_\ell} e^\ell_{jj} = \sum_{j=1}^{m_k} r_k e^k_{11} s_k f e^k_{jj} \in \mathrm{span}\{r_k e^k_{1j}\}_{j=1}^{m_k}. \qquad \square$$

Corollary 7.10.15 — The following are equivalent for a Cauchy complete linear category.

(fSS1) $\mathcal{C}$ is finite semisimple.

(fSS2) $\mathcal{C} \cong \mathsf{Vec}^{\boxplus n}$ for some $n \geq 1$ (cf. Example 7.8.3).

(fSS3) $\mathcal{C} \cong \mathsf{Mod}(A)$ for some finite dimensional complex semisimple algebra A.

Proof.

(fSS2)⇔(fSS1): Clearly when S is finite, $\mathsf{Vec}(S) \cong \mathsf{Vec}^{|S|}$.

(fSS2)⇒(fSS3): If $\mathcal{C} \cong \mathsf{Vec}^{\boxplus n}$, set $A = \mathbb{C}^n$ and note $\mathsf{Mod}(A) \cong \mathsf{Vec}^{\boxplus n}$.

(fSS3)⇒(fSS2): This is exactly Example 7.8.2. $\square$

Remark 7.10.16. Suppose $\mathcal{C}$ is a (finite) semisimple category and $X \in \mathcal{C}$ is any object such that $\mathcal{C}_\mathcal{C}(s \to X) \neq 0$ for all $s \in \mathrm{Irr}(\mathcal{C})$. Since $X \cong \bigoplus_{s \in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to X) \otimes s$, we see that $\mathrm{End}_\mathcal{C}(X) \cong \bigoplus_{s \in \mathrm{Irr}(\mathcal{C})} \mathrm{End}(\mathcal{C}(s \to X))$, which is a semisimple algebra whose simple summands correspond to simples $s \in \mathrm{Irr}(\mathcal{C})$. Thus $\mathsf{Mod}(\mathrm{End}_\mathcal{C}(X)) \cong \mathsf{Vec}(S) \cong \mathcal{C}$.

Corollary 7.10.17 — Suppose $\mathcal{C}, \mathcal{D}$ are semisimple and $F : \mathcal{C} \to \mathcal{D}$ is linear.

(1) F is faithful if and only if no simple in $\mathcal{C}$ is sent to $0_\mathcal{D}$.

(2) F is fully faithful if and only if it sends distinct simples to distinct simples.

(3) F is essentially surjective if and only if each simple in $\mathcal{D}$ is isomorphic to the image of a simple object in $\mathcal{C}$.

Proof. (1) is immediate from (7.10.11). Next, for simples $a, b \in \mathrm{Irr}(\mathcal{C})$, we have

$$\mathcal{C}(a \to b) \cong \mathcal{D}(F(a) \to F(b))$$

if and only if F maps distinct simples to distinct simples. Since F is completely determined by where it sends simples by (7.10.11), (2) follows. For the forward direction of (3), it suffices to prove that if $c \in \mathcal{C}$ such that $F(c)$ is simple, then c is simple, which is immediate from (7.10.11). The reverse direction follows from the fact that linear functors preserve direct sums together with (SS3). $\square$

Exercise 7.10.18. Use Corollary 7.10.17 and the Pigeonhole Principle to prove the Rank-Nullity Theorem for finite semisimple categories. That is, if $\mathcal{C}$ is finite semisimple and $F : \mathcal{C} \to \mathcal{C}$ is linear, then the following are equivalent:

(RN1) F is fully faithful.

(RN2) F is essentially surjective.

(RN3) F is an equivalence.

Theorem 7.10.19 (Fundamental Theorem of pre-semisimple categories) — Suppose $\mathcal{C}$ is a pre-semisimple category. The following conditions are equivalent.

(pSS1) $\mathcal{C}$ is semisimple.

(pSS2) $\mathcal{C} \cong \mathsf{Vec}(S)$ for some set S.

(pSS3) $\mathcal{C}$ is Cauchy complete

If moreover $\mathcal{C}$ is finite, then the above conditions are equivalent to:

(pSS4) $\mathcal{C} \cong \mathsf{Vec}^{\boxplus n}$ for some $n \geq 1$.

(pSS5) $\mathcal{C} \cong \mathsf{Mod}(A)$ for some semisimple finite dimensional complex algebra A.

Proof.

(pSS1)$\Leftrightarrow$(pSS2): Since semisimple categories are Cauchy complete by Example 7.7.2 and Corollary 7.10.9, the result is immediate from (SS1)$\Leftrightarrow$(SS2) from Theorem 7.10.14.

(pSS1)$\Leftrightarrow$(pSS3): Immediate from the equivalence (SS1)$\Leftrightarrow$(SS4) from Theorem 7.10.14.

Now assume that $\mathcal{C}$ is finite.

(pSS2)$\Leftrightarrow$(pSS4): Obvious

(pSS4)$\Leftrightarrow$(pSS5): Since $\mathsf{Mod}(A)$ is Cauchy complete by Example 7.7.5, the result follows by (fSS2)$\Leftrightarrow$(fSS3) from Corollary 7.10.15. $\square$

Corollary 7.10.20 — Suppose $\mathcal{C}$ is a finite pre-semisimple category. Then $\mathcal{C}^{¢}$ is finite semisimple. In particular,

- the Yoneda embedding $よ : \mathcal{C} \hookrightarrow \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})$ extends to an equivalence $よ^{¢} : \mathcal{C}^{¢} \to \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})$, and
- there are $a_1, \ldots, a_n \in \mathcal{C}$ such that $\mathcal{C}^{¢} \cong \mathsf{Mod}(L(a_1, \ldots, a_n))$.

Proof. Observe that every endomorphism algebra of $\mathcal{C}^{¢}$ is a finite dimensional complex semisimple algebra by Lemma **??**, so $\mathcal{C}^{¢}$ is semisimple. Enumerate $\mathrm{Irr}(\mathcal{C}^{¢}) = \{s_1, s_2, \ldots, \}$, and pick for each $s_i \in \operatorname{Irr}(\mathcal{C}^{¢})$ an $a_i \in \mathcal{C}$ such that $\mathcal{C}^{¢}(s_i \to a_i) \neq 0$. (Exercise: why does such an $a_i \in \mathcal{C}$ exist?) We know that there is a global bound

$$\dim(Z(L(a_1, \ldots, a_k))) < K \qquad\qquad \forall\, 1 \leq k \leq |\operatorname{Irr}(\mathcal{C}^{¢})|.$$

Claim. $|\operatorname{Irr}(\mathcal{C}^{¢})| \leq K$.

Proof of Claim. Setting $X_k := \bigoplus_{i=1}^{k} a_i \in \mathcal{C}^{¢}$, we have

$$L(a_1, \ldots, a_k) \cong \operatorname{End}_{\mathcal{C}^{¢}}(X_k).$$

Since each s_i admits a non-zero map to X_k for $i = 1, \ldots, k$, decomposing X_k into simples in $\mathcal{C}^{¢}$, we have $k \leq \dim Z(\operatorname{End}_{\mathcal{C}^{¢}}(X_k)) < K$, and the claim follows. □

The first bullet point now follows from the universal property Proposition 7.7.9 of Cauchy completion together with Corollary 7.10.13. The second bullet point follows from Remark 7.10.16. In more detail, X_k for $k = |\operatorname{Irr}(\mathcal{C}^{¢})|$ satisfies the conditions in Remark 7.10.16 so that

$$\mathcal{C}^{¢} \cong \mathsf{Mod}(\operatorname{End}_{\mathcal{C}^{¢}}(X_k)) \cong \mathsf{Mod}(L(a_1, \ldots, a_k)).$$ □

7.11 Abelian categories

In this section, we review the definition of an abelian category and we prove that semisimple categories over $\mathbb{C}$ are automatically abelian. We assume $\mathcal{C}$ is a linear category which admits direct sums.

Definition 7.11.1 — Fix $f, g \in \mathcal{C}(a \to b)$. The *equalizer* of f and g is an object $e \in \mathcal{C}$ together with a map $i : e \to a$ such that $f \circ i = g \circ i$ which satisfies the following universal property:

(Eq) for any object e' and map $i' : e' \to a$ such that $f \circ i' = g \circ i'$, there is a unique map $j : e' \to e$ such that $i' = ij$.

Dually, the *coequalizer* of f and g is an object c together with a map $p\colon b \to c$ such that $p \circ f = p \circ g$ which satisfies the following universal property:

(CoEq) for any object c' and map $p'\colon b \to c'$ such that $p' \circ f = p' \circ g$, there is a unique map $q\colon c \to c'$ such that $p' = q \circ p$.

Exercise 7.11.2. Show that the (co)equalizer of f and g is unique up to unique isomorphism when it exists. That is, show that the space of (co)equalizers of f and g is contractible.

Definition 7.11.3 — Let $f \in \mathcal{C}(a \to b)$. The *kernel* of f is the equalizer of f and zero and is denoted $\ker(f)$. Dually, the *cokernel* of f is the coequalizer of f and zero and is denoted $\mathrm{coker}(f)$.

Warning 7.11.4 — The kernel $\ker(f)$ is really a pair consisting of an object $k \in \mathcal{C}$ and a morphism $i : k \to a$ satisfying the universal property (Eq). When the meaning is clear from type-checking we may refer to either just k or i using the notation $\ker(f)$.

Similarly, the cokernel $\mathrm{coker}(f)$ is a pair consisting of an object $c \in \mathcal{C}$ and a morphism $p : b \to c$ satisfying the universal property (CoEq). The same warning applies to the use of $\mathrm{coker}(f)$ to denote either c or p.

Definition 7.11.5 — The *image* of $f \in \mathcal{C}(a \to b)$ is $\mathrm{im}(f) := \ker(\mathrm{coker}(f))$. Unpacked,

$$\begin{tikzcd}
a \arrow[r, "f"] & b \arrow[r] & \mathrm{coker}(f) \\
& \mathrm{im}(f) \arrow[u, dashed] \arrow[ur, dashed, "0"'] &
\end{tikzcd}$$

Conversely, the *coimage* of f is $\mathrm{coim}(f) := \mathrm{coker}(\ker(f))$. Unpacked,

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[dr, dashed, "0"'] & a \arrow[r, "f"] \arrow[d, dashed] & b \\
& \mathrm{coim}(f) &
\end{tikzcd}$$

Exercise 7.11.6. Suppose $\mathcal{C} = \mathsf{Vec}$ and $f\colon V \to W$ is a linear map. Show that $\mathrm{im}(f)$ corresponds to the image of f in W and $\mathrm{coim}(x)$ corresponds to the quotient $V/\ker(f)$.

Construction 7.11.7 — Let $f \in \mathcal{C}(a \to b)$. There is a canonical map making the following diagram commute

$$\begin{tikzcd}
\ker(f) \arrow[r, "i"] \arrow[dr, "0"'] & a \arrow[r, "f"] \arrow[d] & b \arrow[r, "p"] & \mathrm{coker}(f) \\
& \mathrm{coim}(f) \arrow[r, dashed, "\tilde{f}"] & \mathrm{im}(f) \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

Proof. Since $p \circ f = 0$ by the definition of the cokernel, the universal property of cokernels yields a unique map $f|^{\mathrm{im}(f)} \in \mathcal{C}(a \to \mathrm{im}(f))$ making the middle triangle commute.

$$\begin{tikzcd}
\ker(f) \arrow[r, "i"] \arrow[dr, "0"'] & a \arrow[r, "f"] \arrow[d] \arrow[dr, dashed, "\exists!"'] & b \arrow[r, "p"] & \mathrm{coker}(f) \\
& \mathrm{coim}(f) & \mathrm{im}(f) \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

Similarly, there is a unique map $\ker(f) \to \mathrm{im}(f)$ making the resulting triangle commute.

$$\begin{tikzcd}
\ker(f) \arrow[r, "i"] \arrow[dr, "0"'] \arrow[drr, dashed] & a \arrow[r, "f"] \arrow[d, "\exists!"'] \arrow[dr, "f|^{\mathrm{im}(f)}"] & b \arrow[r, "p"] & \mathrm{coker}(f) \\
& \mathrm{coim}(f) & \mathrm{im}(f) \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

Since both $0 \in \mathcal{C}(\ker(f) \to \mathrm{im}(f))$ and $f|^{\mathrm{im}(f)} \circ i$ satisfy this condition, $f|^{\mathrm{im}(f)} \circ i = 0$. The universal property of the coimage then yields a unique map $\mathrm{coim}(f) \to \mathrm{im}(f)$ making the resulting triangle commute, as desired.

$$\begin{tikzcd}
\ker(f) \arrow[r, "i"] \arrow[dr, "0"'] & a \arrow[r, "f"] \arrow[d] \arrow[dr, "f|^{\mathrm{im}(f)}"] & b \arrow[r, "p"] & \mathrm{coker}(f) \\
& \mathrm{coim}(f) \arrow[r, dashed, "\widetilde{f}"] & \mathrm{im}(f) \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

□

Definition 7.11.8 — A linear category $\mathcal{C}$ which admits direct sums is called *abelian* if it admits both kernels and cokernels, and the canonical map $\widetilde{f} \in \mathcal{C}(\mathrm{coim}(f) \to \mathrm{im}(f))$ is an isomorphism for every morphism f in $\mathcal{C}$.

Remark 7.11.9. The property that $\widetilde{f}$ is an isomorphism can be stated succinctly as 'the kernel of the cokernel is the cokernel of the kernel.'

Example 7.11.10 — When $\mathcal{C}$ is the category of abelian groups, the fact that each $\widetilde{f}$ is an isomorphism is known as the *First Isomorphism Theorem*.

Example 7.11.11 — The category Vec is abelian by the Rank-Nullity Theorem 1.1.20.

Remark 7.11.12. An abelian linear category is always equivalent to a full subcategory of right modules over a unital complex algebra [**?**].

Proposition 7.11.13 — If $\mathcal{C}, \mathcal{D}$ are abelian linear categories, then so is $\mathcal{C} \boxplus \mathcal{D}$.

Proof. A straightforward calculation shows that if $f \boxplus g : c_1 \boxplus d_1 \to c_2 \boxplus d_2$, then $\ker(f \boxplus g) = \ker(f) \boxplus \ker(g)$ and $\mathrm{coker}(f \boxplus g) = \mathrm{coker}(f) \boxplus \mathrm{coker}(g)$, and similarly for images and coimages. It follows that $(f \boxplus g)^{\sim} = \widetilde{f} \boxplus \widetilde{g}$, whence the result follows. □

Since Vec is abelian and every finite semisimple category is equivalent to $\mathsf{Vec}^{\boxplus n}$, we have the following immediate corollary.

Corollary 7.11.14 — Finite semisimple linear categories are abelian.

Theorem 7.11.15 — Semisimple linear categories are abelian.

Proof. It suffices to show $\mathsf{Vec}(S)$ is abelian. If $V, W \in \mathsf{Vec}(S)$ and $f : V \to W$, there is a finite subset $S_0 \subset S$ such that V, W and $f : V \to W$ all live in the full subcategory $\mathsf{Vec}(S_0) \subset \mathsf{Vec}(S)$. Since $\mathsf{Vec}(S_0)$ is abelian by Corollary 7.11.14, $\widetilde{f}$ is an isomorphism in $\mathsf{Vec}(S_0)$ and thus also in $\mathsf{Vec}(S)$. The result follows. $\square$

Exercise 7.11.16 (Abelian categories c.f. [EGNO15, Def. 1.3.1])**.** Prove that a linear category which admits direct sums is abelian if and only for every $f : a \to b$, there is a sequence

$$K \xrightarrow{k} a \xrightarrow{i} I \xrightarrow{j} b \xrightarrow{c} C$$

such that $ji = f$, $(K, k) = \ker(f)$, $(C, c) = \operatorname{coker}(f)$, $(I, i) = \operatorname{coker}(k)$, and $(I, j) = \ker(c)$.