Math 2568 Homework 7
Math 2568 Due: Monday, October 14, 2019

Problem 1

§5.4, Exercise 7. Show that the functions $f_1(t) = \sin t$, $f_2(t) = \cos t$, and $f_3(t) = \cos\left(t + \frac{\pi}{3}\right)$ are linearly dependent vectors in $\mathcal{C}^1$.

The three functions are linearly dependent vectors in $\mathcal{C}^1$ since there exists a nonzero vector $r = (r_1, r_2, r_3)$ such that $r_1 f_1(t) + r_2 f_2(t) + r_3 f_3(t) = 0$. We can find this vector r using trigonometric identities:

$$f_3(t) = \cos\left(t + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)\cos t + \sin\left(\frac{\pi}{3}\right)\sin t = \frac{1}{2}\cos t - \frac{\sqrt{3}}{2}\sin t = \frac{1}{2}f_1(t) - \frac{\sqrt{3}}{2}f_2(t).$$

That is, $\frac{1}{2}f_1(t) + \frac{\sqrt{3}}{2}f_2(t) - f_3(t) = 0$.

Problem 2 (MATLAB)

Determine whether the given sets of vectors are linearly independent or linearly dependent.

§5.4, Exercise 9.(MATLAB)

$$v_1 = (2, 1, 3, 4) \quad v_2 = (-4, 2, 3, 1) \quad v_3 = (2, 9, 21, 22) \qquad (1^*)$$

Answer: The set $\{v_1, v_2, v_3\}$ is linearly dependent.

Solution: The set is linearly dependent if there exist scalars r_1, r_2, and r_3 such that $r_1 v_1 + r_2 v_2 + r_3 v_3 = 0$. Create a matrix `A` whose columns are v_1, v_2 and v_3. Then row reduce `A` to solve the homogeneous system $Ar = 0$. Specifically, row reducing the matrix `A = [v1 v2 v3]` yields

```
ans =
     1     0     5
     0     1     2
     0     0     0
     0     0     0
```

So $-5v_1 - 2v_2 + v_3 = 0$.

Problem 3 (MATLAB)

§5.4, Exercise 12.(MATLAB) Perform the following experiments.

(a) Use MATLAB to choose randomly three column vectors in $\mathbb{R}^3$. The MATLAB commands to choose these vectors are:

```
y1 = rand(3,1)
y2 = rand(3,1)
y3 = rand(3,1)
```

Use the methods of this section to determine whether these vectors are linearly independent or linearly dependent.

(b) Now perform this exercise five times and record the number of times a linearly independent set of vectors is chosen and the number of times a linearly dependent set is chosen.

(c) Repeat the experiment in (b) — but this time randomly choose four vectors in $\mathbb{R}^3$ to be in your set.

(a) The set of commands to perform this experiment is:

```
y1 = rand(3,1);
y2 = rand(3,1);
y3 = rand(3,1);
A = [y1 y2 y3];
rref(A)
```

If the resulting matrix is I_3, then the set is linearly independent.

(b) The most likely outcome is that all five trials result in linearly independent sets.

(c) Every trial yields a linearly dependent set of vectors.

Problem 4

§5.5, Exercise 1. Show that $\mathcal{U} = \{u_1, u_2, u_3\}$ where

$$u_1 = (1, 1, 0) \quad u_2 = (0, 1, 0) \quad u_3 = (-1, 0, 1)$$

is a basis for $\mathbb{R}^3$.

By Theorem 5.5.3, $\mathcal{U}$ is a basis for $\mathbb{R}^3$ if the vectors of $\mathcal{U}$ are linearly independent and span $\mathbb{R}^3$. By Lemma 5.5.4, the dimension of $\mathcal{U}$ is equal to the rank of the matrix whose rows are u_1, u_2, and u_3. Row reduce this matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\dim(\mathcal{U}) = 3 = \dim(\mathbb{R}^3)$, and we need now only show that u_1, u_2, and u_3 are linearly independent, which we can do by row reducing the matrix whose columns are the vectors of $\mathcal{U}$ as follows:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, there is no nonzero solution to the equation $\mathcal{U}r = 0$, so the vectors of $\mathcal{U}$ are linearly independent and $\mathcal{U}$ is a basis for $\mathbb{R}^3$.

Problem 5

§5.5, Exercise 2. Let $S = \text{span}\{v_1, v_2, v_3\}$ where

$$v_1 = (1, 0, -1, 0) \quad v_2 = (0, 1, 1, 1) \quad v_3 = (5, 4, -1, 4).$$

Find the dimension of S and find a basis for S.

Answer: The dimension of S is 2, and vectors v_1 and v_2 form a basis for S.

Solution: Row reduce the matrix A whose rows are v_1, v_2, and v_3. By Lemma 5.6.4, the number of nonzero rows in the reduced matrix is the dimension of S and these rows form a basis for S. So:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 5 & 4 & -1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 6

§5.5, Exercise 4. Show that the set V of all 2×2 matrices is a vector space. Show that the dimension of V is four by finding a basis of V with four elements. Show that the space $M(m, n)$ of all $m \times n$ matrices is also a vector space. What is $\dim M(m, n)$?

The set V is a vector space because the operations of addition and scalar multiplication satisfy the eight properties of vector spaces described in Table 1. For 2×2 matrices, matrix addition is defined for two matrices such that:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

and scalar multiplication is defined for a matrix and a scalar such that:

$$s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix}.$$

So, using these definitions, addition is commutative and associative, and the additive identity is the 2×2 matrix of zeroes. If

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \text{ then } \quad W^{-1} = \begin{pmatrix} -w_{11} & -w_{12} \\ -w_{21} & -w_{22} \end{pmatrix}.$$

Scalar multiplication is associative. There is a multiplicative identity, I_2, and scalar multiplication is distributive both for scalars and for matrices. So V is a vector space. One basis for V is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The set of $m \times n$ matrices is also a vector space, since it also satisfies the eight properties of vector spaces. In this case, the additive identity is the $m \times n$ zero matrix, and the multiplicative identity is I_n. The dimension of the set is mn, since one basis consists of the mn matrices with $a_{ij} = 1$ and all other entries 0, for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Problem 7

§5.5, Exercise 5. Show that the set $\mathcal{P}_n$ of all polynomials of degree less than or equal to n is a subspace of $\mathcal{C}^1$. What is $\dim \mathcal{P}_2$? What is $\dim \mathcal{P}_n$?

The set P_n is a subspace if it is closed under addition and scalar multiplication. Let $x(t) = a_0 + a_1 t + \cdots + a_n t^n$, $y(t) = b_0 + b_1 t + \cdots + b_n t^n$ and $s \in \mathbb{R}$. Then

$$x(t) + y(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n \in P_n.$$
$$cx(t) = c(a_0 + a_1 t + \cdots + a_n t^n) = ca_0 + ca_1 t + \cdots + ca_n t^n \in P_n.$$

The dimension of P_2 is 3, since $x_1 = 1$, $x_2 = t$, and $x_3 = t^2$ form a basis for P_2. The dimension of P_n is $n + 1$.

Problem 8

§5.6, Exercise 5. Let A be a 7×5 matrix with $\text{rank}(A) = r$.

(a) What is the largest value that r can have?

(b) Give a condition equivalent to the system of equations $Ax = b$ having a solution.

(c) What is the dimension of the null space of A?

(d) If there is a solution to $Ax = b$, then how many parameters are needed to describe the set of all solutions?

(a) The largest value that r can have is 5, since the matrix has 5 columns. Thus, the reduced echelon form matrix can have at most 5 pivot points.

(b) The equation $Ax = b$ has a solution if the rank of the augmented matrix $(A|b)$ is r. If $\text{rank}(A|b)$ is greater than r, then there is a pivot in the 6^{th} column and the system is inconsistent, so there is no solution.

(c) The null space has dimension $5 - r$.

(d) The number of parameters needed to describe the solution to $Ax = b$ is $5-r$, since $5 - r$ parameters are needed to describe the solutions to $Ax = 0$, and the solutions to the inhomogeneous system are obtained by adding the solutions of the homogeneous system to one solution of the inhomogeneous system.

Problem 9

§5.6, Exercise 6. Let

$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 1 & 5 & 7 \\ 3 & 4 & 4 & 11 \end{pmatrix}.$$

(a) Find a basis for the subspace $\mathcal{C} \subset \mathbb{R}^3$ spanned by the columns of A.

(b) Find a basis for the subspace $\mathcal{R} \subset \mathbb{R}^4$ spanned by the rows of A.

(c) What is the relationship between $\dim \mathcal{C}$ and $\dim \mathcal{R}$?

Answer: (a) The vectors $(1, 2, 3)$ and $(3, 1, 4)$ form a basis for the subspace $\mathcal{C}$ of $\mathbb{R}^3$ spanned by the columns of A.

(b) The vectors $(1, 3, -1, 4)$ and $(2, 1, 5, 7)$ form a basis for the subspace $\mathbb{R}$ of $\mathbb{R}^4$ spanned by the rowss of A.

(c) $\dim \mathcal{C} = \dim \mathbb{R}$.

Solution: (a) Note that

$$\begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} = \frac{16}{5}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{7}{5}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$
$$\begin{pmatrix} 4 \\ 7 \\ 11 \end{pmatrix} = \frac{17}{5}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

So the two vectors $(1, 2, 3)$ and $(3, 1, 4)$ span $\mathcal{C}$. Since they are linearly independent, these vectors are a basis for $\mathcal{C}$ and $\dim = \mathcal{C} = 2$.

(b) Note that

$$(3, 4, 4, 11) = (1, 3, -1, 4) + (2, 1, 5, 7).$$

Therefore, $\{(1, 3, -1, 4), (2, 1, 5, 7)\}$ is a basis for $\mathcal{R}$ and $\dim \mathcal{R} = 2$

Problem 10

§5.6, Exercise 14. Let $\{v_1, v_2, v_3\}$ and $\{w_1, w_2\}$ be linearly independent sets of vectors in a vector space V. Show that if

$$\text{span}\{v_1, v_2, v_3\} \cap \text{span}\{w_1, w_2\} = \{0\}$$

then

$$\dim(\text{span}\{v_1, v_2, v_3, w_1, w_2\}) = 5$$

Hint: First show that if $v \in \text{span}\{v_1, v_2, v_3\}$, $w \in \text{span}\{w_1, w_2\}$, and $v + w = 0$, then $v = w = 0$.

Solution: First, we verify the claim in the hint. Let $v \in \text{span}\{v_1, v_2, v_3\}$, $w \in \text{span}\{w_1, w_2\}$, and $v + w = 0$. Since $\text{span}\{w_1, w_2, w_3\}$ is closed under scalar multiplication, it follows that $v = -w \in \text{span}\{w_1, w_2\}$. Therefore

$$v \in \text{span}\{v_1, v_2, v_3\} \cap \text{span}\{w_1, w_2\} = \{0\}$$

and $v = w = 0$.

Next, we show that $\{v_1, v_2, v_3, w_1, w_2\}$ is a linearly independent set, which implies that it is a basis for $\text{span}\{v_1, v_2, v_3, w_1, w_2\}$. Suppose that a_1, a_2, a_3 and b_1, b_2 are scalars so that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + b_1 w_1 + b_2 w_2 = 0 \tag{2}$$

Let $v = a_1 v_1 + a_2 v_2 + a_3 v_3$ and $w = b_1 w_1 + b_2 w_2$. Then $v + w = 0$. So, by the hint, $v = w = 0$. Since $\{v_1, v_2, v_3\}$ is a linearly independent set so

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = v = 0$$

implies that $a_1 = a_2 = a_3 = 0$. Similarly, $\{w_1, w_2\}$ is a linearly independent set so

$$b_1 w_1 + b_2 w_2 = w = 0$$

implies that $b_1 = b_2 = 0$. Therefore, the only solution to (2) is $a_1 = a_2 = a_3 = b_1 = b_2 = 0$. Thus, the set $\{v_1, v_2, v_3, w_1, w_2\}$ is linearly independent and therefore a basis for its span. It follows that

$$\dim(\operatorname{span}\{v_1, v_2, v_3, w_1, w_2\}) = 3 + 2 = 5.$$

Problem 11

In Exercises 15-20 decide whether the statement is true or false, and explain your answer.

§5.6, Exercise 15. Every set of three vectors in $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$. **Answer:** False

Solution: The vectors could be linearly independent. For example $\{e_1, e_2, e_1 + e_2\}$ is not a basis for $\mathbb{R}^3$.

Problem 12

In Exercises 15-20 decide whether the statement is true or false, and explain your answer.

§5.6, Exercise 20. If U is a subspace of $\mathbb{R}^3$ of dimension 1 and V is a subspace of $\mathbb{R}^3$ of dimension 2, then $U \cap V = \{0\}$. **Answer:** False **Solution:** $U \cap V$ is always subspace of $\mathbb{R}^3$, but its dimension could be one. For example, if U is the x-axis ($U = \operatorname{span}\{e_1\}$) and V is the xy-plane ($V = \operatorname{span}\{e_1, e_2\}$) then $U \cap V = U$. $U \cap V$ is always subspace of $\mathbb{R}^3$, but its dimension could be one. For example, if U is the x-axis ($U = \operatorname{span}\{e_1\}$) and V is the xy-plane ($V = \operatorname{span}\{e_1, e_2\}$), then $U \cap V = U$.