Math 2568 Homework 8
Math 2568 Due: Monday, October 21, 2019

Problem 1

Determine whether or not each of the given functions $x_1(t)$ and $x_2(t)$ is a solution to the given differential equation.

§4.1, Exercise 2. ODE: $\frac{dx}{dt} = x + e^t$.

Functions: $x_1(t) = te^t$ and $x_2(t) = 2e^t$.

Answer: The function $x_1(t)$ is a solution to the differential equation; the function $x_2(t)$ is not a solution.

Solution: Compute

$$\frac{d}{dt}(x_1) = \frac{d}{dt}(te^t) = te^t + e^t, \quad \text{and} \quad \frac{dx_1}{dt} = x_1 + e^t = te^t + e^t.$$

Thus, $x_1(t)$ is a solution to the differential equation. Then compute

$$\frac{d}{dt}(x_2) = \frac{d}{dt}(2e^t) = 2e^t, \quad \text{and} \quad \frac{dx_2}{dt} = x_2 + e^t = 2e^t + e^t = 3e^t.$$

Thus, $\frac{d}{dt}(x_2) \neq \frac{dx_2}{dt}$, so $x_2(t)$ is not a solution to the differential equation.

Problem 2

§4.1, Exercise 6. Solve the differential equation

$$\frac{dx}{dt} = -3x.$$

At what time t_1 will $x(t_1)$ be half of $x(0)$?

Answer: Using the initial value problem, we find that $\frac{dx}{dt} = -3x$ implies $x(t) = x_0 e^{-3t}$. Given this equation, $x(t_1)$ will be half of $x(0)$ at time $t_1 = -\frac{1}{3}\ln(0.5)$.

Solution: Find this value of t_1 by substituting into the formula for x. That is, use:

$$x_0 e^{-3t_1} = x(t_1) = \frac{1}{2}x_0$$

which implies

$$e^{-3t_1} = \frac{1}{2}.$$

Then solve for t_1.

Problem 3

Consider the uncoupled system of differential equations (4.3.2). For each choice of a and d, determine whether the origin is a saddle, source, or sink.

§4.3, Exercise 3. $a = 1$ and $d = -1$.

Answer: The origin is a saddle.

Solution: This uncoupled system is of the form

$$\begin{array}{rcl} \frac{dx}{dt}(t) & = & Ax(t) \\ \frac{dy}{dt}(t) & = & Dy(t) \end{array}$$

If $AD < 0$, then the origin is a saddle. If $A < 0$ and $D < 0$, then the origin is a sink. If $A > 0$ and $D > 0$, then the origin is a source. In this case, $AD = -1 < 0$.

Problem 4

§4.3, Exercise 6. Let $(x(t), y(t))$ be the solution (4.3.3) of (4.3.2) with initial condition $(x(0), y(0)) = (x_0, y_0)$, where $x_0 \neq 0 \neq y_0$.

(a) Show that the points $(x(t), y(t))$ lie on the curve whose equation is:

$$y_0^a x^d - x_0^d y^a = 0.$$

(b) Verify that if $a = 1$ and $d = 2$, then the solution lies on a parabola tangent to the x-axis.

The solutions $x(t)$ and $y(t)$ are:

$$\begin{array}{rcl} x(t) & = & x_0 e^{At} \\ y(t) & = & y_0 e^{Dt} \end{array}.$$

We show that the point $(x(t), y(t))$ lies on the curve $y_0^A x^D - x_0^D y^A = 0$ as follows. Substitute the formulas for $x(t)$ and $y(t)$ into the equation to obtain

$$y_0^A \left(x_0 e^{At}\right)^D - x_0^D \left(y_0 e^{Dt}\right)^A = x_0^D y_0^A e^{ADt} - x_o^D y_0^A e^{ADt} = 0.$$

If $A = 1$ and $D = 2$, then the solutions lie on the curve $0 = y_0 x^2 - x_0^2 y$, which can be rewritten as $y = \frac{y_0}{x_0^2} x^2$. Since x_0 and y_0 are constants, this curve is a parabola tangent to the x-axis.

Problem 5 (MATLAB)

§4.3, Exercise 10.(MATLAB) Suppose that $a = d < 0$. Verify experimentally using pplane9 that all trajectories approach the origin along straight lines. Try to prove this conjecture?

If $A = D < 0$, the equations for the system will be

$$\begin{array}{rcl} \frac{dx}{dt}(t) & = & Ax(t) \\ \frac{dy}{dt}(t) & = & Ay(t) \end{array}.$$

Therefore,

$$x(t) = x_0 e^{At} \quad \text{and} \quad y(t) = y_0 e^{At}.$$

Solve the first equation for e^{At} and substitute into the second, obtaining

$$y(t) = \frac{y_0}{x_0} x(t).$$

Since $\frac{y_0}{x_0}$ is a constant, all trajectories are straight lines. Since $A < 0$, all trajectories go toward the origin as t increases.

Problem 6 (MATLAB)

§4.4, Exercise 1.(MATLAB) Choose the linear system in pplane9 and set $a = 0$, $b = 1$, and $c = -1$. Then find values d such that except for the origin itself all solutions appear to

(a) spiral into the origin;

(b) spiral away from the origin;

(c) form circles around the origin;

Solution:

(a) All trajectories converge on the origin when $D < 0$, as shown in Figure 1a, which graphs the system with $D = -1$;

(b) All trajectories move away from the origin when $D > 0$, as shown in Figure 1b, which graphs the system with $D = 1$

(c) Trajectories form circles around the origin when $D = 0$, as shown in Figure 1c.

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/3-5-a01a.eps exfigure/3-5-a01b.eps exfigure/3-5-a01c.eps

$D = -1$
Figure 1a

$D = 1$
Figure 1b

$D = 0$
Figure 1c

Problem 7

Determine which of the function pairs $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions to the given system of ordinary differential equations.

§4.4, Exercise 6. The ODE is:

$$\begin{aligned} \dot{x} &= 2x - 3y \\ \dot{y} &= x - 2y. \end{aligned}$$

The pairs of functions are:

$$(x_1(t), y_1(t)) = e^t(3, 1) \quad \text{and} \quad (x_2(t), y_2(t)) = (e^{-t}, e^{-t}).$$

Answer: Both function pairs are solutions to the given system.

Solution: To determine whether $(x_1(t), y_1(t)) = (3e^t, e^t)$ is a solution to the system, compute the left hand sides of the equations:

$$\frac{dx_1}{dt}(t) = \frac{d}{dt}(3e^t) = 3e^t \quad \text{and} \quad \frac{dy_1}{dt}(t) = \frac{d}{dt}(e^t) = e^t.$$

Then compute the right hand sides of the equations:

$$2x_1(t) - 3y_1(t) = 2(3e^t) - 3e^t = 3e^t \quad \text{and} \quad x_1(t) - 2y_1(t) = 3e^t - 2e^t = e^t.$$

Since the left hand side of each equation equals the right hand side, the equations are consistent, and the pair of functions is a solution.

Similarly, to determine whether $(x_2(t), y_2(t)) = (e^{-t}, e^{-t})$ is a solution to the system, compute the left hand sides of the equations:

$$\frac{dx_2}{dt}(t) = \frac{d}{dt}(e^{-t}) = -e^{-t} \quad \text{and} \quad \frac{dy_2}{dt}(t) = \frac{d}{dt}(e^{-t}) = -e^{-t}.$$

Then compute the right hand sides of the equations:

$2x_2(t) - 3y_2(t) = 2e^{-t} - 3e^{-t} = -e^{-t}$ and $x_2(t) - 2y_2(t) = e^{-t} - 2e^{-t} = -e^{-t}$.

Since the left hand side of each equation equals the right hand side, the equations are consistent, and the pair of functions is a solution.

Problem 8

§4.5, Exercise 2. Show that all solutions to the system of linear differential equations

$$\begin{aligned} \frac{dx}{dt} &= 3x \\ \frac{dy}{dt} &= -2y \end{aligned}$$

are linear combinations of the two solutions

$$U(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V(t) = e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The system is uncoupled, so we can solve each equation independently, using the initial value problem to obtain:

$$\begin{aligned} x(t) &= x_0 e^{3t} \\ y(t) &= y_0 e^{-2t}. \end{aligned}$$

All solutions are of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{3t} \\ y_0 e^{-2t} \end{pmatrix} = x_0 \left(e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + y_0 \left(e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

So all solutions are linear combinations of

$$U(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V(t) = e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Problem 9

§4.5, Exercise 3. Consider

$$\frac{dX}{dt}(t) = CX(t) \tag{1}$$

where

$$C = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}.$$

Let

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and let

$$Y(t) = e^{2t} v_1 \quad \text{and} \quad Z(t) = e^{-t} v_2.$$

(a) Show that $Y(t)$ and $Z(t)$ are solutions to (1).

(b) Show that $X(t) = 2Y(t) - 14Z(t)$ is a solution to (1).

(c) Use the principle of superposition to verify that $X(t) = \alpha Y(t) + \beta Z(t)$ is a solution to (1).

(d) Using the general solution found in part (c), find a solution $X(t)$ to (1) such that

$$X(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Solution:

(a) In order to determine that $Y(t)$ is a solution to (1), substitute $Y(t)$ into both sides of the equation $\frac{dX}{dt} = CX$:

$$\frac{dY}{dt} = \frac{d}{dt}\left(e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{d}{dt}\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 0 \end{pmatrix};$$

$$CY(t) = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 0 \end{pmatrix}.$$

Similarly, show that $Z(t)$ is a solution:

$$\frac{dZ}{dt} = \frac{d}{dt}\left(e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \frac{d}{dt}\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix};$$

$$CZ(t) = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}.$$

(b) Again, verify that $X(t) = 2Y(t) - 14Z(t)$ is a solution to (1) by substituting into both sides of the equation and noting that the values are equal:

$$\frac{dX}{dt} = \frac{d}{dt}\left(2e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 14e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \frac{d}{dt}\begin{pmatrix} 2e^{2t} - 14e^{-t} \\ 14e^{-t} \end{pmatrix} = \begin{pmatrix} 4e^{2t} + 14e^{-t} \\ -14e^{-t} \end{pmatrix};$$

$$CX(t) = C\left(2Y(t) - 14Z(t)\right) = C\left(\begin{pmatrix} 2e^{2t} \\ 0 \end{pmatrix} - \begin{pmatrix} 14e^{-t} \\ -14e^{-t} \end{pmatrix}\right) = \begin{pmatrix} 4e^{2t} + 14e^{-t} \\ -14e^{-t} \end{pmatrix}.$$

(c) As demonstrated in Section 3.4, if $Y(t)$ and $Z(t)$ are both solutions to (1), then $X(t) = \alpha Y(t) + \beta Z(t)$ is also a solution to (1).

(d) **Answer:**

$$X(t) = 2e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution: Note that

$$X(t) = \alpha Y(t) + \beta Z(t) = \alpha e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a solution to (1). Substitute the value $X(0) = (3, -1)^t$ into the equation to find a solution with that initial condition:

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = X(0) = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We now have the linear system:

$$\begin{array}{rcrcr} 3 & = & \alpha & + & \beta \\ -1 & = & & & -\beta \end{array}$$

which we can solve to find $\alpha = 2$ and $\beta = 1$.

Problem 10

§4.5, Exercise 5. Let

$$C = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Show that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are eigenvectors of C. What are the corresponding eigenvalues?

Answer: Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The vector v_1 is an eigenvector of C with corresponding eigenvalue $a + b$, and v_2 is an eigenvector with eigenvalue $a - b$.

Solution: Calculate

$$Cv_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix} = (a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$Cv_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a-b \\ b-a \end{pmatrix} = (a-b)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Problem 11

§4.5, Exercise 6. Let

$$C = \begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix}.$$

Show that C has no real eigenvectors.

A vector (x, y) is an eigenvector of C if

$$C \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

that is, if

$$(C - \lambda I_2) \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

In this case,

$$\begin{pmatrix} 1-\lambda & 2 \\ -3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

This equation will have a nonzero solution (x, y) only if

$$\begin{pmatrix} 1-\lambda & 2 \\ -3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is not row equivalent to the identity matrix. Row reducing the matrix yields

$$\begin{pmatrix} 1 & \frac{2}{1-\lambda} \\ 0 & -1-\lambda+\frac{6}{1-\lambda} \end{pmatrix}$$

so C has an eigenvector when

$$-1-\lambda+\frac{6}{1-\lambda} = 0,$$

that is, when $\lambda^2 = -5$. Therefore, C has no real eigenvectors.

Problem 12

§4.6, Exercise 1. For which values of λ is the matrix

$$\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix}$$

not invertible? **Note:** These values of λ are just the eigenvalues of the matrix $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Answer: The matrix is not invertible when $\lambda = 5$ or $\lambda = -1$.

Solution: Corollary 3.8.3 states that a matrix is not invertible if and only if the determinant is zero; in this case, if

$$(1-\lambda)(3-\lambda) - (2)(4) = \lambda^2 - 4\lambda - 5 = 0.$$