More detail on [Pen20, Lem. 3.26] **David Penneys**

We provide more detail on the proof of [Pen20, Lem. 3.26]. We thank Giovanni Ferrer and Brett Hungar for their careful reading of the manuscript, leading to this note.

Let $(\mathcal{C}, \varphi)$ be a (semisimple) pivotal tensor category. The following definition is based on André Henriques' notion of *modular distortion* for bimodules over a $|rmII_1$ factor [BCE+19, BCE+20].

Definition 1. We say an object $c \in \mathcal{C}$ has *constant distortion* $\delta \in \mathbb{C}^\times$ if

$$\operatorname{tr}_L(f) = \delta \cdot \operatorname{tr}_R(f) \qquad \forall\, f \in \operatorname{End}_{\mathcal{C}}(c). \tag{1}$$

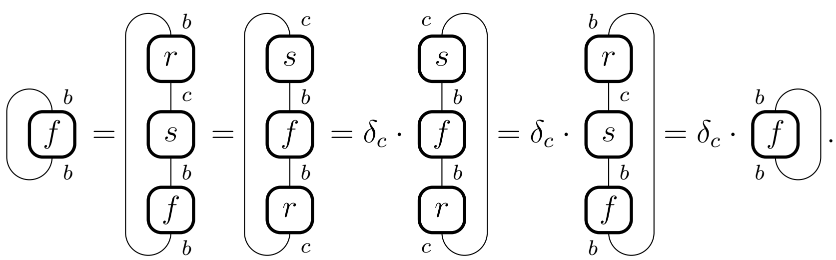
Observe that every simple object has constant distortion.

Lemma 2.

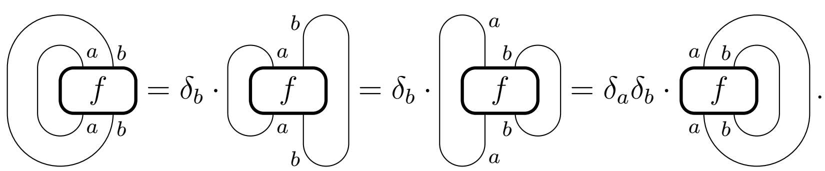
(1) If $c \in \mathcal{C}$ has constant distortion δ_c, then so does every subobject $b \subseteq c$.

(2) If $a, b \in \mathcal{C}$ have constant distortion δ_a, δ_b respectively, then $a \otimes b$ has constant distortion $\delta_a\delta_b$.

Proof. To prove (1), let $r : c \to b$ and $s : b \to c$ such that $r \circ s = \operatorname{id}_b$. Then for all $f \in \operatorname{End}_{\mathcal{C}}(b)$, suppressing the pivotal structure φ,



To prove (2), we observe that for all $f \in \operatorname{End}_{\mathcal{C}}(a \otimes b)$,



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By Lemma 2 above, we get a (most likely non-faithful) $\mathbb{C}^\times$-grading on $\mathcal{C}$ given by $\mathcal{C} = \bigoplus_{z \in \mathbb{C}^\times} \mathcal{C}_z$ where $\mathcal{C}_z$ is the semisimple subcategory of $\mathcal{C}$ whose objects have constant distortion z. Observe $\mathcal{C}_w \otimes \mathcal{C}_z \subseteq \mathcal{C}_{wz}$, and if $c \in \mathcal{C}_z$, then $c^\vee \in \mathcal{C}_{z^{-1}}$.

Denote by G the subgroup of $\mathbb{C}^\times$ such that $\mathcal{C}_z \neq 0$ so that $\mathcal{C}$ is faithfully graded by G. By [Pen20, Rem. 3.17], there is a surjective group homomorphism from the universal grading group $U_{\mathcal{C}} \twoheadrightarrow G$. Composing with the inclusion map $G \hookrightarrow \mathbb{C}^\times$ gives a group homomorphism $U_{\mathcal{C}} \to \mathbb{C}^\times$. In summary, we have the following proposition.

Proposition 3. *Let $(\mathcal{C}, \varphi)$ be a pivotal (semisimple) tensor category. The map $\delta : \operatorname{Irr}(\mathcal{C}) \to \mathbb{C}^\times$ given by $c \mapsto \delta_c := \dim_L(c)/\dim_r(c)$ gives a group homomorphim from the universal grading group $U_{\mathcal{C}}$ to $\mathbb{C}^\times$. In particular, $\mathcal{C}_e$ is spherical.*

Now suppose $(\mathcal{C}, \varphi)$ is a pivotal (semisimple) multitensor category with unit decomposition $1 = \bigoplus_{i=1}^{r} 1_i$. We write $\mathcal{C}_{ij} = 1_i \otimes \mathcal{C} \otimes 1_j$ and $c_{ij} = 1_i \otimes c \otimes 1_j$ for $c \in \mathcal{C}$.

Definition 4. We say $c \in \mathcal{C}$ has constant distortion $\Delta \in M_r(\mathbb{C}^\times)$ if c_{ij} has constant distortion Δ_{ij} for all $1 \leq i, j, \leq r$. Here, we have used a slight abuse of notation; for $f_{ij} \in \operatorname{End}_{\mathcal{C}}(c_{ij})$, $\operatorname{tr}_L(f_{ij}) \in \operatorname{End}_{\mathcal{C}}(1_j)$ and $\operatorname{tr}_R(f_{ij}) \in \operatorname{End}_{\mathcal{C}}(1_i)$. We may identify $\operatorname{End}_{\mathcal{C}}(1_i) \cong \mathbb{C}$ and $\operatorname{End}_{\mathcal{C}}(1_j) \cong \mathbb{C}$ by mapping the identity to $1_{\mathbb{C}}$; under this isomorphism, we require (1) to hold.

We omit the proof of the following lemma, which is similar to the proofs of Lemma 2.

Lemma 5.

(1) If $c \in \mathcal{C}$ has constant distortion Δ_c, then so does every subobject $b \subseteq c$.

(2) If $a \in \mathcal{C}_{ij}$ and $b \in \mathcal{C}_{jk}$ have constant distortion Δ_a, Δ_b respectively (which have exactly one non-zero entry), then $a \otimes b \in \mathcal{C}_{ik}$ has constant distortion $\Delta_a \Delta_b$ (which again has exactly one non-zero entry).

Similar to above, we get a (most likely non-faithful) $\mathcal{G}_r \times \mathbb{C}^\times$-grading on $\mathcal{C}$ given by $\mathcal{C} = \bigoplus (\mathcal{C}_{ij})_z$, where $\mathcal{G}_r$ is the groupoid with r objects and a unique isomorphism between any two objects, and $(\mathcal{C}_{ij})_z$ is the semsimple subcategory of $\mathcal{C}_{ij}$ whose objects have constant distortion z. Observe that $(\mathcal{C}_{ij})_w \otimes (\mathcal{C}_{jk})_z \subseteq (\mathcal{C}_{ik})_{wz}$, and if $c \in (\mathcal{C}_{ij})_z$, then $c^\vee \in (\mathcal{C}_{ji})_{z^{-1}}$.

Denote by $\mathcal{G}$ the subgroupoid of $\mathcal{G}_r \times \mathbb{C}^\times$ such that $\mathcal{C}$ is faithfully graded by $\mathcal{G}$. Observe that the map

$$(\mathcal{C}_{ij})_z \ni c \longmapsto z \in \mathbb{C}^\times$$

descends to a well-defined groupoid homomorphism $\mathcal{G} \to \mathbb{C}^\times$. By [Pen20, Rem. 3.17], there is a surjective groupoid homomorphism from the universal grading groupoid $\mathcal{U}_{\mathcal{C}} \twoheadrightarrow \mathcal{G}$. Composing these two homomorphisms gives a groupoid homomorphism $\mathcal{U}_{\mathcal{C}} \to \mathbb{C}^\times$. In summary, we have the following proposition.

Proposition 6. *Let $(\mathcal{C}, \varphi)$ be a pivotal (semisimple) multitensor category. The map $\Delta : \operatorname{Irr}(\mathcal{C}) \to \mathbb{C}^\times$ given by $c \mapsto \delta_c := \dim_L(c)/\dim_R(c)$ gives a groupoid homomorphism from the universal grading groupoid $\mathcal{U}_{\mathcal{C}}$ to $\mathbb{C}^\times$. In particular, for any idempotent $e \in \mathcal{U}_{\mathcal{C}}$, $\mathcal{C}_e$ is spherical.*

References

[BCE+19] Marcel Bischoff, Ian Charlesworth, Samuel Evington, Luca Giorgetti, David Penneys, and André Henriques, *Modular distortion for* II_1 *factor bimodules*, Subfactors and Applications (Organized by Dietmar Bisch, Terry Gannon, Vaughan Jones, and Yasuyuki Kawahigashi, eds.), Oberwolfach Rep., vol. 49, 2019, DOI:10.4171/OWR/2019/49, pp. 42–46.

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