Exercises in quantum algebra

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Chapter 1

Introduction

These notes are being prepared during the Autumn 2018 semester at The Ohio State University for a topics course on Quantum Algebra. In lieu of an introduction, for the time being, we provide the proposed syllabus for the course.

In this course, we will learn the main tools, techniques, and examples in the field of Quantum Algebra. A good functional definition of Quantum Algebra is 'the mathematics derived from the Jones polynomial.' Topics may include:

- diagrammatic quantum algebras (Examples: Temperley-Lieb-Jones algebras, Hecke algebras, Planar rook algebras, annular variants; Techniques: Jones-Wenzl idempotents, Jones' index rigidity theorem, Markov traces on towers of algebras)

- tensor categories (Examples: $\mathrm{Vec}(G, \omega)$, $\mathrm{Rep}(H)$ for a Hopf algebra, Construction and classification of the type A_n categories from the TLJ algebras, Tambara-Yamagami; Techniques: idempotent completion, projection categories of an algebra, inclusions of multi-matrix algebras and the basic construction)

- properties and structures of tensor categories (Examples: rigid, pivotal, spherical, braided, balanced, ribbon, fusion, modular; Techniques: graphical calculus and planar algebras, Drinfeld center/quantum double construction)

- knot polynomials (Examples: Jones polynomial, Kauffman bracket, HOMFLY; Techniques: skein theory, R-matrices, Khovanov homology)

- topological field theory (Examples: Turaev-Viro, Reshetikhin-Turaev; Techniques: 6j and theta symbols)

- algebras in tensor categories (Examples: construction of the D_{2n} and E_6, E_8 tensor categories from type A_k categories; Techniques: (de)equivariantization, boson condensation)

- unitary tensor categories (Examples: unitary versions of previous examples; Techniques: finite dimensional operator algebras, positivity)

- topological phases of matter (Examples: Toric code, bosonic topological orders; Techniques: Levin-Wen string net models, string-net condensation)

References

There will be no official book for the course, but helpful references include:

- Adams, Colin C. The knot book. An elementary introduction to the mathematical theory of knots. Revised reprint of the 1994 original. American Mathematical Society, Providence, RI, 2004. xiv+307 pp. ISBN: 0-8218-3678-1 MR2079925

- Bakalov, Bojko; Kirillov, Alexander, Jr. Lectures on tensor categories and modular functors. University Lecture Series, 21. American Mathematical Society, Providence, RI, 2001. x+221 pp. ISBN: 0-8218-2686-7 MR1797619

- Etingof, Pavel; Gelaki, Shlomo; Nikshych, Dmitri; Ostrik, Victor. Tensor categories. Mathematical Surveys and Monographs, 205. American Mathematical Society, Providence, RI, 2015. xvi+343 pp. ISBN: 978-1-4704-2024-6 MR3242743

- Jones, Vaughan F. R. Subfactors and knots. CBMS Regional Conference Series in Mathematics, 80. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1991. x+113 pp. ISBN: 0-8218-0729-3 MR1134131

- Turaev, Vladimir; Virelizier, Alexis. Monoidal categories and topological field theory. Progress in Mathematics, 322. Birkhäuser/Springer, Cham, 2017. xii+523 pp. ISBN: 978-3-319-49833-1; 978-3-319-49834-8 MR3674995

- Wang, Zhenghan. Topological quantum computation. CBMS Regional Conference Series in Mathematics, 112. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2010. xiv+115 pp. ISBN: 978-0-8218-4930-9 MR2640343

Part of the allure for teaching this course will be to prepare my own lecture notes and make them publicly available online.

Target audience

This course targets graduate students from mathematics or physics with an interest in tensor categories, quantum groups, hopf algebras, operator algebras, skein theory, knot theory, representation theory, topological field theory, or mathematical physics. This course may also be suitable for extremely advanced undergraduate students.

Prerequisites

No prerequisites are required beyond a (deep) understanding of basic linear algebra and finite dimensional inner product spaces. Familiarity with categories will be very useful, although not absolutely necessary.

Chapter 2

Temperley-Lieb (TL)

We will try to stay as example-focused as possible in the course. A fundamental example in Quantum Algebras is Temperley-Lieb (TL), which is sometimes referred to as Temperley-Lieb-Jones. One might mean many things by TL:

- a tower of finite dimensional (C*/von Neumann) algebras
- a sequence of diagrammatic algebras of (oriented?) string diagrams
- a(n idempotent complete?) tensor (C*) category
- a pivotal tensor (C*) category
- the pivotal tensor (C*) category $\mathsf{Rep}(\mathcal{U}_q(\mathfrak{su}_2))$
- a (C*) 2-category
- a (subfactor) planar algebra

So when someone says "Temperley-Lieb," your first question should be, "What do you mean by that?"

2.1 The TL algebras

In this section, we start with the TL algebras, which are finite dimensional $*$-algebras.

2.1.1 Jones' algebraic Temperley-Lieb algebras

The following abstract $*$-algebras were defined in [Jon83].

Definition 2.1.1. For $n \geq 0$ and $d = [2] = q + q^{-1}$ which uniquely determines $q \in Q \cup -Q$ by Exercise 4.1.2, we define Jones' algebraic TL algebra $TLJ_n(d)$ as the unital $*$-algebra generated by $1, e_1, \ldots, e_{n-1}$ subject to the following relations:

(J1) $e_i^2 = e_i = e_i^*$ for all $i = 1, \ldots, n-1$

(J2) $e_i e_j = e_j e_i$ for all $|i-j| > 1$, and

(J3) $e_i e_{i\pm 1} e_i = d^{-2} e_i$.

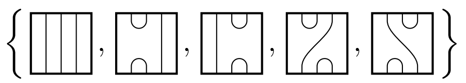
Exercise 2.1.2. Use the relations (J1) – (J3) to prove that any word in $e_1, \ldots, e_n$ is equal to a word with at most one e_n.

Exercise 2.1.3. Prove that $\dim(TLJ_n(d)) \leq \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.
Hint: Use Exercise 2.1.2.

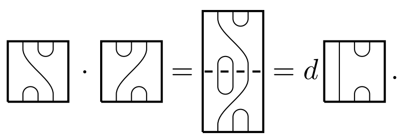
2.1.2 Kauffman's diagrammatic Temperley-Lieb algebras

In his skein-theoretic description of the Jones polynomial [Kau87], Kauffman provided a diagrammatic description of the Temperley-Lieb-Jones algebras.

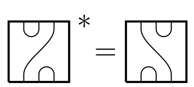
Definition 2.1.4. For $n \geq 0$ and $d = [2] = q + q^{-1}$ which uniquely determines $q \in Q \cup -Q$ by Exercise 4.1.2, we define $TLK_n(d)$ to be the complex vector space whose standard basis is the set of non-intersecting string diagrams (up to isotopy) on a rectangle with n boundary points on the top and bottom. For example, the basis for $TLK_3(d)$ is given by



On $TLK_n(d)$, we define a multiplication by (the bilinear extension of) stacking boxes, removing the middle line segment, and smoothing the strings, and removing any closed loops and multiplying by a factor of d, e.g.

 (2.1)

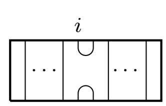
We define an involution by (the anti linear extension of) reflection about a horizontal line, e.g.

 (2.2)

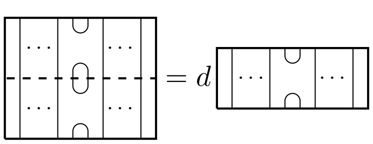
The multiplication and the adjoint make $TLK_n(d)$ a complex $*$-algebra.

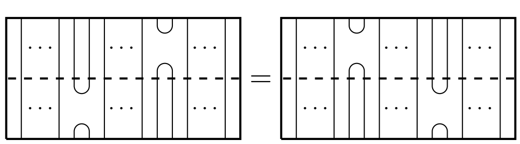
Exercise 2.1.5. Prove that $\dim(TLK_n(d)) = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.

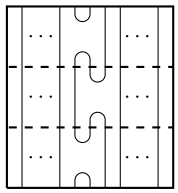
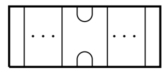
Exercise 2.1.6. Prove that for $i = 1, \ldots, n-1$, the elements

$E_i :=$  $\in TLK_n(d)$

satisfy the following relations:

(K1) $E_i^2 =$  $= dE_i = dE_i^*$,

(K2) $E_iE_j =$  $= E_jE_i$ if $|i-j| > 1$, and

(K3) $E_iE_{i\pm 1}E_i =$  $=$  $= E_i$.

Compare these relations with (J1) – (J3).

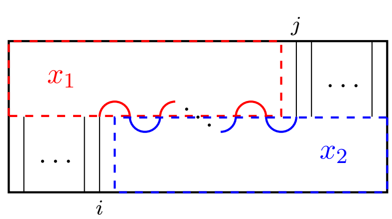
2.1.3 Isomorphism between algebraic TLJ and diagrammatic TLK

We now construct for $n \geq 0$ and $d = [2] = q + q^{-1}$ a $*$-algebra isomorphism $TLJ_n(d) \cong TLK_n(d)$. Note that by Exercise 2.1.6, the map $e_i \mapsto d^{-1}E_i$ extends to a well-defined unital $*$-algebra homomorphism $\Phi_n : TLJ_n(d) \to TLK_n(d)$, since the relations (J1) – (J3) are satisfied by dE_i for $i = 1, \ldots, n-1$.

Proposition 2.1.7. *The $\Phi_n : TLJ_n(d) \to TLK_n(d)$ are unital $*$-algebra isomorphisms.*

Proof. By Exercises 2.1.3 and 2.1.5, we have $\dim(TLJ_n(d)) \leq \dim(TLK_n(d)) = \frac{1}{n+1}\binom{2n}{n}$. By the Rank-Nullity theorem, it suffices to show that Φ_n is surjective. We proceed by strong induction on n. The base case $n = 0$ is trivial. Suppose that Φ_k is surjective for all $0 \leq k < n$. Let $x \in TLK_n(d)$ be a standard basis element.

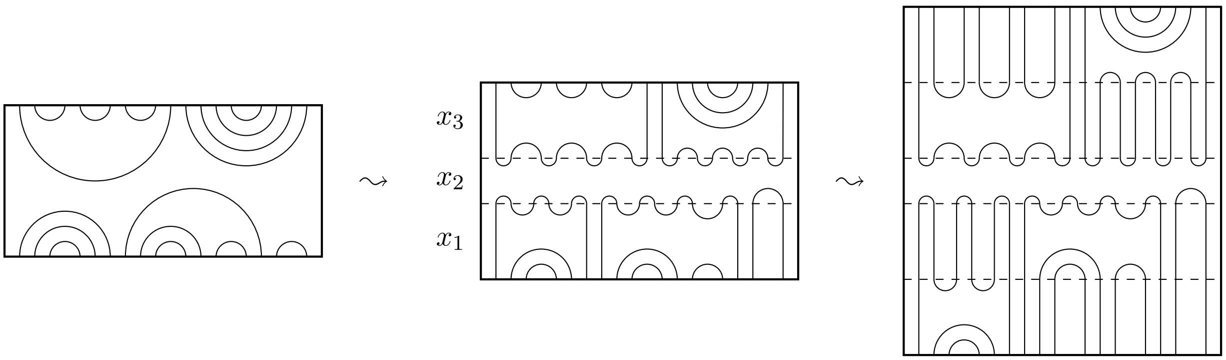
<u>Case 1:</u> Suppose x has a *through string*, i.e., a string which connects the i-th lower boundary point to the j-th upper boundary point. Notice that $i \equiv j \mod 2$; without loss of generality, we assume $i < j$. Performing isotopy on the diagram x, we divide it up as follows:

$x =$ 

Notice that the part in red, denoted x_1, is a diagrammatic basis element in $TLK_{j-1}(d)$, and the part in blue, dentoed x_2, is a diagrammatic basis element in $TLK_{n-i}(d)$. Since $i, j \geq 1$, by the induction hypothesis, both x_1 and x_2 can be expressed as products of the E_k. Then x_2 shifted by i strings to the right is exactly the product of those E_k shifted by i, i.e., E_{k+i}. Since x is the product of x_1 and x_2 shifted by i strings to the right, and both of these latter diagrammatic basis elements are products of the E_k, so is x. Thus $x \in \mathrm{im}(\Phi_n)$.

<u>Case 2:</u> Suppose x has no through strings. By applying isotopy, we pull all the *outermost cups and caps* on the top and bottom toward the center of the diagram, and wiggle the strings as in Case 1 to divide the diagram into the product of three diagrams x_1, x_2, and x_3, where x_1 and x_3 are each a horizontal concatenation of basis elements in $TLK_i(d)$ for i strictly smaller than n, and x_2

is visibly a product of the odd E_k. We provide an explicit example below:



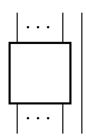
Applying isotopy again to shift the smaller basis elements which are horizontally concatenated within x_1 and x_3 up and down as in the rightmost diagram above, we can express each of x_1 and x_3 as products of the E_k, and thus x is a product of the E_k. We conclude $x \in \mathrm{im}(\Phi_n)$, and we are finished. $\square$

Notation 2.1.8. From this point forward, we simply write $TL_n(d)$ to denote either $TLJ_n(d)$ or $TLK_n(d)$, which we identify under the unital $*$-algebra isomorphisms Φ_n.

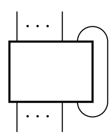
2.1.4 Planar operations on TL algebras

As the TL algebras afford a diagrammatic description, we get a powerful planar calculus.

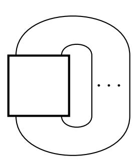
Definition 2.1.9 (Linear operations)**.** The right inclusion tangle is a unital, injective $*$-algebra homomorphism

$i_n :=$  $: TL_n(d) \to TL_{n+1}(d).$

The conditional expectation tangle is a surjective $*$-map of $\mathbb{C}$-vector spaces

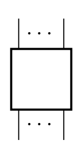
$\mathcal{E}_{n+1} :=$  $: TL_{n+1}(d) \to TL_n(d).$

The trace tangle is a linear $*$-map of $\mathbb{C}$-vector spaces

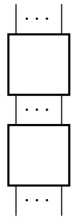
$\mathrm{Tr}_n :=$  $: TL_n(d) \to TL_0(d). \qquad (2.3)$

Note that $TL_0(d) \cong \mathbb{C}$ as a $*$-algebra via the map which sends the empty diagram to $1_{\mathbb{C}}$. Using Tr_n, we can define a sesquilinear form on $TL_n(d)$ by $\langle x, y\rangle_n := \mathrm{Tr}_n(xy^*)$.

Of course, the identity map $\mathrm{id}_n : TL_n(d) \to TL_n(d)$ is given by the following diagram:

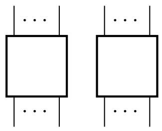


Definition 2.1.10 (Quadratic operations)**.** We already saw that multiplication was given by vertically stacking diagrams. We can also draw a tangle for multiplication as follows:



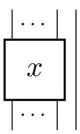
$$: TL_n(d) \times TL_n(d) \to TL_n(d).$$

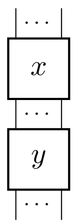
The tensor product tangle takes elements which may live in distinct TL algebras and horizontally concatenates them

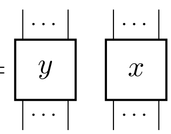


$$: TL_m(d) \times TL_n(d) \to TL_{m+n}(d).$$

Notation 2.1.11. When we apply one of these operations to an $x \in TL_n(d)$ (or possibly two elements from two distinct TL algebras), we denote the output by labeling the tangle with the input(s). For example,

$i_n(x) =$ 

$\in TL_{n+1}(d)$ $\qquad$ $xy =$ 

$\in TL_n(d)$ $\qquad$ $y \otimes x =$ 

$\in TL_{m+n}(d).$

Exercise 2.1.12. Prove the following relations amongst the maps $i_n, \mathcal{E}_{n+1}$, Tr_n, and id_n by drawing diagrams.

(1) $\mathcal{E}_{n+1} \circ i_n = d \operatorname{id}_n$,

(2) $\operatorname{Tr}_{n+1} = \operatorname{Tr}_n \circ \mathcal{E}_{n+1}$,

(3) $(i_n \circ i_{n-1} \circ \mathcal{E}_n(x))E_n = E_n i_n(x) E_n$ for all $x \in TL_n(d)$,

(4) $\operatorname{Tr}_n(xy) = \operatorname{Tr}_n(yx)$ for all $x, y \in TL_n(d)$,

(5) (Markov property) $\operatorname{Tr}_{n+1}(i_n(x) \cdot E_n) = \operatorname{Tr}_n(x)$ for all $x \in TL_n(d)$, and

(6) $\operatorname{Tr}_n(\mathcal{E}_{n+1}(x) \cdot y) = \operatorname{Tr}_{n+1}(x \cdot i_n(y))$ for all $x \in TL_{n+1}(d)$ and $y \in TL_n(d)$.

2.2 The Kauffman bracket

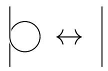
We now show how the Temperley-Lieb algebras can be used to construct a polynomial invariant of knots and links. We begin by defining the Kauffman bracket for knot projections [].

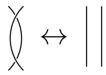
Recall that a *knot* is an embedding $S^1 \hookrightarrow \mathbb{R}^3$, and a *link* is an embedding $\coprod_{i=1}^n S^1 \hookrightarrow \mathbb{R}^3$ where $n \in \mathbb{N}$. A *knot/link projection* is the image in $\mathbb{R}^2$ under a generic regular projection, which avoids various bad behaviors, like triple intersections and kinks.

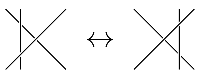
To define the Kauffman bracket and the Jones polynomial, we will use the following theorems of Reidemeister and Markov.

2.2.1 Reidemeister moves

Theorem 2.2.1 (Reidemeister [Rei27]). *Two knot/link projections represent isotopic knots in* $\mathbb{R}^3$ *if and only if they are related by a finite number of the* Reidemeister moves*:*

(R1) 

(R2) 

(R3) 

2.2.2 The Kauffman bracket

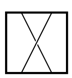
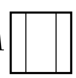
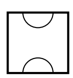
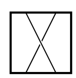
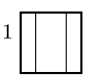
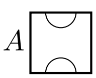
Exercise 2.2.2.

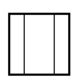
(1) Prove that for each $d \in \mathbb{R} \setminus \{0\}$, there is a unique $q \in \mathbb{C}$ with non-negative imaginary part such that $q + q^{-1} = d$.

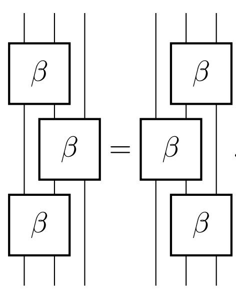
(2) Prove that for each $q \in \mathbb{C}$ with non-negative imaginary part, there is a unique $A \in \mathbb{C}$ with non-negative imaginary part such that $A^2 = q$.

(3) Deduce that for each $d \in \mathbb{R} \setminus \{0\}$, there is a unique $A \in \mathbb{C}$ with non-negative imaginary part such that $d = A^2 - A^{-2}$.

Definition 2.2.3. Suppose $d = -A^2 - A^{-2}$. Define the crossings $\beta^{\pm 1}$ in TL_2 by

$\beta =$  $:= A$  $+ A^{-1}$  $\qquad \beta^{-1} =$  $:= A^{-1}$  $+ A$  . (2.4)

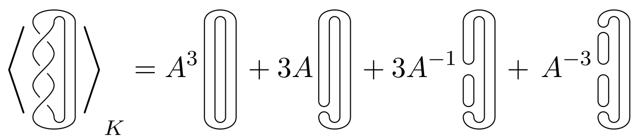
Exercise 2.2.4. Prove that $\beta\beta^{-1} = \beta^{-1}\beta = \mathrm{id}_2 :=$  $\in TL_2$.

Exercise 2.2.5. Prove that  .

Definition 2.2.6. Given a link ℓ, we define an element $\langle \ell \rangle_K \in TL_0(d)$ called the *Kauffman bracket* of ℓ by replacing the crossings by $\beta^{\pm 1}$ as in (2.4).[1] Here, we identify $TL_0(d) = \mathbb{C}[A, A^{-1}]$, polynomials in A and A^{-1}. By Exercises 2.2.4 and 2.2.5, we see that $\langle \ell \rangle_K$ is *invariant* under applying (R2) and (R3) to ℓ anywhere locally. Thus the Kauffman bracket is almost an invariant of knots and links, modulo (R1).

[1]This differs from Kauffman's original definition of the bracket polynomial by a normalization. Kauffman normalized so that the uknot has bracket equal to 1, whereas we normalize so that the unknot has bracket equal to δ.

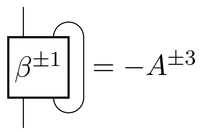
Example 2.2.7. We calculate the Kauffman bracket of a trefoil knot as follows:



$$= -A^{-9} + A^{-1} + A^{3} + A^{7}.$$

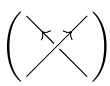
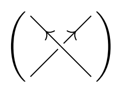
This proves a trefoil is not isotopic to its mirror image.

2.2.3 The writhe factor

Exercise 2.2.8. Show that 

. Deduce that $\langle \ell \rangle_K$ is not invariant under (R1).

Definition 2.2.9. Let $\vec{\ell}$ be an *oriented* link. For each crossing in a projection of $\vec{\ell}$, we define the *sign* of the crossing as follows:

sign  $:= 1$ sign  $:= -1$

We define the *writhe factor* $\mathrm{wr}(\vec{\ell})$ to be the number of crossings, counted with their signs.

Exercise 2.2.10. Let $\vec{\ell}$ be an oriented link and let ℓ be the link obtained from forgetting the orientation. Show that

$$V_{\vec{\ell}}(A) := d^{-1}(-A)^{-3\,\mathrm{wr}(\vec{\ell})} \cdot \langle \ell \rangle_K \tag{2.5}$$

is invariant under (R1), (R2), and (R3).

Definition 2.2.11. The *Jones polynomial* of $\vec{\ell}$ is $V_{\vec{\ell}}(A)$, as defined in Exercise 2.2.10.

2.3 Jones' construction of his polynomial

We now go through Jones' original approach to his knot polynomial using Markov traces on Temperley-Lieb algebras [Jon85].

2.3.1 Artin's braid groups

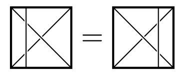
Definition 2.3.1. The algebraic braid group AB_n is the group generated by $\beta_1, \ldots, \beta_{n-1}$ subject to the relations

(B1) $\beta_i \beta_j = \beta_j \beta_i$ for $|i - j| > 1$ and

(B2) $\beta_i \beta_{i\pm1} \beta_i = \beta_{i\pm1} \beta_i \beta_{i\pm1}$.

Exercise 2.3.2. Show that AB_2 is isomorphic to $\mathbb{Z}$, but that AB_3 contains a group isomorphic to the free group $\mathbb{F}_2$.

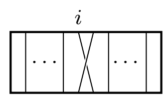
Definition 2.3.3. The diagramamtic braid group DB_n is the group whose elements consist of string diagrams with n boundary points on the lower and upper sides of a rectangle, and the lower points are paired to the upper points by smooth strings which only intersect at a finite number of points, where we indicate which string passes over the other as in a knot/link projection. Moreover, the strings are not allowed to have any critical points. All such diagrams are considered up to isotopy and Reidemeister moves (R2) and (R3). For example, the following elements of DB_3 are equal:



We multiply in DB_n by stacking boxes and smoothing out strings, similar to multiplication in TL_n, which is manifestly associative.

Exercise 2.3.4. Prove that DB_n is a group under the above multiplication. That is, find the identity element, and show every element has an inverse.

Exercise 2.3.5. Consider the distinguished elements of DB_n given by

$b_i :=$  .

Prove that the elements $b_1, \ldots, b_{n-1} \in DB_n$ satisfy Relations (R2) and (R3). Deduce there is a well-defined group homomorphism $\Phi_n : AB_n \to DB_n$.

Exercise 2.3.6. Show that every element of B_n can be written as a product of $b_1, \ldots, b_{n-1}$ from Exercise 2.3.5. Deduce that Φ_n from Exercise 2.3.5 is surjective.

We will not prove the following theorem as it would take us too far afield.

Theorem 2.3.7 (Artin [Art25])**.** *The group homomorphism $\Phi_n : AB_n \to DB_n$ from Exercise 2.3.5 is an isomorphism.*

Notation 2.3.8. From this point forward, we simply write B_n to denote either AB_n or DB_n, which we identify under the group isomorphisms Φ_n.

Exercise 2.3.9. Show that the map $\Psi : B_n \to TL_n(d)$ given by

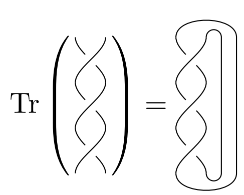
$$e \mapsto 1 \qquad \beta_i \mapsto A\,\mathrm{id}_n + A^{-1}E_i \qquad \beta_i^{-1} \mapsto A^{-1}\,\mathrm{id}_n + AE_i$$

where $d = -A^2 - A^{-2}$ (which uniquely determines A by Exercise 2.2.2) preserves (B1) and (B2). Deduce that Ψ extends to a well-defined unital $*$-algebra homomorphism $\Psi : \mathbb{C}[B_n] \to TL_n(d)$, where the $*$ on the group algebra is the conjugate-linear extension of inversion.

Exercise 2.3.10. Determine when $\Psi(\beta_i)$ is a unitary in $U(TL_n(d))$ for $i - 1, \ldots, n-1$.

2.3.2 Markov's theorem

Given a braid b, we obtain a link ℓ by *closing/capping/tracing* the braid to the right. For example, we can represent a trefoil knot as follows:



Theorem 2.3.11 (Markov [Mar35])**.** *Every link is the closure of a braid. Moreover, two braids give the same link under closure if and only if they are related by a finite number of the following two moves:*

(M1) *If $b \in B_n$, we can swap $b \leftrightarrow aba^{-1}$ for some braid $a \in B_n$.*

(M2) *If $b \in B_n$, we can swap $b \leftrightarrow b\beta_n^{\pm 1}$, the n-th generator of B_n.*

Exercise 2.3.12. Prove that we get the same link under taking the closure of a braid under either (M1) or (M2).

2.3.3 Construction of the Jones polynomial

We now construct the Jones polynomial as follows.

Definition 2.3.13. Suppose $\vec{\ell}$ is an oriented link. Write $\vec{\ell} = \operatorname{Tr}(\vec{b})$ for some braid $b \in B_n$ where $\vec{b}$ is obtained from b by orienting all strands from bottom to top. Define

$$V_{\vec{\ell}}(A) := d^{-1}(-A^3)^{-\exp(b)} \cdot \operatorname{Tr}_{TL_n(d)}(\Psi(b)) \tag{2.6}$$

where $d = -A^2 - A^{-2}$, $\exp(b)$ is the *exponent sum* of b as a word in $\beta_1, \ldots, \beta_{n-1}$, and $\Psi : \mathbb{C}[B_n] \to TL_n(d)$ the unital $*$-algebra homomorphism from Exercise 2.3.9.

Exercise 2.3.14. Show that $\exp(b)$ is exactly the writhe factor of $\operatorname{Tr}(\vec{b})$.

Proposition 2.3.15. *The formula* (2.6) *for $V_{\vec{\ell}}$ is well-defined, i.e., it does not depend on the choice of b. Moreover, it agrees with* (2.5)*.*

Proof. It is sufficient to show (2.6) agrees with (2.5), which is straightforward. However, for the sake of pedagogy, we will show that (2.6) is well-defined by showing it is invariant under the Markov moves (M1) and (M2).

(M1): This is immediate from $\exp(a) = -\exp(a^{-1})$ for all $a \in B_n$, together with the facts that Ψ is a homomorphism and Tr is a trace:

$$\operatorname{Tr}(\Psi(aba^{-1})) = \operatorname{Tr}(\Psi(a)\Psi(b)\Psi(a)^{-1}) = \operatorname{Tr}(\Psi(a)^{-1}\Psi(a)\Psi(b)) = \operatorname{Tr}(\Psi(b)).$$

(M2): We prove that $B_n \ni b \leftrightarrow b\beta_n \in B_{n+1}$ does not change (2.6), and the proof for $b \leftrightarrow b\beta_n^{-1}$ is similar. Note that $\exp(b\beta_n) = 1 + \exp(b)$. Expanding $\Psi(\beta_n) = A\,\mathrm{id}_n + A^{-1}E_n$, we have

$$\begin{aligned}
(-A^3)^{-\exp(b\beta_n)} \cdot \operatorname{Tr}_{TL_{n+1}(d)}(\Psi(b\beta_n)) &= (-A^3)^{-1-\exp(b)} \cdot (A \operatorname{Tr}_{TL_{n+1}(d)}(\Psi(b)) + A^{-1} \operatorname{Tr}_{TL_n(d)}(\Psi(b)E_n)) \\
&= (-A^3)^{-1-\exp(b)} \cdot (Ad + A^{-1}) \cdot \operatorname{Tr}_{TL_n(d)}(\Psi(b)) \\
&= (-A^3)^{-1-\exp(b)} \cdot (A^3) \cdot \operatorname{Tr}_{TL_n(d)}(\Psi(b)) \\
&= (-A^3)^{-\exp(b)} \cdot \operatorname{Tr}_{TL_n(d)}(\Psi(b)).
\end{aligned}$$

This completes the proof. $\square$

Chapter 3

Towers of algebras

3.1 Finite dimensional multimatrix and von Neumann algebras

One of the main techniques in the untiary viewpoint of Quantum Algebra is towers of algebras. The next two sections provide the prerequisites.

3.1.1 Basic facts about $M_n(\mathbb{C})$

For this section, $*$ denotes the conjugate transpose operation on $M_n(\mathbb{C})$.

Exercise 3.1.1. Show that if $a \in M_n(\mathbb{C})$ commutes with all $b \in M_n(\mathbb{C})$, then $a = \lambda 1$ for some $\lambda \in \mathbb{C}$.

Exercise 3.1.2. Suppose $p \in M_n(\mathbb{C})$ is a *minimal projection*, i.e., $pM_n(\mathbb{C})p = \mathbb{C}p$. Show that there are $v_1, \ldots, v_n \in M_n(\mathbb{C})$ such that $\sum_{i=1}^n v_i p v_i^* = 1$. Show that in addition, we may choose $v_1, \ldots, v_n$ so that $v_i^* v_i = p$ for all $i = 1, \ldots, n$, so that the v_i are *partial isomertries.*

Exercise 3.1.3. Prove that $M_n(\mathbb{C})$ has no non-trivial 2-sided ideals.

Exercise 3.1.4. Use Exercise 3.1.3 to show that any (not necessarily unital) $*$-algebra map out of $M_n(\mathbb{C})$ into another complex $*$-algebra is either injective or the zero map.

The matrix algebra $M_n(\mathbb{C})$ acts on the inner product (Hilbert) space $\mathbb{C}^n$ with inner product given by $\langle \eta, \xi \rangle := \sum_{j=1}^n \eta_j \overline{\xi_j}$.

Definition 3.1.5. An element $a \in M_n(\mathbb{C})$ is called *normal*, if $aa^* = a^*a$.

Exercise 3.1.6 (Spectral Theorem)**.** Show that the following are equivalent for $a \in M_n(\mathbb{C})$.

(1) a is normal.

(2) There is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors for a.

(3) There is a unitary $u \in M_n(\mathbb{C})$ ($uu^* = u^*u = 1$) such that u^*au is diagonal.

Definition 3.1.7 (Functional calculus)**.** Suppose $a \in M_n(\mathbb{C})$ is normal. Let $\operatorname{Spec}(a)$ denote the *spectrum* of a, which is the set of eigenvalues. For $\lambda \in \operatorname{Spec}(a)$, let $E_\lambda \subset \mathbb{C}^n$ denote the corresponding eigenspace, and let $p_\lambda \in M_n(\mathbb{C})$ be the orthogonal projection onto E_λ. Note that

$$a = \sum_{\lambda \in \operatorname{Spec}(a)} \lambda p_\lambda,$$

as both operators agree on an orthonormal basis of $\mathbb{C}^n$, namely the orthonormal basis consisting of eigenvectors for a from Exercise 3.1.6. For $f : \mathrm{Spec}(a) \to \mathbb{C}$, we define

$$f(a) := \sum_{\lambda \in \mathrm{Spec}(a)} f(\lambda) p_\lambda \in M_n(\mathbb{C}).$$

Exercise 3.1.8. Suppose $a \in M_n(\mathbb{C})$ is normal, and let $C(\mathrm{Spec}(a))$ denote the unital $*$-algebra of $\mathbb{C}$-valued functions on $\mathrm{Spec}(a)$.

(1) Show that $C(\mathrm{Spec}(a)) \ni f \mapsto f(a) \in M_n(\mathbb{C})$ is a unital $*$-algebra homomorphism.

(2) (Spectral mapping) Prove that $\mathrm{Spec}(f(a)) = f(\mathrm{Spec}(a))$.

Definition 3.1.9. An element $a \in M_n(\mathbb{C})$ is called *positive*, denoted $a \geq 0$, if for every $\xi \in \mathbb{C}^n$, $\langle a\xi, \xi \rangle \geq 0$.

Exercise 3.1.10. Show that the following are equivalent for $a \in M_n(\mathbb{C})$.

(1) $a \geq 0$.

(2) a is normal ($aa^* = a^*a$) and all eigenvalues of a are non-negative.

(3) There is a $b \in M_n(\mathbb{C})$ such that $b^*b = a$.

(4) There is a $b \in M_{n\times k}(\mathbb{C})$ for some $k \in \mathbb{N}$ such that $b^*b = a$.

Exercise 3.1.11 ($**$, [Pal01, Thm. 9.1.45])**.**

(1) Show that any involution $\dagger$ on $M_n(\mathbb{C})$ is of the form $a^\dagger = ha^*h^{-1}$ for some invertible $h \in M_n(\mathbb{C})$ such that $h = h^*$.

(2) Show that $(M_n(\mathbb{C}), \dagger) \cong (M_n(\mathbb{C}), *)$ as involutive algebras if and only if the corresponding h for $\dagger$ is positive or negative definite.

3.1.2 Finite dimensional complex multimatrix algebras

In this section, A will always denote a finite dimensional complex $*$-algebra.

Definition 3.1.12. A linear functional $\varphi : A \to \mathbb{C}$ is called:

- a *trace* or *tracial* if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$.
- *positive* if $\varphi(a^*a) \geq 0$ for all $a \in A$.
- a *state* if φ is positive and $\varphi(1) = 1$.
- *faithful* if φ is positive and $\varphi(a^*a) = 0$ implies $a = 0$.

Exercise 3.1.13. Prove that $M_n(\mathbb{C})$ has a unique trace such that $\mathrm{tr}(1) = 1$. In this case, prove that tr is positive (so tr is a state) and faithful.

Exercise 3.1.14. Let $A = \mathbb{C}^2$ with coordinate-wise multiplication and $(a, b)^* := (\overline{b}, \overline{a})$. Prove that A has no states.

Exercise 3.1.15 (∗)**.** Prove that for any state φ on $M_n(\mathbb{C})$, there exists $d \in M_n(\mathbb{C})$ with $d \geq 0$ and $\operatorname{tr}(d) = 1$ such that $\varphi(a) = \operatorname{tr}(da)$ for all $a \in M_n(\mathbb{C})$. Prove that φ is a faithful if and only if d is also invertible.
The matrix d is called the density matrix *of φ with respect to* tr*.*

Exercise 3.1.16. Suppose tr is a trace on a multimatrix algebra. Show that:

(1) tr is positive if and only if $\operatorname{tr}(p) \geq 0$ for all projections $p \in A$ ($p = p^* = p^2$).

(2) tr is positive and faithful if and only if $\operatorname{tr}(p) > 0$ for all projections $p \in A$.

Definition 3.1.17. A finite dimensional complex ∗-algebra A is called a *multimatrix algebra* if it is ∗-isomorphic to a ∗-algebra of the form

$$M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

The row vector $n_A := (n_1, \ldots, n_k)$ is called the *dimension row vector* of A. For $1 \leq i \leq k$, we denote by $p_i \in A$ the minimal central projection corresponding to the summand $M_{n_i}(\mathbb{C})$, so that $p_i A p_i \cong M_{n_i}(\mathbb{C})$.

Exercise 3.1.18. Assume the notation of Definition 3.1.17. Suppose $p \in A$ is an orthogonal projection. The *central support* $z(p)$ of p is the smallest central projection such that $p \leq z(p)$. Show that $z(p)$ is the sum of the p_i such that $pp_i \neq 0$.

Exercise 3.1.19. Find a bijective correspondence between faithful tracial states on a finite dimensional complex multimatrix algebra with dimension row vector $n_A = (n_1, \ldots, n_k)$ and column vectors $\lambda \in (0,1)^j$ such that $n_A \lambda = 1$. Under this correspondence, what does the entry λ_i signify?

3.1.3 Finite dimensional operator algebras (∗)

Let H denote a finite dimensional inner product (Hilbert) space. Denote by $B(H)$ the unital ∗-algebra of linear operators on H, where ∗ is the adjoint operation.

Exercise 3.1.20. Show that a choice of orthonormal basis of H gives a unitary linear map $u : H \to \mathbb{C}^n$ ($uu^* = \operatorname{id}_{\mathbb{C}^n}$ and $u^*u = \operatorname{id}_H$) such that $x \mapsto uxu^*$ is a unital ∗-algebra isomorphism $B(H) \to M_n(\mathbb{C})$, where the ∗ on the latter is conjugate transpose.

Exercise 3.1.21 (∗)**.** Show that a finite dimensional unital ∗-algebra is a C∗ algebra if and only if it has a faithful tracial state. Deduce that a multimatrix algebra is a C∗ algebra.

Definition 3.1.22. Suppose H is a finite dimensional inner product (Hilbert) space, and denote by $B(H)$ the linear operators on H. For a subset $S \subset B(H)$, the *commutant* of S is $S' := \{x \in B(H) | xs = sx \text{ for all } s \in S\}$

Exercise 3.1.23. Show that if $S \subset T \subset B(H)$, then $T' \subset S'$.

Exercise 3.1.24. Show that if $S \subset B(H)$, then $S' = S'''$.

Exercise 3.1.25 (∗∗)**.** Show that if $A \subset B(H)$ is a unital ∗-subalgebra, then $A = A''$.
Hint: See [Jon15, Thm. 3.2.1].

Exercise 3.1.26 (∗∗)**.**

(1) Show that a finite dimensional von Neumann algebra is a multimatrix algebra.

(2) Show that a finite dimensional C*-algebra is a multimatrix algebra.

Exercise 3.1.27 ($*$)**.** Suppose A is a finite dimensional unital complex $*$-algebra. In this exercise, we will show that A being a C*-algebra is equivalent to the condition

$$a^*a = 0 \Longrightarrow a = 0. \tag{3.1}$$

(1) Recall that the *Jacobson radical* of A is

$$J(A) = \{b \in A | 1 + abc \text{ is invertible } \forall a, c \in A\}.$$

Show that every element of $J(A)$ is nilpotent.
Hint: If $a \in A$, there is a polynomial $p \in \mathbb{C}[x]$ such that $p(a) = 0$.

(2) Show that if there is a non-zero $a \in J(A)$, then there is a non-zero $b \in J(A)$ such that $b^*b = 0$.
Hint: Reduce to the case $a = a^*$ and use part (1).

(3) Use the Artin-Wedderburn Theorem to deduce that (3.1) implies A is a multimatrix algebra.

(4) Show that (3.1) implies each full matrix algebra summand of A is preserved under $*$.
Hint: Consider the minimal central idempotents $\{p_i\}_{i=1}^n$. Show that $\{p_i^*\}_{i=1}^n$ are also minimal central idempotents, so $p_i^* = p_j$ for some $j = 1, \ldots, n$. Then apply (3.1).

(5) Consider the involution of A restricted to a single full matrix algebra summand $M_n(\mathbb{C})$ of A. By Exercise 3.1.11, there is a self-adjoint $h \in M_n(\mathbb{C})$ such that $x^* = hx^\dagger h$ for all $x \in M_n(\mathbb{C})$, where $\dagger$ denotes the usual conjugate-transpose in $M_n(\mathbb{C})$. Show that (3.1) implies h is positive or negative definite.
Hint: If h is not positive or negative definite, choose $-\infty < r < 0$ and $0 < s < \infty$ such that $r, s \in \operatorname{Spec}(h)$, and pick eigenvectors $v, w \in \mathbb{C}^n$ for h corresponding to r, s respectively. Then find a non-zero $x \in M_n(\mathbb{C})$ such that $hx^\dagger h^{-1}x = 0$. (For example, take x to have one non-zero column, which is a linear combination of v, w.)

(6) Prove A is a C*-algebra if and only if (3.1) holds.

3.1.4 The GNS construction

Suppose A is a multimatrix algebra and φ is a faithful state.

Exercise 3.1.28. Show that $\langle a, b\rangle := \varphi(b^*a)$ defines a positive definite inner product on A (thought of as a $\mathbb{C}$-vector space).

Definition 3.1.29. We define $L^2(A, \varphi)$ to be A as an inner product (Hilbert) space with the inner product from Exercise 3.1.28. We denote the image of $1 \in A$ in $L^2(A, \varphi)$ by Ω, so $a\Omega$ is the image of $a \in A$.

Exercise 3.1.30. Prove that if $a \in A$, the map given by $b\Omega \mapsto ab\Omega$ defines a left multiplication operator $\lambda_a \in B(L^2(A, \varphi))$. Prove that the adjoint of this operator is λ_{a^*} given by $b\Omega \mapsto a^*b\Omega$.

Exercise 3.1.31. Prove that if $a \in A$, the map given by $b\Omega \mapsto ba\Omega$ defines a right multiplication operator $\rho_a \in B(L^2(A, \varphi))$. Calculate the adjoint of ρ_a. When does $\rho_a^* = \rho_{a^*}$?

Exercise 3.1.32. Suppose φ is a faithful state on $M_n(\mathbb{C})$. Prove that the commutant of the *left* $M_n(\mathbb{C})$ action on $L^2(M_n(\mathbb{C}), \varphi)$ is the *right* $M_n(\mathbb{C})$ action.

Exercise 3.1.33. Suppose $\langle \cdot, \cdot \rangle$ is a positive definite inner product on the vector space $M_{m\times n}(\mathbb{C})$. Prove that the commutant of the *left* $M_m(\mathbb{C})$ action on $M_{m\times n}(\mathbb{C})$ is the *right* $M_n(\mathbb{C})$ action.

Exercise 3.1.34. Suppose φ is a faithful state on A. Use Exercise 3.1.33 to prove that the commutant of the *left* A acting on $L^2(A, \varphi)$ is the *right* action of A on $L^2(A, \varphi)$.

Exercise 3.1.35 ($*$)**.** Show that a finite dimensional complex $*$-algebra is a multimatrix algebra if and only if it has a faithful state.
Hint: For the forward direction, use Exercise 3.1.19. For the reverse direction, if A has a faithful state, then the image of A inside the linear operators on $L^2(A, \varphi)$ is a unital $*$-subalgebra, and thus a finite dimensional von Neumann algebra by Exercise 3.1.25. The result now follows by (1) of Exercise 3.1.26.

3.1.5 Inclusions of multimatrix algebras

Definition 3.1.36. Consider a multimatrix algebra B and a $*$-subalgebra $A \subset B$ such that A is also a multimatrix algebra (so A is unital). We call the inclusion $A \subset B$ *unital* if the unit of A is also the unit of B.

Exercise 3.1.37. Give examples of unital and non-unital inclusions of multimatrix algebras.

Exercise 3.1.38 ($*$)**.** Show that $M_k(\mathbb{C})$ isomorphic to a unital $*$-subalgebra of $M_n(\mathbb{C})$ if and only if $k \mid n$. Then show that up to unitary conjugation in $M_n(\mathbb{C})$, the isomorphism above is given by

$$M_k(\mathbb{C}) \ni x \mapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix} \in M_n(\mathbb{C})$$

where x is repeated on the diagonal j times where $jk = n$.

Consider a unital inclusion of multimatrix algebras $A \subset B$. Suppose B has dimension row vector $n_B = (n_1, \ldots, n_\ell)$ and A has dimension row vector $m_A = (m_1, \ldots, m_k)$. Denote the minimal central projections of A by $p_1, \ldots, p_k$ and the minimal central projections of B by $q_1, \ldots, q_\ell$. Consider for $1 \leq i \leq k$ and $1 \leq j \leq \ell$ the $*$-homomorphism $\varphi_{ij} : M_{m_i}(\mathbb{C}) \to M_{n_j}(\mathbb{C})$ given by

$$\begin{aligned} M_{m_i}(\mathbb{C}) \hookrightarrow A \hookrightarrow B \twoheadrightarrow M_{n_j}(\mathbb{C}). \\ x \mapsto p_i x = p_i x \mapsto p_i q_j x. \end{aligned}$$

That is, $\varphi_{ij}(x) := p_i q_j x \in B$. Note that φ_{ij} need not be unital, but note that by Exercise 3.1.4, $\varphi_{i,j}$ is either injective or zero.

Exercise 3.1.39. Show that if we consider φ_{ij} as a map $p_i A \to p_i q_j B p_i q_j$, then φ_{ij} is a unital $*$-homomorphism.

Definition 3.1.40. By Exercises 3.1.38 and 3.1.39 there is a non-negative integer $\Lambda_{ij} \in \mathbb{N}_{\geq 0}$ such that up to unitary conjugation in B, $\varphi_{ij}(x)$ consists of Λ_{ij} copies of x along the diagonal of $p_i q_j B p_i q_j$. We define the *inclusion matrix* of $A \subseteq B$ by $\Lambda_A^B := (\Lambda_{i,j}) \in M_{k\times\ell}(\mathbb{C})$. When there is only one inclusion, we suppress the sub- and superscripts and simply write Λ.

Exercise 3.1.41. Show that since $A \subset B$ is a unital inclusion of multimatrix algebras ($1_B \in A$), we must have $m_A \Lambda_A^B = n_B$.

Exercise 3.1.42. Show that if $A \subseteq B \subseteq C$ are all unital inclusions of multimatrix algebras, then $\Lambda_A^C = \Lambda_A^B \Lambda_B^C$.

Definition 3.1.43. The *Bratteli diagram* of the inclusion $A \subset B$ is the bipartite graph Γ with:

- k even vertices labelled by the integers $m_1, \ldots, m_k$,
- ℓ odd vertices labelled by the integers $n_1, \ldots, n_\ell$, and
- Λ_{ij} edges from the i-th even vertex to the j-th odd vertex.

That is, Γ is the bipartite graph with adjacency matrix Λ_A^B whose even and odd vertices are labelled by the entries of the dimension row vectors of A and B respectively.

Exercise 3.1.44 $(*)$**.** Let B be a multimatrix algebra. Prove that up to unitary conjugation in B, any unital $*$-subalgebra $A \subset B$ is completely determined by its Bratteli diagram.

Exercise 3.1.45. Suppose λ_A and λ_B are trace column vectors for A and B satisfying $m_A\lambda_A = n_B\lambda_B$ respectively as in Exercise 3.1.19. Assume the entries of λ_A and λ_B are all strictly positive. Denote by tr_A and tr_B the corresponding faithful tracial states on A and B. Prove that $\operatorname{tr}_B|_A = \operatorname{tr}_A$ if and only if $\Lambda\lambda_B = \lambda_A$.

3.1.6 Connected inclusions

We continue the notation of the previous section for an inclusion $A \subset B$ with dimension row vectors $m_A = (m_1, \ldots, m_k)$ and $n_B = (n_1, \ldots, n_\ell)$ respectively.

Definition 3.1.46. The inclusion $A \subset B$ is called *connected* if the graph Γ is connected.

Exercise 3.1.47 $(*)$**.** Prove that Γ is connected if and only if $Z(A) \cap Z(B) = \mathbb{C}$.

Exercise 3.1.48 $(*)$**.** Show that if $A \subset B$ is connected, there is a unique $d > 0$ and unique trace vector λ_B such that $m_B\lambda_B = 1$ and $\Lambda^T\Lambda\lambda_B = d^2\lambda_B$. Then deduce:

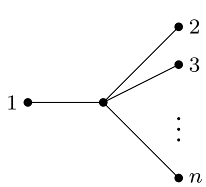
(1) If $\lambda_A := \Lambda\lambda_B$, then $\Lambda^T\lambda_A = d^2\lambda_B$.

(2) $\begin{pmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{pmatrix} \begin{pmatrix} \lambda_A \\ d\lambda_B \end{pmatrix} = d \begin{pmatrix} \lambda_A \\ d\lambda_B \end{pmatrix}.$

Hint: Use the Frobenius-Perron Theorem.

Definition 3.1.49. If $A \subset B$ is connected, the scalar d from Exercise 3.1.48 is called the *Frobenius Perron eigenvalue*. The trace vector λ_B is called a *Frobenius Perron eigenvector*.

Exercise 3.1.50. Compute the Frobenius-Perron eigenvalue and an associated eigenvector for the following graphs:

(1) 

(2) 

Exercise 3.1.51. Show that if Λ is a proper subgraph of the finite graph Γ, then the Frobenius-Perron eigenvalue of Λ is strictly less than the Frobenius-Perron eigenvalue of Γ.
Hint: Use the Rayleigh quotient for the adjacency matrix, which is self-adjoint.

Exercise 3.1.52. Use Exercise 3.1.51 to classify all connected bipartite graphs with Frobenius-Perron eigenvalue strictly less than 2.
Hint: See [AMP15, p.11].

3.2 Jones' basic construction in finite dimensions

For this section, $A \subset B$ will always denote a unital inclusion of finite dimensional complex multi-matrix algebras.

3.2.1 Conditional expectations

Pick faithful tracial states tr_B on B and tr_A on A.

Definition 3.2.1. A *conditional expectation* $E : B \to A$ is a linear map such that

- ($A - A$ bilinear) $E(axb) = aE(x)b$ for all $x \in B$ and $a, b \in A$.
- (unital) $E(1) = 1$, and

A conditional expectation is called:

- *trace preserving* if $\operatorname{tr}_A(E(x)) = \operatorname{tr}_B(x)$ for all $x \in B$.
- *faithful* if $E(x^*x) = 0$ implies $x = 0$.

Exercise 3.2.2. Prove that $E|_A = \operatorname{id}_A$ and $E^2 = E$, i.e., for all $x \in B$, $E(E(x)) = E(x)$. Deduce that if $E : B \to A$ is trace preserving, then $\operatorname{tr}_B|_A = \operatorname{tr}_A$.

Exercise 3.2.3. Show that if $E, F : B \to A$ are two conditional expectations such that for all $a \in A$ and $b \in B$, $\operatorname{tr}_A(aE(b)) = \operatorname{tr}_A(aF(b))$, then $E = F$. Deduce that there is at most *one* trace preserving conditional expectation $B \to A$.
Hint: Show that for all $b \in B$ and $a \in A$, $\langle E(b)\Omega, a\Omega\rangle = \langle F(b)\Omega, a\Omega\rangle$ in $L^2(A, \operatorname{tr}_A)$.

Exercise 3.2.4. Suppose E is trace preserving. Show that $E(x^*) = E(x)^*$ for all $x \in B$.
Hint: First prove that $\operatorname{tr}_B(x^*) = \overline{\operatorname{tr}_B(x)}$ for all $x \in B$. Then show $\langle E(x)^*\Omega, a\Omega\rangle = \langle E(x^*)\Omega, a\Omega\rangle$ for all $x \in B$ and $a \in A$.

Exercise 3.2.5. Suppose E is trace preserving. Show that for any $x \in B$, $E(x^*x) \geq 0$.
Hint: Compute $\langle E(x^*x)a\Omega, a\Omega\rangle$ in $L^2(A, \operatorname{tr}_A)$.

Exercise 3.2.6. Consider the subspace $A\Omega \subset L^2(B, \operatorname{tr}_B)$. Let $e_A \in B(L^2(B, \operatorname{tr}_B))$ be the orthogonal projection onto $A\Omega$. Define $E : B \to A$ by $E(b) = a$ where $a \in A$ is the unique element such that $e_A(b\Omega) = a\Omega$. Prove that E is a faithful conditional expectation. Prove that E is trace preserving if and only if $\operatorname{tr}_B|_A = \operatorname{tr}_A$.

Exercise 3.2.7. Continue the notation of Exercise 3.2.6.

(1) Show that for all $b \in B$, $E(b)e_A = e_A b e_A$.

(2) Show that for all $b \in B$, we have $b \in A$ if and only if $e_A b = b e_A$.

Exercise 3.2.8. Compute the unique trace preserving conditional expectation for the following unital inclusions:

(1) The inclusion $M_k(\mathbb{C}) \hookrightarrow M_{nk}(\mathbb{C})$ with the unique normalized traces.

(2) The connected inclusion $A = M_n(\mathbb{C}) \oplus M_k(\mathbb{C}) \hookrightarrow M_{n+k}(\mathbb{C}) = B$ with trace vector on A given by $(\frac{1}{n+k}, \frac{1}{n+k})$ and the unique normalized trace on B.
Note: First verify that $n_A \lambda_A = 1$ where n_A denotes the dimension row vector of A.

(3) The connected inclusion $A = \mathbb{C} \oplus \mathbb{C} \hookrightarrow M_2(\mathbb{C}) \oplus \mathbb{C} = B$ with Bratteli diagram and trace vectors for A and B given by

$$\Lambda := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \lambda_A := \begin{pmatrix} \phi^{-2} \\ \phi^{-1} \end{pmatrix} \qquad \lambda_B := \begin{pmatrix} \phi^{-2} \\ \phi^{-3} \end{pmatrix}$$

where $\phi := \frac{1+\sqrt{5}}{2}$ (so $\phi^2 = 1 + \phi$ and $\phi^{n+2} = \phi^n + \phi^{n+1}$ for all $n \in \mathbb{Z}$).
Note: First verify that $n_A \lambda_A = 1 = n_B \lambda_B$ where n_A, n_B denotes the dimension row vector of A, B respectively.

3.2.2 The basic construction

Suppose $A \subset B$ is a unital inclusion of multimatrix algebras and tr_B is a faithful normal trace on B. Define $\mathrm{tr}_A = \mathrm{tr}_B|_A$, and let $e_A \in B(L^2(B, \mathrm{tr}_B))$ be the orthogonal projection with range $L^2(A, \mathrm{tr}_A) = A\Omega$ as in Exercise 3.2.6. Let $E : B \to A$ be the canonical trace-preserving conditional expectation, which is defined by $E(b)\Omega := e_A(b\Omega)$ for all $b \in B$.

Exercise 3.2.9. Show that $e_A b e_A = E(b) e_A$ for all $b \in B$.

Definition 3.2.10. The *basic construction* of $A \subset B$ is the unital $*$-subalgebra $\langle B, e_A \rangle \subset B(L^2(B, \mathrm{tr}_B))$ generated by B and e_A.

Exercise 3.2.11. Prove that $\langle B, e_A \rangle$ equals $B + Be_AB := \mathrm{span}\,\{a + b e_A c | a, b, c \in B\} \subset B(L^2(B, \mathrm{tr}_B))$.

Exercise 3.2.12 (**)**.** Prove that $\langle B, e_A \rangle$ equals $Be_AB := \mathrm{span}\,\{a e_A b | a, b \in B\} \subset B(L^2(B, \mathrm{tr}_B))$.
Note: We will try to use Exercise 3.2.11 instead of this exercise as much as possible.

Definition 3.2.13. The *modular conjugation* is the map $J : L^2(B, \mathrm{tr}_B) \to L^2(B, \mathrm{tr}_B)$ given by the anti-linear extension of $Jb\Omega := b^*\Omega$.

Exercise 3.2.14. Prove that for all $a, b \in B$, $\langle Ja\Omega, b\Omega \rangle = \langle Jb\Omega, a\Omega \rangle$.

Exercise 3.2.15. Use Exercise 3.2.4 to show that $Je_A = e_A J$ on $L^2(B, \mathrm{tr}_B)$.

Exercise 3.2.16. Recall $A' = \left\{x \in B(L^2(B, \mathrm{tr}_B)) \middle| xa = ax \text{ for all } a \in A\right\}$. Show that $JA'J = (JAJ)'$.

Exercise 3.2.17. Show that $\langle B, e_A \rangle = JA'J$.

Using this last exercise, we see that the basic construction algebra naturally arises as the missing algebra in the following picture.

$$\begin{tikzcd}
?? \arrow[dr] & & A' \arrow[dl] \\
B \arrow[r] & L^2(B, \mathrm{tr}_B) & JBJ = B' \arrow[l] \\
A \arrow[ur] & & JAJ \arrow[ul]
\end{tikzcd}$$

Exercise 3.2.18. Show that $A \subseteq B$ is connected if and only if $B \subseteq \langle B, e_A \rangle$ is connected.

3.2.3 Morita equivalence

[[todo.]]

3.2.4 The inclusion $B \subset \langle B, e_A \rangle$

For this section, $A \subseteq B$ is a unital inclusion of multimatrix algebras whose dimension row vectors are given respectively by $n^A = (n_1^A, \dots, n_k^A)$ and $n^B = (n_1^B, \dots, n_\ell^B)$, and Λ_A^B is the inclusion matrix. We denote the minimal central projections of A and B by $p_1, \dots, p_k$ and $q_1, \dots, q_\ell$ respectively. Let λ^A, λ^B be the trace column vectors for A and B respectively, which satisfy the normalization condition $n^A \lambda^A = 1 = n^B \lambda^B$. This means the i-th entry of λ^A is the trace of a minimal projection in $p_i A$, and similarly for B. Since $\mathrm{tr}_B|_A = \mathrm{tr}_A$, we have $\lambda^A = \Lambda_A^B \lambda^B$. Using either Exercise 3.1.25 or 3.2.17, Jones' basic construction $\langle B, e_A \rangle$ is a finite dimensional von Neumann algebra, and thus a multimatrix algebra by Exercise 3.1.26.

Theorem 3.2.19. *The inclusion matrix for $B \subseteq \langle B, e_A \rangle$ can be canonically identified with $(\Lambda_A^B)^T$.*

We give three separate proofs of this important, non-trivial theorem. Each proof develops and expands the ideas in the previous proof making fewer assumptions. The first is a very quick K-theoretic proof which uses Exercise 3.2.12. The second translates the first proof into operator algebras, which gives a conceptual explanation of the proof that appears in [JS97, Lem. 3.2.2(iv)]. The third is a bare-bones approach based on ideas provided to us by Srivatsa Srinivas during his undergraduate research project Summer 2018.

Proof 1 of Thm. 3.2.19. It follows from Exercise 3.2.12 that the unital $*$-algebras A and $\langle B, e_A \rangle$ are Morita-equivalent via the Morita-equivalence bimodule ${}_{\langle B, e_A \rangle} B_A$. Hence there is a canonical isomorphism $Z(\langle B, e_A \rangle) \cong Z(A)$. We denote the minimal central projections of $\langle B, e_A \rangle$ by $r_1, \dots, r_k$ which respectively correspond to the minimal central projections $p_1, \dots, p_k$ of A.

Now $(\Lambda_A^B)_{ij}$ is the number of summands of $p_i B q_j$ as an $A - B$ bimodule. Tensoring with ${}_{\langle B, e_A \rangle} B_A$, we see this is the same number of summands of

$${}_{\langle B, e_A \rangle} B_A \otimes_A p_i B q_j \cong r_i \langle B, e_A \rangle q_j$$

as a $\langle B, e_A \rangle - B$ bimodule. By taking the conjugate bimodule, we have that $(\Lambda_A^B)_{ij}$ is equal to the number of summands of $q_j \langle B, e_A \rangle r_i$ as a $B - \langle B, e_A \rangle$ bimodule, which is exactly the entry $(\Lambda_B^{\langle B, e_A \rangle})_{ji}$. □

Proof 2 of Thm. 3.2.19. First, since $\langle B, e_A\rangle = JA'J$ on $B(L^2(B, \operatorname{tr}_B))$, we see that the map $z \mapsto JzJ$ gives a canonical isomorphism $Z(A) \to Z(\langle B, e_Z\rangle)$. Now in operator algebraic language, the can identify endomorphism space of $p_iBq_j = p_iq_jB$ as an $A-B$ bimodule as

$$\operatorname{End}_{A-B}(p_iq_jB) \cong \underbrace{(p_iq_jA)'}_{(\text{left } A\text{-action})'} \cap \underbrace{p_iq_jBp_iq_j}_{(\text{right } B\text{-action})'} \subset B(L^2(p_iq_jB)),$$

where the faithful tracial state on p_iq_jB is given by $x \mapsto \operatorname{tr}_B(p_iq_jx)/\operatorname{tr}_B(p_iq_j)$. Since this endomorphism algebra is a full matrix algebra corresponding to the $A-B$ isotypic component of p_iq_jB in B, the number of summands is given by taking the square root of the dimension of the endomorphism algebra:

$$(\Lambda_A^B)_{ij} = \dim(\operatorname{End}_{A-B}(p_iq_jB))^{1/2} = \dim((p_iq_jA)' \cap p_iq_jBp_iq_j)^{1/2}.$$

By symmetry, we have

$$(\Lambda_B^{\langle B,e_A\rangle})_{ji} = \dim(\operatorname{End}_{B-\langle B,e_A\rangle}(q_jJp_iJB))^{1/2} = \dim((q_jJp_iJB)' \cap q_jJp_iJ\langle B, e_A\rangle q_jJp_iJ)^{1/2}.$$

Now both $(p_iq_jA)' \cap p_iq_jBp_iq_j$ and $(q_jJp_iJB)' \cap q_jJp_iJ\langle B, e_A\rangle q_jJp_iJ$ are von Neumann subalgebras of $B(L^2(B, \operatorname{tr}_B))$, and we see that the unital $*$-algebra anti-homomorphism $\operatorname{Ad}(J) : B \to B' = JBJ$ given by $x \mapsto Jx^*J$ satisfies

$$\begin{aligned}
J\left((p_iq_jA)'\right)J &= (Jp_iq_jAJ)' = (Jp_iJq_jJAJ)' = Jp_iJq_j\langle B, e_A\rangle Jp_iJq_j \\
J\left(p_iq_jBp_iq_j\right)J &= Jp_iJq_j(JBJ)q_jJp_iJ = Jp_iJq_j(B')q_jJp_iJ = (q_jJp_iJB)'.
\end{aligned}$$

Hence $\operatorname{Ad}(J)$ provides a unital $*$-algebra anti-isomorphism

$$\operatorname{End}_{A-B}(p_iq_jB) \cong \operatorname{End}_{B-\langle B,e_A\rangle}(q_jJp_iJB)$$

viewed as von Neumann subalgebras of $B(L^2(B, \operatorname{tr}_B))$, and thus they have the same dimension. We conclude $(\Lambda_A^B)_{ij} = (\Lambda_B^{\langle B,e_A\rangle})_{ji}$. □

Proof 3 of Thm. 3.2.19. Fix a unital $*$-isomorphism

$$\alpha : M_{n_1^A}(\mathbb{C}) \oplus \cdots \oplus M_{n_r^A}(\mathbb{C}) \xrightarrow{\cong} A. \tag{3.2}$$

Without loss of generality, we may assume the inclusion $A \subseteq B$ is block diagonal on the summands of $\alpha^{-1}(A)$ such that all copies of $M_{n_i^A}(\mathbb{C})$ appear before any copies of $M_{n_{i'}^A}(\mathbb{C})$ if $i < i'$.

Denote the row vector $m := (m_1, \ldots, m_k) := n^B(\Lambda_A^B)^T$, and note that $n^B(n^B)^T = n^B(\Lambda_A^B)^T(n^A)^T = m(n^A)^T$, which implies $\sum_{j=1}^{\ell}(n_j^B)^2 = \sum_{i=1}^{k} n_i^A m_i$. Let $H = \bigoplus_{i=1}^{k} M_{n_i^A \times m_i}(\mathbb{C})$ with inner product

$$\langle (x_i)_{i=1}^k, (y_i)_{i=1}^k\rangle_H := \sum_{i=1}^{k} \frac{\lambda_i^A}{n_i^A} \operatorname{Tr}(x_iy_i^*), \text{[[probably incorrect!]]}$$

where Tr denotes the non-normalized trace on each summand $\alpha(M_{n_i^A}(\mathbb{C})) \subseteq A$. Notice that A has a faithful left action on H given by $\pi_a(x_i) := (a_ix_i)_{i=1}^k$ where $a_i := \alpha^{-1}(a)_i \in M_{n_i^A}(\mathbb{C})$. Moreover, by Exercise 3.1.33, the commutant $\pi(A)' \cap B(H)$ of the left A-action on H is the right action of $\bigoplus_{i=1}^{k} M_{m_i}(\mathbb{C})$.

Now fix $1 \le i \le k$. For $j = 1, \ldots, \ell$, p_iq_jB can be viewed as Λ_{ij} matrices of dimension $n_i^A \times n_j^B$. Denote these matrices by $(p_iq_jB)_k$, $1 \le k \le \Lambda_{ij}$. We concatenate these matrices *horizontally* indexing lexicographically over $1 \le j \le \ell$ and $1 \le k \le \Lambda_{ij}$:

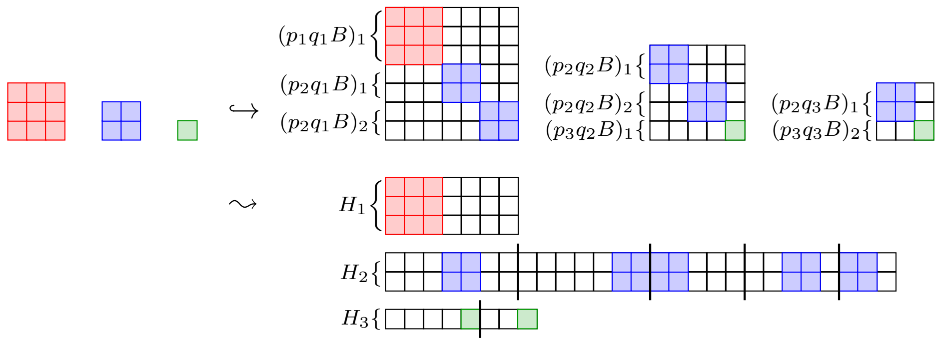
$$H_i := \begin{pmatrix} (p_iq_1B)_1 & \cdots & (p_iq_1B)_{\Lambda_{i1}} & (p_iq_2B)_1 & \cdots & (p_iq_2B)_{\Lambda_{i2}} & (p_iq_\ell B)_1 & \cdots & (p_iq_\ell B)_{\Lambda_{i\ell}} \end{pmatrix}.$$

That is, H_i is *equal* to $M_{n_i^A \times m_i}(\mathbb{C})$ as a vector space.

For the explicit inclusion $M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) \overset{\cong}{\hookrightarrow} A \subseteq B = M_7(\mathbb{C}) \oplus M_5(\mathbb{C}) \oplus M_3(\mathbb{C})$ with inclusion matrix

$$\Lambda_A^B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$

we can visualize H_1, H_2, H_3 graphically as follows:



We now define a linear transformation $u : L^2(B, \operatorname{tr}_B) \to H$ component-wise by

$$[u(b\Omega)]_i := \begin{pmatrix} (p_iq_1b)_1 & \cdots & (p_iq_1b)_{\Lambda_{i1}} & \cdots & (p_iq_\ell b)_1 & \cdots & (p_iq_\ell b)_{\Lambda_{i\ell}} \end{pmatrix} \in H_i.$$

is a unitary operator which intertwines the left A-actions: $uab\Omega = \pi_a \cdot ub\Omega$ for all $a \in A$ and $b \in B$. Hence the commutant $A' \cap B(L^2(B, \operatorname{tr}_B))$ is unitally $*$-isomorphic to $\bigoplus_{i=1}^k M_{m_i}(\mathbb{C})$. Now H also carries a right B-action given by

$$\rho_c \cdot \begin{pmatrix} (p_iq_1b)_1 & \cdots & (p_iq_\ell b)_{\Lambda_{i\ell}} \end{pmatrix} := \begin{pmatrix} (p_iq_1bc)_1 & \cdots & (p_iq_\ell bc)_{\Lambda_{i\ell}} \end{pmatrix},$$

which commutes with the left A-action by construction. Moreover, $u : L^2(B, \operatorname{tr}_B) \to H$ is manifestly right B-linear, i.e., u is an $A - B$ bilinear unitary. Hence we have a unital $*$-isomorphism

$$B(L^2(B, \operatorname{tr}_B)) \supseteq A' \cap B \xrightarrow{\operatorname{Ad}(u)} \pi(A)' \cap \rho(B) \subseteq B(H).$$

But we calculated earlier that

$$\pi(A)' = \bigoplus_{i=1}^k 1_{n_i^A} \otimes M_{m_i}(\mathbb{C}) \subseteq \bigoplus_{i=1}^k M_{n_i^A}(\mathbb{C}) \otimes M_{m_i}(\mathbb{C}) = B(H),$$

[[finish.]] □

We get the following strengthening of Exercise 3.2.18.

Corollary 3.2.20. *The Bratteli diagram for $B \subseteq \langle B, e_A \rangle$ is the reflection of the Bratteli diagram for $A \subseteq B$.*

We now investigate the topic of compatible traces on $\langle B, e_A \rangle$. Suppose $\operatorname{tr}_{\langle B, e_A \rangle}$ is a faithful state on $\langle B, e_A \rangle$ with corresponding trace vector $\lambda^{\langle B, e_A \rangle}$. By Exercise 3.1.45 and Theorem 3.2.19, we know that $\operatorname{tr}_{\langle B, e_A \rangle} |B = \operatorname{tr}_B$ if and only if $\Lambda_B^{\langle B, e_A \rangle} \lambda^{\langle B, e_A \rangle} = (\Lambda_A^B)^T \lambda^{\langle B, e_A \rangle} = \lambda^B$.

Now suppose the inclusion $A \subseteq B$ is connected, so that the inclusion $B \subseteq \langle B, e_A \rangle$ is also connected. We still assume that $\operatorname{tr}_B |_A = \operatorname{tr}_A$ so $\Lambda_A^B \lambda^B = \lambda^A$.

Proposition 3.2.21 ([Jon83, Thm. 3.3.2])**.** *The following are equivalent for $d > 0$:*

- *(Markov property)* $\operatorname{tr}_{\langle B,e_A\rangle}(xe_A) = d^{-2}\operatorname{tr}_B(x)$ *for all* $x \in B$*, and*
- *(Frobenius-Perron property)* $\lambda^{\langle B,e_A\rangle} = d^{-2}\lambda^A$

Thus by Exercise 3.1.48, the unique Frobenius-Perron eigenvalue d gives the unique $\lambda^{\langle B,e_A\rangle}$ satisfying the above conditions.

Proof. Recall the minimal central projections of $\langle B, e_A\rangle$ are given by Jp_iJ, $i = 1,\ldots,k$. We claim that if r_i is a minimal projection in p_iA, then e_Ar_i is a minimal projection in $Jp_iJ\langle B, e_A\rangle$. Indeed, r_ie_A is easily seen to be a projection which commutes with Jp_iJ, and we calculate for all $x \in B$,

$$r_ie_AJp_iJ(x\Omega) = r_ie_Axp_i\Omega = r_iE(xp_i)\Omega = r_iE(x)p_i\Omega = r_iE(x)\Omega = r_ie_A(x\Omega).$$

To see minimality of r_ie_A, we calculate for $x, y \in B$ that

$$(r_ie_A)x(r_ie_A) = r_iE(x)r_ie_A \in \mathbb{C}r_ie_A \qquad \text{and} \qquad (r_ie_A)xe_Ay(r_ie_A) = r_iE(x)e(y)r_ie_A \in \mathbb{C}r_ie_A$$

by minimality of $r_i \in A$.

Suppose $\operatorname{tr}_{\langle B,e_A\rangle}$ satisfies the Markov property. Then for each $1 \leq i \leq k$,

$$\lambda_i^{\langle B,e_A\rangle} = \operatorname{tr}_{\langle B,e_A\rangle}(r_ie_A) = d^{-2}\operatorname{tr}_A(r_i) = d^{-2}\lambda_i^A$$

where $r_i \leq p_i \in A$ is a minimal projection. Hence $\lambda^{\langle B,e_A\rangle} = d^{-2}\lambda^A$.

Conversely, suppose $\lambda^{\langle B,e_A\rangle} = d^{-2}\lambda^A$. Then the Markov property $\operatorname{tr}_{\langle B,e_A\rangle}(ae_A) = d^{-2}\operatorname{tr}_A(a)$ folds for every minimal projection in A, and thus for every element of A. Then for all $x \in B$, we have

$$\operatorname{tr}_{\langle B,e_A\rangle}(xe_A) = \operatorname{tr}_{\langle B,e_A\rangle}(e_Axe_A) \underset{\text{(Ex. 3.2.9)}}{=} \operatorname{tr}_{\langle B,e_A\rangle}(E(x)e_A) = d^{-2}\operatorname{tr}_A(E(x)) = d^{-2}\operatorname{tr}_B(x).$$

The last equality holds since E is the unique trace-preserving conditional expectation. □

3.3 Markov towers

In this section, we study a tower of finite dimensional tracial von Neumann algebras called a *Markov tower*. The definition of a Markov tower can obtained from the definition of Popa's *λ-sequences of commuting squares* from [Pop95] by forgetting one of the towers, analogous to the way one defines a module for an algebraic object by replacing one argument of the algebraic operation with an element from the module.

3.3.1 Markov towers and their elementary properties

Definition 3.3.1. A *Markov tower* $M_\bullet = (M_n, \operatorname{tr}_n, e_{n+1})_{n\geq 0}$ consists of a sequence $(M_n, \operatorname{tr}_n)_{n\geq 0}$ of finite dimensional von Neumann algebras, such that M_n is unitally included in M_{n+1}, each M_n has a faithful normal tracial states such that $\operatorname{tr}_{n+1}|_{M_n} = \operatorname{tr}_n$ for all $n \geq 0$, and there is a sequence of *Jones projections* $e_n \in M_{n+1}$ for all $n \geq 1$, such that:

(M1) The projections (e_n) satisfy the Temperley-Lieb-Jones relations (J1) – (J3) for a fixed constant $d > 0$ called the *modulus* of the Markov tower.

(M2) For all $x \in M_n$, $e_n x e_n = E_n(x)e_n$, where $E_n : M_n \to M_{n-1}$ is the canonical faithful trace-preserving conditional expectation from Exercise 3.2.6.

(M3) For all $n \geq 1$, $E_{n+1}(e_n) = d^{-2}$.

(M4) (pull down) For all $n \geq 1$, $M_{n+1}e_n = M_n e_n$.

We call a Markov tower *connected* if $\dim(M_0) = 1$.

Remark 3.3.2. One should think of the preceding definition as obtained from Popa's definition of λ-*sequence* [Pop95] and removing one of the two sequences of algebras, together with the commuting square condition. Compare the existence of Jones projections, (M1), and (M2) with (1.3.2), and (M3) and (M4) with (1.3.3') from [Pop95] respectively.

Remark 3.3.3. Observe that $M_n e_n M_n$ is a 2-sided ideal in M_{n+1} for all $n \geq 1$ if and only if the pull down condition holds. Indeed, if the pull down condition holds, then $M_{n+1}M_n e_n M_n \subseteq M_{n+1}e_n M_n = M_n e_n M_n$; the same argument holds on the right by first taking adjoints. Conversely, if $M_n e_n M_n$ is a 2-sided ideal, then $M_{n+1}e_n = (M_{n+1}e_n)e_n \subseteq (M_n e_n M_n)e_n = M_n e_n$.

Exercise 3.3.4. Prove that a Markov tower satisfies the following elementary properties for $n \geq 1$.

(EP1) The map $M_n \ni y \mapsto ye_n \in M_{n+1}$ is injective.

(EP2) For all $x \in M_{n+1}$, $d^2 E_{n+1}(xe_n)$ is the unique element $y \in M_n$ such that $xe_n = ye_n$ [PP86, Lem. 1.2].

(EP3) The traces tr_{n+1} satisfy the following *Markov property* with respect to M_n and e_n: for all $x \in M_n$, $\operatorname{tr}_{n+1}(xe_n) = d^{-2}\operatorname{tr}_n(x)$.

(EP4) $e_n M_{n+1} e_n = M_{n-1}e_n$.

(EP5) $X_{n+1} := M_n e_n M_n$ is a 2-sided ideal of M_{n+1}, and thus M_{n+1} splits as a direct sum of von Neumann algebras $X_{n+1} \oplus Y_{n+1}$. (In [GdlHJ89, Thm. 4.1.4 and Thm. 4.6.3], Y_{n+1} is the so-called 'new stuff'.) By convention, we define $Y_0 = M_0$ and $Y_1 = M_1$, so that $X_0 = (0)$ and $X_1 = (0)$.

(EP6) The map $ae_n b \mapsto ap_n b$ gives a $*$-isomorphism from $X_{n+1} = M_n e_n M_n$ to $\langle M_n, p_n \rangle = M_n p_n M_n$, the Jones basic construction of $M_{n-1} \subseteq M_n$ acting on $L^2(M_n, \operatorname{tr}_n)$.

(EP7) Under the isomorphism $X_{n+1} \cong M_n p_n M_n$, the canonical non-normalized trace Tr_{n+1} on the Jones basic construction algebra $M_n p_n M_n$ satisfying $\operatorname{Tr}_{n+1}(ap_n b) = \operatorname{tr}_n(ab)$ for $a, b \in M_n$ equals $d^2 \operatorname{tr}_{n+1}|_{X_{n+1}}$.

(EP8) If $y \in Y_{n+1}$ and $x \in X_n$, then $yx = 0$ in M_{n+1}. Hence $E_{n+1}(Y_{n+1}) \subseteq Y_n$. ("The new stuff comes only from the old new stuff" [GdlHJ89].)

(EP9) If $Y_n = (0)$, then $Y_k = (0)$ for all $k \geq n$.

Remark 3.3.5. The foregoing observations all hold in the case where the M_n are arbitrary tracial von Neumann algebras. In this section, we restrict our attention to the finite dimensional case to obtain a principal graph for a Markov tower.

Notice that by (EP6), the Bratteli diagram for the inclusion $M_n \subset M_{n+1}$ consists of the reflection of the Bratteli diagram for the inclusion $M_{n-1} \subset M_n$, together with possibly some new edges and vertices corresponding to simple summands of Y_{n+1}. By (EP8), the new vertices at level $n+1$ only connect to the vertices that were new at level n. This leads to the following definition.

Definition 3.3.6. The *principal graph* of the Markov tower $M_\bullet$ consists of the *new* vertices at every level n of the Bratteli diagram, together with all the edges connecting them. A Markov tower is said to have *finite depth* if the principal graph is finite.

It follows that a Markov tower has finite depth if and only if there is $n \in \mathbb{N}$ such that $Y_n = (0)$, as in (EP9). Let $M_\bullet$ be a Markov tower with finite depth, and take the minimal integer $n \in \mathbb{N}$ such that $Y_n = (0)$. Notice that for $k < n$, the Bratteli diagram of $M_k \subseteq M_{k+1}$ contains the reflection of the Bratteli diagram of $M_{k-1} \subseteq M_k$, along with additional vertices and edges which are part of the principal graph. At the base, all of the Bratteli diagram for $M_0 \subseteq M_1$ is part of the principal graph. We can therefore 'unravel' the Bratteli diagram for $M_n \subseteq M_{n+1}$ to obtain the principal graph for the Markov tower $M_\bullet$.

Exercise 3.3.7. Show that if a Markov tower $M_\bullet$ has finite depth and $n \in \mathbb{N}$ is such that $Y_n = (0)$, then for $k \geq n$, there is a canonical graph isomorphism between the principal graph of $M_\bullet$ and the Bratteli diagram for $M_k \subseteq M_{k+1}$.

Definition 3.3.8. The principal graph Γ of a Markov tower $M_\bullet$ has a *quantum dimension function* $\dim : V(\Gamma) \to \mathbb{R}_{>0}$ given as follows. Let $v \in V(\Gamma)$, and let $p \in M_k$ be a minimal projection with k minimal corresponding to the vertex v. We define $\dim(v) := d^k \operatorname{tr}_k(p)$, and we note this dimension is independent of the choice of $p \in M_k$ representing v. Moreover, the quantum dimension function dim satisfies the Frobenius-Perron property

$$d \cdot \dim(v) = \sum_{w \sim v} \dim(w) \tag{3.3}$$

where we write $w \sim v$ to mean w is connected to v, and the above sum is taken with multiplicity.

Chapter 4

Jones' modulus rigidity theorem

Recall that the TL $*$-algebras $TL_n(d)$ had distinguished traces Tr_n. Using these traces, the adjoint, and the multiplication, we get *sesquilinear forms* on the $TL_n(d)$ defined by

$$\langle x, y\rangle_n := \mathrm{Tr}_n(y^* x).$$

In this section, we prove the following surprising result from [Jon83], which was expressed in the language of index for subfactors.

Theorem 4.0.1 (Jones' modulus rigidity)**.** *The sesquilinear forms $\langle \cdot\,, \cdot\rangle_n$ are positive semi-definite for all $n \geq 0$ if and only if*

$$d \in \underbrace{\{2\cos(\pi/k)|k \geq 3\}}_{\text{semi}-\text{definite}} \cup \underbrace{[2, \infty)}_{\text{definite}}. \tag{4.1}$$

We will prove this result in three steps. First, we prove the forward direction which proves the modulus restriction in Theorem 4.1.11. Second, we will start at the special moduli and show the forms are positive semi-definite in Theorem 4.2.2. Finally, we will sketch the proof that the forms are positive definite in the generic range in Theorem 4.3.3.

Exercises and sections marked $(*)$ below are more advanced and can be skipped on first read through. Exercises and sections marked $(**)$ are even more advanced, and they are very difficult relative to the exposition!

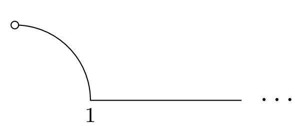
4.1 The modulus restriction

In this section, we will show that if the sesquilinear forms are all positive definite, then d satisfies (4.1).

4.1.1 Quantum Integers

Here is some cursory information on quantum integers. Since we will mostly present a unitary viewpoint in the course, of great importance is the adjoint, which means we will work only with particular loop parameters for TL.

Definition 4.1.1. Consider $Q := \left\{e^{i\theta}\middle|\theta \in (0, \frac{\pi}{2})\right\} \cup [1, \infty)$



For $q \in Q \cup -Q$, we define quantum n by

$$[n] = [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Exercise 4.1.2. Prove that $q \mapsto [2] = q + q^{-1}$ is a bijective map $Q \cup -Q \to \mathbb{R} \setminus \{0\}$. Hence writing $d := [2] = q + q^{-1}$ uniquely determines $q \in Q \cup -Q$.

Exercise 4.1.3. Prove that the quantum integers satisfy the following relations:

- $[1] = 1$
- $[2][1] = [2] = q + q^{-1}$
- $[2][n] = [n+1] + [n-1]$.

Exercise 4.1.4. Show that the map $q \mapsto -q$ fixes all even quantum integers $[2n]$ and negates all odd quantum integers $[2n+1]$.

Exercise 4.1.5. Show that:

- If $q = e^{i\theta}$, then $[2] = q + q^{-1} = 2\cos(\theta)$.
- $[n] = 0$ if and only if q is a $(2n)$-th root of unity.

The following lemma will be very important for the modulus restriction in Theorem 4.1.11.

Lemma 4.1.6. *Suppose $q = e^{i\theta}$ for some $\theta \in (0, \frac{\pi}{2})$, where $\theta \neq \frac{2\pi}{2n}$ for some $n \geq 3$, i.e., q is not a primitive $(2n)$-th root of unity for some $n \geq 3$. Let $k \geq 2$ be minimal such that $\theta > \frac{2\pi}{2(k+1)}$. Then $[1], [2] \ldots, [k] > 0$, but $[k+1] < 0$.*

Proof. Note that since $q = e^{i\theta}$,

$$[j] = \frac{e^{ij\theta} - e^{-ij\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(j\theta)}{\sin(\theta)}.$$

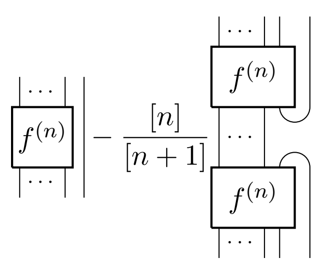
Since $\sin(\theta) > 0$, we only care about $\sin(j\theta)$. Since $\frac{\pi}{k} > \theta > \frac{\pi}{k+1}$, we have that $\sin(1\theta), \ldots, \sin(k\theta) > 0$, but $\sin((k+1)\theta) < 0$. $\square$

4.1.2 The Jones-Wenzl projections

The following idempotents were first defined in [Jon83]. The recurrence relation first appeared in [Wen87].

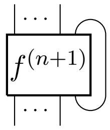
Definition 4.1.7. Let $f^{(0)} \in TL_0(d)$ be the empty diagram. Let $f^{(1)} \in TL_1(d)$ be the strand, i.e., $f^{(1)} =$ . If $[n+1] \neq 0$, we inductively define the $(n+1)$-th *Jones-Wenzl projection*

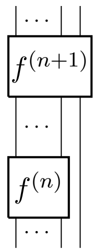
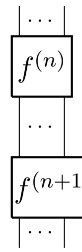
$$f^{(n+1)} = i_n(f^{(n)}) - \frac{[n]}{[n+1]} i_n(f^{(n)}) E_n i_n(f^{(n)}) =$$



Proposition 4.1.8. *Suppose $n \geq 0$ and $[1], \cdots, [n+1] \neq 0$ so that $f^{(0)}, f^{(1)}, \ldots, f^{(n+1)}$ are well-defined. Then $f^{(n+1)}$ satisfies the following properties:*

(JW1) *$f^{(n+1)}$ is an orthogonal projection, i.e., $f^{(n+1)} = (f^{(n+1)})^* = (f^{(n+1)})^2$,*

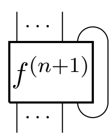
(JW2) $\mathcal{E}_{n+1}(f^{(n+1)}) =$  $= \dfrac{[n+2]}{[n+1]} f^{(n)}$,

(JW3) $(i_n(f^{(n)}))f^{(n+1)} =$  $= f^{(n+1)}(i_n(f^{(n)}) =$  $= f^{(n+1)}$.

Proof. We proceed by induction on n. The base case $n = 0$ is straightforward as $f^{(1)}$ is the strand and $f^{(0)}$ is the empty diagram. Suppose the result holds for $f^{(n)}$.

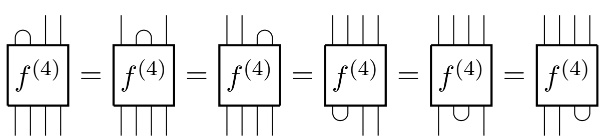
(JW1): That $(f^{(n+1)})^* = f^{(n+1)}$ follows from the fact that $(f^{(n)})^* = f^{(n)}$ by (JW1) for $f^{(n)}$, which holds by the induction hypothesis. We now calculate using (JW2) and (JW3) for $f^{(n)}$, which hold by the induction hypothesis, that $f^{(n+1)} = (f^{(n+1)})^2$.

(JW2): By (JW1) for $f^{(n)}$, which holds by the induction hypothesis, we see that

$\mathcal{E}_{n+1}(f^{(n+1)}) =$  $= \left([2] - \dfrac{[n]}{[n+1]}\right) f^{(n)} = \dfrac{[n+2]}{[n+1]} f^{(n)}.$

(JW3): By the definition of $f^{(n+1)}$, this property follows directly from $(f^{(n)})^2 = f^{(n)}$ by (JW1) for $f^{(n)}$, which holds by the induction hypothesis. □

Exercise 4.1.9. Deduce that when $f^{(0)}, \ldots, f^{(n)}$ are well-defined, $f^{(n)}$ is *rectangularly uncappable*, that is, capping any two strings on the top or bottom of $f^{(n)}$ gives zero, e.g.,

 $= 0$.

Exercise 4.1.10. Deduce that when $f^{(0)}, \ldots, f^{(n)}$ are well-defined, $\operatorname{tr}_n(f^{(n)}) = [n+1]$.

4.1.3 Proof of the modulus restriction

Theorem 4.1.11 (Jones' modulus restriction)**.** *Suppose $\langle \cdot, \cdot \rangle_j$ is positive semidefinite for all $j \geq 0$. Then either $q \geq 1$, or q is a primitive $(2n)$-th root of unity for some $n \geq 3$. Hence*

$$d = [2] = q + q^{-1} \in \left\{ 2\cos\left(\frac{\pi}{n}\right) \middle| n \geq 3 \right\} \cup [2, \infty).$$

Proof. If q is not of this form, then let k be as in Lemma 4.1.6. We see that since $[1], [2], \ldots, [k] \neq 0$, $f^{(k)}$ is well-defined. However,

$$\langle f^{(k)}, f^{(k)} \rangle_k = \operatorname{Tr}_k(f^{(k)}) \underset{\text{Ex. 4.1.10}}{=} [k+1] < 0,$$

which is a contradiction. □

4.2 Representations of TL algebras via towers of algebras

Starting with a connected inclusion $A_0 \subseteq A_1$ of multimatrix algebras with dimension row vectors n^0, n^1, trace column vectors λ^0, λ^1, and inclusion matrix $\Lambda := \Lambda_{A_0}^{A_1}$ satisfying

$$n^0\lambda^0 = 1 = n^1\lambda^1 \qquad n^0\Lambda = n^1 \qquad \Lambda\lambda^1 = \lambda^0,$$

we saw that the Jones basic construction $A_2 := \langle A_1, e_1\rangle$ is another multimatrix algebra with dimension row vector n^2, and the inclusion $A_1 \subset A_2$ has inclusion matrix Λ^T. If moreover λ^0 and λ^1 are the unique trace column vectors such that

$$\begin{pmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{pmatrix}\begin{pmatrix} \lambda^0 \\ d\lambda^1 \end{pmatrix} = d\begin{pmatrix} \lambda^0 \\ d\lambda^1 \end{pmatrix}$$

where $d > 0$ is the unique Frobenius-Perron eigenvalue, we get a trace column vector $\lambda^2 := d^{-2}\lambda^0$ satisfying

$$n^1\lambda^1 = 1 = n^2\lambda^2 \qquad n^1\Lambda^T = n^2 \qquad \Lambda^T\lambda^2 = \lambda^1.$$

We may take another Jones basic construction to obtain the multimatrix algebra $A_3 := \langle A_2, e_2\rangle$ with dimension row vector n^3, and the inclusion $A_2 \subset A_3$ has inclusion matrix Λ. We may again take the Frobenius-Perron trace column vector $\lambda^3 := d^{-2}\lambda^1$.

Iterating this process, we obtain the *Jones tower* of multimatrix algebras $(A_n)_{n\geq 0}$, each equipped with a faithful tracial state tr_n such that $\operatorname{tr}_n|_{A_{n-1}} = \operatorname{tr}_{n-1}$, and $A_{n+1} = \langle A_n, e_n\rangle = J_nA'_{n-1}J_n \subset B(L^2(A_n, \operatorname{tr}_n))$ is the Jones basic construction of $A_{n-1} \subset A_n$.

$$A_0 \subset A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \overset{e_3}{\subset} A_4 \overset{e_4}{\subset} \cdots$$

Observe that since tr_n corresponds to the Frobenius-Perron trace vector, for every $n \in \mathbb{N}$ we have

$$\operatorname{tr}_{n+1}(xe_n) = d^{-2}\operatorname{tr}_n(x) \iff E_{n+1}(e_n) = d^{-2} \tag{4.2}$$

where $E_{n+1} : A_{n+1} \to A_n$ is the unique trace preserving conditional expectation by Exercise 3.2.3.

Theorem 4.2.1. *The projections $e_1, \ldots, e_{n-1}$ satisfy the Temperley-Lieb-Jones relations (J1) – (J3).*

Proof. Clearly (J1) holds as all the e_i are defined to be orthogonal projections. Suppose $2 \leq i+1 < j$. Then $e_i \in A_{i+1} \subsetneq A_j \subset A_{j+1} \ni e_j$, and $e_jxe_j = E_{A_{j-1}}^{A_j}(x)e_j$ for all $x \in A_j$. In particular, e_j commutes with $A_{i+1} \ni e_i$ by Exercise 3.2.7, and (J2) holds.

For $n \geq 1$, we have $e_{n+1}e_ne_{n+1} = E_{n+1}(e_n)e_{n+1} = d^{-2}e_n$ by (4.2) as e_{n+1} implements E_{n+1} by Exercise 3.2.9. We calculate for all $x, y \in A_n$,

$$\begin{aligned} e_ne_{n+1}e_n(x\Omega) &= e_ne_{n+1}(e_nx\Omega) = e_nE_{n+1}(e_nx)\Omega = e_nE_{n+1}(e_n)x\Omega = d^2e_n(x\Omega) \text{ and} \\ e_ne_{n+1}e_n(xe_ny\Omega) &= e_ne_{n+1}(E_n(x)e_ny\Omega) = e_nE_{n+1}(E_n(x)e_ny)\Omega \\ &= e_nE_n(x)E_{n+1}(e_n)y\Omega = d^{-2}e_nE_n(x)y\Omega = d^{-2}e_n(xe_ny\Omega). \end{aligned}$$

Since $A_{n+1} = A_n + A_ne_nA_n$ by Exercise 3.2.11, we conclude $e_ne_{n+1}e_n = d^{-2}e_n$, and (J3) holds. $\square$

By Theorem 4.2.1, for every $n \in \mathbb{N}$, we have a unital $*$-algebra homomorphism $\Psi_n : TL_n(d) \to A_n \subset B(L^2(A_n, \operatorname{tr}_n))$. Moreover, these homomorphisms are easily seen to commute with the inclusions $TL_n(d) \hookrightarrow TL_{n+1}(d)$ and $A_n \hookrightarrow A_{n+1}$.

$$\begin{tikzcd} TL_0(d) \arrow[r, hook] \arrow[d, "\Psi_0"] & TL_1(d) \arrow[r, hook] \arrow[d, "\Psi_1"] & TL_2(d) \arrow[r, hook] \arrow[d, "\Psi_2"] & TL_3(d) \arrow[r, hook] \arrow[d, "\Psi_3"] & TL_4(d) \arrow[r, hook] \arrow[d, "\Psi_4"] & \cdots \\ A_0 \arrow[r, hook] & A_1 \arrow[r, hook] & A_2 \arrow[r, hook] & A_3 \arrow[r, hook] & A_4 \arrow[r, hook] & \cdots \end{tikzcd} \tag{4.3}$$

Theorem 4.2.2 (Jones' TL positivity theorem, discrete range)**.** *Suppose $A_0 \subset A_1$ is an inclusion of multimatrix algebras with Bratteli diagram*

$$\Lambda_n := \overset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!- \cdots -\!\!\!-\!\!\overset{n-1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{n}{\bullet} \tag{4.4}$$

where $n \geq 2$, so the Frobenius-Perron eigenvalue of Λ is $d = 2\cos(\pi/(n+1))$ by Exercise 3.1.50. Let $(A_n, \operatorname{tr}_n)_{n\geq 0}$ be the Jones tower corresponding to the Frobenius-Perron traces on $A_0 \subset A_1$. For all $n \geq 0$, $\operatorname{tr}_{A_n} \circ \Psi_n = d^{-n} \operatorname{Tr}_n$, where Tr_n was defined in (2.3) on $TL_n(d)$. Hence $\langle \cdot, \cdot \rangle_n$ is positive on $TL_n(d)$ for all $n \geq 0$.

Proof. We proceed by induction on n. Note that $TL_0 = \mathbb{C}1 \hookrightarrow \mathbb{C}1_{A_0}$, and both traces are normalized, so they must agree. Suppose the result holds for $n \geq 0$. To avoid confusion, we will work with the diagrammatic version of $TL_{n+1}(d)$ with Kauffman's diagrammatic generators $E_1, \ldots, E_n$ from Exercise 2.1.6, and we reserve $e_1, \ldots, e_n$ for the Jones projections in A_{n+1}. Fix a word $w \in TL_{n+1}(d)$ in $E_1, \ldots, E_n$. By Exercise 2.1.2, there are $x, y \in TL_n(d)$ such that $w = xE_n y$. We now calculate that

$$\begin{aligned}
(\operatorname{tr}_{n+1} \circ \Psi_{n+1})(w) &= \operatorname{tr}_{n+1}(\Psi_n(x)\Psi_{n+1}(E_n)\Psi_n(y)) = d \operatorname{tr}_{n+1}(\Psi_n(x) e_n \Psi_n(y))\\
&= d \operatorname{tr}_{n+1}(\Psi_n(y)\Psi_n(x) e_n) \underset{(4.2)}{=} d^{-1} \operatorname{tr}_{n+1}(\Psi_n(y)\Psi_n(x))\\
&= d^{-1}(\operatorname{tr}_n \circ \Psi_n)(yx) \underset{\text{(IH)}}{=} d^{-n-1} \operatorname{Tr}_n(yx) \underset{\text{(Ex. 2.1.12)}}{=} d^{-n-1} \operatorname{Tr}_{n+1}(yxE_n)\\
&= d^{-n-1} \operatorname{Tr}_{n+1}(xE_n y) = d^{-n-1} \operatorname{Tr}_{n+1}(w),
\end{aligned}$$

where the (IH) denotes where we used the induction hypothesis. Hence $\operatorname{tr}_{A_{n+1}} \circ \Psi_{n+1} = d^{-n-1} \operatorname{Tr}_{n+1}$ by linearity. We are finished. ∎

[[kernel of the form?]]

4.3 Subfactors and the generic range (∗∗)

In this section, we sketch the proof of the TL positivity theorem for the generic theorem using II_1 factors. We provide as parsimonious an exposition as possible.

Definition 4.3.1. A II_1 *factor* is an infinite dimensional von Neumann algebra M with a normal faithful tracial state tr_M such that $Z(M) = M' \cap M = \mathbb{C}1_M$. A II_1 *subfactor* is a unital inclusion $N \subset M$ of II_1 factors. The *index* $[M : N]$ of $N \subset M$ is the *von Neumann dimension* $\dim_{N-}(L^2(M, \operatorname{tr}_M))$.

Here are some of the most useful properties of II_1 (sub)factors:

- A II_1 factor has a *unique* trace.
- They have no minimal projections.
- Two projections $p, q \in M$ are *(Murray-von Neumann) equivalent*[1] if and only if they have the same trace.
- The trace of a projection can take any value in $[0, 1]$, and all values are obtained.

[1]Projections $p, q \in M$ are equivalent if there is a $v \in M$ such that $vv^* = p$ and $v^*v = q$.

- Given a finite index subfactor $N \subseteq M$, the unique trace on the basic construction $\langle M, e_N \rangle$ satisfies the Markov property $\operatorname{tr}_{\langle M, e_N\rangle}(xe_N) = d^{-2} \operatorname{tr}_M(x)$ for all $x \in M$ where $d^2 = [M : N]$.

- Iterating the Jones basic construction for a finite index II_1 subfactor $M_0 \subset M_1$ gives a Jones tower of II_1 factors $(M_n, \operatorname{tr}_n)$ with Jones projections $e_n \in M_{n+1}$ which satisfy the Temperley-Lieb-Jones relations (J1) – (J3).

Definition 4.3.2. The *hyperfinite* II_1 factor is $R := (\mathbb{C}[S_\infty])'' \subset B(\ell^2 S_\infty)$ where S_∞ is the group of finite permutations of $\mathbb{N}$. The II_1 factor R has the property that for every projection $p \in R \setminus \{0\}$, there is an isomorphism of von Neumann algebras $pRp \cong R$.

Recall that the map $t \mapsto t^{-1} + (1-t)^{-1}$ gives a homeomorphism $[1/2, 1) \to [2, \infty)$. Thus given $d \geq 2$, there is a projection $p \in R$ such that $\operatorname{tr}(p) = t$ and $d = t^{-1} + (1-t)^{-1}$. Picking N isomorphism $\theta : pRp \to (1-p)R(1-p)$, the subfactor

$$M_0 :=\cong \left\{ \begin{pmatrix} x & 0 \\ 0 & \theta(x) \end{pmatrix} \middle| x \in R \right\} \subset R =: M_1$$

has index $d^2 \geq 4$. Taking the Jones tower $(M_n, \operatorname{tr}_n)$, we get *injective* unital $*$-homomorphisms $\Psi_n : TL_n(d) \to M_n$ as in (4.3) for all $n \geq 0$. [[explain why it's injective?]]

Repeating the proof of Theorem 4.2.2 *mutatis mutandis* proves the following theorem.

Theorem 4.3.3 (Jones' TL positivity theorem, generic range)**.** *Suppose $d \geq 2$, and let $M_0 \subseteq M_1$ be a (hyperfinite) II_1 subfactor of index d^2. For all $n \geq 0$, $\operatorname{tr}_{A_n} \circ \Psi_n = d^{-n} \operatorname{Tr}_n$, where Tr_n was defined in (2.3) for $TL_n(d)$. Hence $\langle \cdot, \cdot \rangle_n$ is positive definite on $TL_n(d)$ for all $n \geq 0$.*

4.4 Temperley-Lieb forms a Markov tower

Exercise 4.4.1. Prove that the Temperley-Lieb algebras $TL_n^\dagger(d)$ where d is as in (4.1) with the usual Jones projections and Markov traces form a Markov tower $TL_\bullet^\dagger(d)$.

Exercise 4.4.2. Show that if $q = \exp(2\pi i/(2(n+1)))$ for $n \geq 2$ so $d = 2\cos(\pi/(n+1))$, then the principal graph $TL_\bullet^\dagger(d)$ is Λ_n from (4.4).

Chapter 5

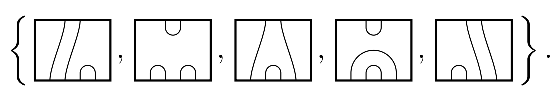
Linear categories

Recall that a category $\mathcal{C}$ has a collection of objects $a, b, c, \ldots$, and between any two objects, a collection of morphisms $\mathcal{C}(a \to b)$. Given three objects $a, b, c \in \mathcal{C}$ and two morphisms $f \in \mathcal{C}(a \to b)$ and $g \in \mathcal{C}(b \to c)$, there is a composite morphism $g \circ f \in \mathcal{C}(a \to c)$. Composition is required to be associative, and for every $a \in \mathcal{C}$, there is a distinguished identity morphism $\mathrm{id}_a \in \mathcal{C}(a \to a)$. The identities are required to satisfy $f \circ \mathrm{id}_a = f = \mathrm{id}_b \circ f$ for all $f \in \mathcal{C}(a \to b)$.

In this section, our categories are *linear*, i.e., for $a, b \in \mathcal{C}$, $\mathcal{C}(a \to b)$ is a complex vector space.

5.1 The TL category

Definition 5.1.1. The TL category $\mathcal{TL}(d)$ has objects the non-negative integers $n \geq 0$. The morphism space $\mathcal{TL}(d)(m \to n)$ is the $\mathbb{C}$ vector space whose standard basis is the set of string diagrams (up to isotopy) on a rectangle with m boundary points on the bottom and n boundary points on the top. For example, the basis for $\mathcal{TL}(d)(4 \to 2)$ is given by



Composition is defined similarly to the multiplication in $TLK(d)$; we stack boxes, smooth strings, and remove closed loops for multiplicative factors of d as in (2.1). For all $m, n \geq 0$, there is an antilinear map $\dagger : \mathcal{TL}(d)(m \to n) \to \mathcal{TL}(d)(n \to m)$ given by the antilinear extension of vertical reflection about a horizontal line as in (2.2).

Notice that we have a canonical unital $*$-algebra isomorphism $\mathcal{TL}(d)(n \to n) \cong TL_n(d)$.

Exercise 5.1.2. Show that

$$\dim(\mathcal{TL}(d)(m \to n)) = \begin{cases} 0 & \text{if } m \not\equiv n \mod 2 \\ C_{\frac{m+n}{2}} & \text{if } m \equiv n \mod 2 \end{cases}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number.

Definition 5.1.3. For $x, y \in \mathcal{TL}(d)(m \to n)$, we define $\langle x, y \rangle_{m \to n} := \mathrm{Tr}_m(y^\dagger \circ x)$, as $y^\dagger \circ x \in \mathcal{TL}(d)(m \to m) \cong TL_m(d)$. The set of *negligibles* of the form $\langle \cdot\, , \cdot \rangle_{m \to n}$ is given by

$$\mathcal{N}_{m \to n} := \{x \in \mathcal{TL}(d)(m \to n) | \langle x, x \rangle_{m \to n} = 0\}.$$

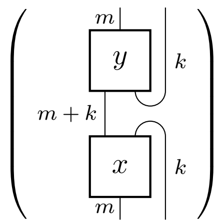
Exercise 5.1.4. Show that for all $x \in \mathcal{TL}(d)(m \to n)$, $y \in \mathcal{TL}(d)(n \to p)$, and $z \in \mathcal{TL}(d)(m \to p)$, we have

$$\langle y, z \circ x^\dagger \rangle_{n \to p} = \langle y \circ x, z\rangle_{m \to p} = \langle x, y^\dagger \circ z\rangle_{m \to n}.$$

Thus the forms $\langle \cdot\,, \cdot \rangle_{m \to n}$ equip the category $\mathcal{TL}(d)$ with a 2-*Hilbert space* structure [Bae97].

Exercise 5.1.5. Show that if $n = m + 2k$ so $m + k = (m+n)/2$, then for all $x, y \in \mathcal{TL}(d)(m \to n)$, we have

$$\langle x, y\rangle_{m \to n} = \mathrm{Tr}_{m+k}$$



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Exercise 5.1.6. Prove that $\langle \cdot\,, \cdot \rangle_{m \to n}$ is positive for all $m, n \geq 0$ if and only if d is as in (4.1), i.e.,

$$d \in \{2\cos(\pi/k) | k \geq 3\} \cup [2, \infty).$$

For the remainder of this section, we assume that d is as in (4.1) so that $\langle \cdot\,, \cdot \rangle_{m \to n}$ is positive for all $m, n \geq 0$ by Exercise 5.1.6.

Exercise 5.1.7. Show that for *any* positive sesquilinear form $\langle \cdot\,, \cdot \rangle$ on a vector space V, the Cauchy-Schwarz inequality holds:

$$|\langle u, v\rangle| \leq \langle u, u\rangle \cdot \langle v, v\rangle. \tag{5.1}$$

Proposition 5.1.8. *If $x \in \mathcal{TL}(d)(\ell \to m)$, $y \in \mathcal{N}_{m \to n}$, and $z \in \mathcal{TL}(d)(n \to p)$, then $y^\dagger \in \mathcal{N}_{n \to m}$ and $z \circ y \circ x \in \mathcal{N}_{\ell \to p}$.*

Proof. First, we calculate

$$\langle y^\dagger, y^\dagger \rangle_{n \to m} = \mathrm{Tr}_n(y \circ y^\dagger) \underset{\text{(Ex. 5.1.4)}}{=} \mathrm{Tr}_m(y^\dagger \circ y) = \langle y, y\rangle_{m \to n} = 0,$$

so $y^\dagger \in \mathcal{N}_{n \to m}$. Next, we calculate that

$$\begin{aligned} 0 \leq \langle z \circ y \circ x, z \circ y \circ x\rangle_{\ell \to p} &\underset{\text{(Ex. 5.1.4)}}{=} \langle z^\dagger \circ z \circ y \circ x \circ x^\dagger, y\rangle_{m \to n} \\ &\underset{(5.1)}{\leq} \langle z^\dagger \circ z \circ y \circ x \circ x^\dagger, z^\dagger \circ z \circ y \circ x \circ x^\dagger \rangle_{m \to n} \cdot \langle y, y\rangle_{m \to n} = 0. \end{aligned}$$

Hence $z \circ y \circ x \in \mathcal{N}_{\ell \to p}$. $\square$

Definition 5.1.9. We define $TL_n^\dagger(d) := TL_n(d)/\mathcal{N}_{n \to n}$. Notice that $\mathrm{tr}_n := d^{-n}\,\mathrm{Tr}_n$ is a faithful tracial state on $TL_n^\dagger(d)$, which is a finite dimensional C*/W*/multimatrix algebra by Exercise 3.1.35.

Exercise 5.1.10 $(*)$**.** Compute how $TL_n^\dagger(d)$ decomposes as a sum of multimatrix algebras. Then compute the Bratteli diagram for the tower of algebras $(TL_n^\dagger(d))_{n \geq 0}$.

Definition 5.1.11. Let $\mathcal{TL}^\dagger(d)$ be the category whose objects are the non-negative integers $n \geq 0$ and whose morphism spaces $\mathcal{TL}^\dagger(d)(m \to n)$ are the spaces $\mathcal{TL}(d)(m \to n)/\mathcal{N}_{m \to n}$. We define composition as follows. For $x + \mathcal{N}_{m \to n} \in \mathcal{TL}^\dagger(d)(m \to n)$ and $y + \mathcal{N}_{n \to p} \in \mathcal{TL}^\dagger(d)(n \to p)$, we define $(y + \mathcal{N}_{n \to p}) \circ (x + \mathcal{N}_{m \to n}) := y \circ x + \mathcal{N}_{m \to p}$. For $x + \mathcal{N}_{m \to n} \in \mathcal{TL}^\dagger(d)(m \to n)$, we define $(x + \mathcal{N}_{m \to n})^\dagger := x^\dagger + \mathcal{N}_{n \to m} \in \mathcal{TL}^\dagger(d)(n \to m)$.

Exercise 5.1.12. Use Proposition 5.1.8 to show that $\mathcal{TL}^\dagger(d)$ is a well-defined linear $\dagger$-category.

5.2 C* categories

Definition 5.2.1. An *involution* $\dagger$ on a (linear) category $\mathcal{C}$ is a conjugate-linear map $\dagger : \mathcal{C}(a \to b) \to \mathcal{C}(b \to a)$ for all $a, b \in \mathcal{C}$ such that

- For all $x \in \mathcal{C}(a \to b)$ and $y \in \mathcal{C}(b \to c)$, $(y \circ x)^\dagger = x^\dagger \circ y^\dagger$, and
- For all $x \in \mathcal{C}(a \to b)$, $x^{\dagger\dagger} = x$.

The pair $(\mathcal{C}, \dagger)$ is called a (linear) *dagger category* or a $\dagger$-*category*.

Exercise 5.2.2. Prove that an involution $\dagger$ on a linear category $\mathcal{C}$ satisfies $\mathrm{id}_a^\dagger = \mathrm{id}_a$ for all $a \in \mathcal{C}$.

Definition 5.2.3. A dagger category $(\mathcal{C}, \dagger)$ is called a C* *category* if

(C*) For all $a, b \in \mathcal{C}$, the *linking algebra* [GLR85]

$$\mathcal{L}(a,b) := \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(b \to a) \\ \mathcal{C}(a \to b) & \mathcal{C}(b \to b) \end{pmatrix}$$

is a C* algebra where multiplication is given by

$$\begin{pmatrix} w_1 & x_1 \\ y_1 & z_1 \end{pmatrix} \cdot \begin{pmatrix} w_2 & x_2 \\ y_2 & z_2 \end{pmatrix} := \begin{pmatrix} w_1 \circ w_2 + x_1 \circ y_2 & w_1 \circ x_2 + x_1 \circ z_2 \\ y_1 \circ w_2 + z_1 \circ y_2 & y_1 \circ x_2 + z_1 \circ z_2 \end{pmatrix}$$

and adjoint is the $\dagger$-tranpose.[1]

Exercise 5.2.4. Show that if $(\mathcal{C}, \dagger)$ is a C* category, then for any $c_1, \ldots, c_n \in \mathcal{C}$, the linking algebra

$$\mathcal{L}(c_1, \ldots, c_n) := \bigoplus_{i=1}^{n} \mathcal{C}(c_i \to c_j)$$

with the obvious $\dagger$-algebra structure is a unital C* algebra.

Proposition 5.2.5. *When $(\mathcal{C}, \dagger)$ is a dagger category with finite dimensional hom spaces, then $(\mathcal{C}, \dagger)$ is C* if and only if the following two conditions hold:*

(C*1) *For all $a \in \mathcal{C}$, $\mathcal{C}(a \to a)$ is a unital C* algebra with involution $\dagger$.*

(C*2) *For all $x \in \mathcal{C}(a \to b)$, there is a $y \in \mathcal{C}(a \to a)$ such that $x^\dagger \circ x = y^\dagger \circ y$.*

Proof. It is clear that (C*) implies both (C*1) and (C*2). Suppose (C*1) and (C*2) hold. Since we assumed the endomorphism spaces are finite dimensional, by (C*1) and Exercise 3.1.21, there are faithful tracial states tr_a and tr_b on $\mathcal{C}(a \to a)$ and $\mathcal{C}(a \to b)$ respectively. We claim that

$$\mathrm{tr}_{a,b} := \frac{1}{2}\mathrm{tr}_a + \frac{1}{2}\mathrm{tr}_b$$

is a faithful tracial state on the linking algebra $\mathcal{L}(a, b)$. We calculate

$$\begin{aligned} \mathrm{tr}_{a,b}\left(\begin{pmatrix} w^\dagger & y^\dagger \\ x^\dagger & z^\dagger \end{pmatrix} \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix}\right) &= \mathrm{Tr}_{a,b}\left(\begin{pmatrix} w^\dagger \circ w + y^\dagger \circ y & w^\dagger \circ x + y^\dagger \circ z \\ x^\dagger \circ w + z^\dagger \circ y & x^\dagger \circ x + z^\dagger \circ z \end{pmatrix}\right) \\ &= \mathrm{tr}_a(w^\dagger \circ w) + \mathrm{tr}_a(y^\dagger \circ y) + \mathrm{tr}_b(x^\dagger \circ x) + \mathrm{tr}_b(z^\dagger \circ z) \end{aligned} \quad (5.2)$$

[1] Being a C* algebra is a *property* of a unital complex $*$-algebra, and not extra structure. When the hom spaces of $\mathcal{C}$ are finite dimensional, this is exactly the condition that $\mathcal{L}(a, b)$ is a multimatrix algebra under its involution by Exercises 3.1.21 and 3.1.26.

Now for $x \in \mathcal{C}(b \to a)$ and $y \in \mathcal{C}(a \to b)$, there are $u \in \mathcal{C}(b \to b)$ and $v \in \mathcal{C}(a \to a)$ such that $x^\dagger \circ x = u^\dagger \circ u$ and $y^\dagger \circ y = v^\dagger \circ v$ by (C*2). Hence (5.2) is equal to

$$\mathrm{tr}_a(w^\dagger \circ w) + \mathrm{tr}_a(v^\dagger \circ v) + \mathrm{tr}_b(u^\dagger \circ u) + \mathrm{tr}_b(z^\dagger \circ z) \geq 0,$$

and this quantity equals zero if and only if $w = v = 0$ and $u = z = 0$. Notice that $u = 0$ if and only if $x = 0$ and $v = 0$ if and only if $y = 0$. Hence $\mathrm{tr}_{a,b}$ is a faithful tracial state, and $\mathcal{L}(a, b)$ is a C* algebra by Exercise 3.1.21. □

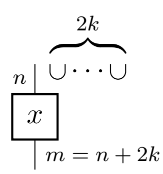
Proposition 5.2.6. *When d is as in (4.1), the $\dagger$-category $\mathcal{TL}^\dagger(d)$ is* C*.

Proof. Note that $d^{-n}\,\mathrm{Tr}_n$ is a faithful tracial state on the finite dimensional unital $\dagger$-algebra $\mathcal{TL}^\dagger(d)(n \to n)$, which is a C* algebra by Exercise 3.1.21; hence (C*1) holds. Suppose now that $x + \mathcal{N}_{m\to n} \in \mathcal{TL}^\dagger(d)(m \to n)$, where we may assume $x \neq 0$ so $m \equiv n \mod 2$. Suppose $z + \mathcal{N}_{m\to m} \in \mathcal{TL}^\dagger(d)(m \to m)$ is arbitrary. We calculate the GNS-inner product using the faithful tracial state $d^{-m}\,\mathrm{Tr}_m$ on $\mathcal{TL}^\dagger(d)(m \to m) = TL_m^\dagger$:

$$\langle x^\dagger \circ x \circ z + \mathcal{N}_{m\to m}, z + \mathcal{N}_{m\to m}\rangle = \langle x^\dagger \circ x \circ z, z\rangle_{m\to m} \underset{\text{Ex. 5.1.4}}{=} \langle x \circ z, x \circ z\rangle_{m\to n} \geq 0.$$

We conclude that $x^\dagger \circ x + \mathcal{N}_{m\to m}$ is a positive operator in $B(\mathcal{TL}^\dagger(d)(m \to m), d^{-m}\,\mathrm{Tr}_m)$, and is thus of the form $y^\dagger \circ y + \mathcal{N}_{m\to m}$ for some $y \in \mathcal{TL}_{m\to m}$ by the existence of a square root for a positive operator in a C* algebra together with (C*1). □

Remark 5.2.7. In the event that $m = n + 2k$ for $k \geq 0$, notice that we can add k cups to the top of $x \in \mathcal{TL}(d)(m \to n)$ to obtain a morphism in $\mathcal{TL}(d)(m \to m)$. Thus setting

$y := d^{-k}$


we have $y^\dagger \circ y = x^\dagger \circ x$.

5.3 Functors and natural transformations

Just as linear maps are the functions between vector spaces which preserve vector space structure, functors are maps between categories which preserve categorical structure.

Definition 5.3.1. Suppose $\mathcal{C}$ and $\mathcal{D}$ are two categories. A *functor*a $F : \mathcal{C} \to \mathcal{D}$ consists of

- an assignment of an object $F(c) \in \mathcal{D}$ to each object $c \in \mathcal{C}$, and
- an assignment of a morphism $F(f) \in \mathcal{D}(F(a) \to F(b))$ for each $f \in \mathcal{C}(a \to b)$ such that $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ for all $a \in \mathcal{C}$ and $F(g \circ f) = F(g) \circ F(f)$ for composable morphisms f, g in $\mathcal{C}$.

Definition 5.3.2. If $\mathcal{C}$ and $\mathcal{D}$ are linear categories, a functor $F : \mathcal{C} \to \mathcal{D}$ is called *linear* if for all $a, b \in \mathcal{C}$, the map $\mathcal{C}(a \to b) \to \mathcal{D}(F(a) \to F(b))$ given by $f \mapsto F(f)$ is a linear transformation.

Exercise 5.3.3. Suppose $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are (linear) functors. Define the composite (linear) functor $G \circ F : \mathcal{C} \to \mathcal{E}$. Then show that composition of functors is associative.

Remark 5.3.4. In Exercise 5.3.3, we denote composition of functors as composition of functions, which is read *right to left*. We will see later in Exercises 5.3.6 and 5.3.7 that it is more advantageous to denote the composite as $F \otimes G : \mathcal{C} \to \mathcal{E}$ reading *left to right*.

Definition 5.3.5. Suppose $F, G : \mathcal{C} \to \mathcal{D}$ are two functors. A *natural transformation* $\theta : F \Rightarrow G$ consists of a morphism $\theta_c \in \mathcal{D}(F(c) \to G(c))$ for each $c \in \mathcal{C}$ such that for every $a, b \in \mathcal{C}$ and every $f \in \mathcal{C}(a \to b)$, the following diagram commutes:

$$\begin{tikzcd} F(a) \arrow[r, "F(f)"] \arrow[d, "\theta_a"] & F(b) \arrow[d, "\theta_b"] \\ G(a) \arrow[r, "G(f)"] & G(b). \end{tikzcd} \tag{5.3}$$

Exercise 5.3.6. Suppose $F, G, H : \mathcal{C} \to \mathcal{D}$ are functors and $\theta : F \Rightarrow G$ and $\phi : G \Rightarrow H$ are natural transformations. Show that $\phi \circ \theta : F \Rightarrow H$ given by $(\phi \circ \theta)_c := \phi_c \circ \theta_c$ gives a well-defined natural transformation. We call $\phi \circ \theta$ the *vertical composite* of ϕ and θ.

Exercise 5.3.7. Suppose $F, H : \mathcal{C} \to \mathcal{D}$ and $G, K : \mathcal{D} \to \mathcal{E}$ are functors and $\theta : F \Rightarrow H$ and $\phi : G \Rightarrow K$ are natural transformations. Show that $\theta \otimes \phi : F \otimes G \Rightarrow H \otimes K$ given by

$$(\theta \otimes \phi)_c := \phi_{H(c)} \circ G(\theta_c) = K(\theta_c) \circ \phi_{F(c)}$$

gives a well-defined natural transformation. We call $\theta \circ \phi$ the *horizontal composite* of θ and ϕ.

Exercise 5.3.8. Show that $(\phi_1 \otimes \phi_2) \circ (\theta_1 \otimes \theta_2) = (\phi_1 \circ \theta_1) \otimes (\phi_2 \circ \theta_2)$ whenever these expressions type-check.

Exercise 5.3.9. Show that categories, functors, and natural transformations forms a *2-category*.

5.3.1 Adjoint functors

[[todo]]

5.3.2 Linear †-functors and bounded natural transformations for C* categories

[[todo]]

5.4 Cauchy completion

In this section, we discuss the notions of direct sums, idempotent completion, and Cauchy completion for linear categories. We also discuss the notions of finite orthogonal direct sums, projection completion, and C* Cauchy completion for C* categories.

5.4.1 Universal properties of coproduct, product, biproduct, and direct sum

Suppose $\mathcal{C}$ is a linear category.

Definition 5.4.1. Given a collection of objects $\{c_i\}_{i\in I}$, an object $\coprod_{i\in I} c_i$ together with morphisms $\iota_j : c_j \to \coprod_{i\in I} c_i$ for all $j \in I$ is called the *coproduct* of $\{c_i\}_{i\in I}$ if for any $d \in \mathcal{C}$ and morphisms

$j_i : c_i \to d$ for $i \in I$, there is a unique morphism $f : \coprod_{i\in I} c_i \to d$ such that the following diagram commutes for all $i \in I$:

$$\begin{tikzcd} c_i \arrow[r, "\iota_i"] \arrow[dr, "j_i"'] & \coprod_{i\in I} c_i \arrow[d, dashed, "\exists! f"] \\ & d \end{tikzcd} \tag{5.4}$$

Exercise 5.4.2. Why do we call $(\coprod_{i\in I} c_i, (\iota_j)_{j\in I})$ *the* coproduct of $\{c_i\}_{i\in I}$?

Exercise 5.4.3. Reverse the arrows in Definition 5.4.1 to give the definition of what it means for $(\prod_{i\in I} c_i, (\pi_j)_{j\in I})$ to be the *product* of $\{c_i\}_{i\in I}$. Why do we call it *the* product?

Definition 5.4.4. Given a *finite* collection of objects $c_1, \ldots, c_n$, an object $\bigoplus_{i=1}^n c_i$ with morphisms $\iota_j : c_j \to \bigoplus_{i=1}^n$ and $\pi_j : \bigoplus_{i=1}^n c_i \to c_j$ is called the *direct sum* of $c_1, \ldots, c_n$ if

($\oplus$1) $\pi_i \circ \iota_j = \delta_{i=j} \operatorname{id}_{c_j}$ for all $i, j = 1, \ldots, n$, and

($\oplus$2) $\sum_{j=1}^n \iota_j \circ \pi_j = \operatorname{id}_{\bigoplus_{i=1}^n c_i}$.

Exercise 5.4.5. Suppose that $(\bigoplus_{i=1}^n c_i, (\iota_j)_{j=1}^n, (\pi_j)_{j=1}^n)$ is the direct sum of $c_1, \ldots, c_n$. Show that $(\bigoplus_{i=1}^n c_i, (\iota_j)_{j=1}^n)$ is the coproduct of $c_1, \ldots, c_n$ and $(\bigoplus_{i=1}^n c_i, (\pi_j)_{j=1}^n)$ is the product of $c_1, \ldots, c_n$.

Exercise 5.4.6. Given a finite collection of objects $c_1, \ldots, c_n$, an object $\boxplus_{i=1}^n c_i$ with morphisms $\iota_j : c_j \to \boxplus_{i=1}^n c_i$ and $\pi_j : \boxplus_{i=1}^n c_i \to c_j$ is called the *biproduct* of $c_1, \ldots, c_n$ if $(\boxplus_{i=1}^n c_i, (\iota_j)_{j=1}^n)$ is the coproduct and $(\boxplus_{i=1}^n c_i, (\pi_j)_{j=1}^n)$ is the product of $c_1, \ldots, c_n$ respectively.

- Let V be a finite dimensional vector space with basis $\{\pi_1, \ldots, \pi_n\}$, and define $\iota_j : \mathbb{C} \to V$ by $\lambda \mapsto \lambda v_j$. Fix a basis $\{\pi_1, \ldots, \pi_n\}$ for $V^\vee = \operatorname{Hom}(V \to \mathbb{C})$. Show that $(\boxplus_{i=1}^n c_i, (\iota_j)_{j=1}^n, (\pi_j)_{j=1}^n)$ is the biproduct of $c_1, \ldots, c_n$.
- Show that $(\boxplus_{i=1}^n c_i, (\iota_j)_{j=1}^n, (\pi_j)_{j=1}^n)$ is the direct sum if and only if $\{\pi_1, \ldots, \pi_n\}$ is the dual basis of $\{\pi_1, \ldots, \pi_n\}$.
- Show that for every biproduct $(\boxplus_{i=1}^n c_i, (\iota_j)_{j=1}^n, (\pi_j)_{j=1}^n)$, we can find $\iota_j' : c_j \to \boxplus_{i=1}^n c_i$ such that $(\boxplus_{i=1}^n c_i, (\iota_j')_{j=1}^n, (\pi_j)_{j=1}^n)$ is the direct sum.
- Instead of changing the ι_j, change the π_j to get a direct sum.

Exercise 5.4.7. Suppose that $(\bigoplus_{i=1}^n c_i, (\iota_j)_{j=1}^n, (\pi_j)_{j=1}^n)$ is the direct sum of $c_1, \ldots, c_n$. Find a canonical isomorphism

$$\mathcal{C}\left(\bigoplus_{i=1}^n c_i \to \bigoplus_{j=1}^n c_j\right) \cong \bigoplus_{i,j=1}^n \mathcal{C}(c_i \to c_j), \tag{5.5}$$

where the direct sum on the right hand side is in the category of $\mathbb{C}$ vector spaces. The right hand side carries the obvious matrix-multiplication composition.
Hint: write an element f of the left hand side of (5.5) as $\sum_{i,j=1}^n \pi_j \circ (\iota_j \circ f \circ \pi_i) \circ \iota_i$. Notice that $(f_{ij} := \iota_j \circ f \circ \pi_i)$ defines an element of the right hand side of (5.5).

Exercise 5.4.8. Suppose $\mathcal{C}, \mathcal{D}$ are linear categories and $c, a_1, \ldots, a_n \in \mathcal{C}$. Use Exercise 5.4.7 to show that the property $c \cong \bigoplus_{i=1}^n a_i$ is preserved by all linear functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$.

Exercise 5.4.9. Find a linear functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ such that $c \cong \coprod_{i=1}^n a_i$, but $\mathcal{F}(c) \ncong \coprod_{i=1}^n \mathcal{F}(a_i)$. What additional condition could you impose on $\mathcal{F}$ to make sure that $c \cong \coprod_{i=1}^n a_i$ is preserved by all linear functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$?

Exercise 5.4.10. Repeat Exercise 5.4.9 for products.

Definition 5.4.11. Given a linear category $\mathcal{C}$, the *additive envelope* of $\mathcal{C}$ is the linear category $\mathsf{Add}(\mathcal{C})$ whose objects are *formal finite direct sums* $\bigoplus_{i=1}^n a_i$ for $a_1, \ldots, a_n \in \mathcal{C}$, and whose morphism sets are given by matrices of operators:

$$\mathsf{Add}(\mathcal{C})\left(\bigoplus_{j=1}^n b_j \to \bigoplus_{i=1}^m a_i\right) := \{(x_{ij}) | x_{ij} \in \mathcal{C}(b_j \to a_i)\} \tag{5.6}$$

where composition is given by $(x_{ij}) \circ (y_{jk}) := (\sum_j x_{ij} \circ y_{jk})$.

Observe that $c \mapsto (c)$ for $c \in \mathcal{C}$ and $x \mapsto (x)$ for $x \in \mathcal{C}(a \to b)$ is a fully faithful linear functor $\mathcal{C} \hookrightarrow \mathsf{Add}(\mathcal{C})$.

Exercise 5.4.12. Suppose $\mathcal{C}, \mathcal{D}$ are linear categories, and assume $\mathcal{D}$ admits all finite direct sums. Show that any linear functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ factors uniquely through $\mathsf{Add}(\mathcal{C})$, i.e., there is a unique linear functor $\mathsf{Add}(\mathcal{F}) : \mathsf{Add}(\mathcal{C}) \to \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathsf{Add}(\mathcal{C}) \arrow[dr, "\exists! \mathsf{Add}(\mathcal{F})"] & \\ \mathcal{C} \arrow[u, hook] \arrow[r, "\mathcal{F}"] & \mathcal{D} \end{tikzcd} \tag{5.7}$$

Exercise 5.4.13. Use Exercise 5.4.12 to show that if $\mathcal{C}$ admits all finite direct sums, then $\mathcal{C}$ is equivalent to $\mathsf{Add}(\mathcal{C})$.

[[say something about infinite direct sums?]]

5.4.2 Finite orthogonal direct sums in C* categories

Let $\mathcal{C}$ be a C* category. In this section, we restict our attention to *finite* orthogonal direct sums to avoid talking about convergence.

Definition 5.4.14. Given $\{c_1, \ldots, c_n\} \subset \mathcal{C}$, an object $\bigoplus_{i=1}^n c_i$ together with morphisms $\iota_j : c_j \to \bigoplus_{i \in I} c_i$ for $1 \leq j \leq n$ is called the *orthogonal direct sum* of $c_1, \ldots, c_n$ if $(\bigoplus_{i=1}^n c_i, (\iota_j)_{j=1}^n, (\iota_j^\dagger)_{j=1}^n)$ is the direct sum of $c_1, \ldots, c_n$, which holds if and only if the following two conditions hold:

($\perp\oplus$1) $\iota_j^\dagger \circ \iota_j = \mathrm{id}_{c_j}$ for all $1 \leq j \leq n$, i.e., each ι_j is an *isometry*, and

($\perp\oplus$2) $\sum_{j=1}^n \iota_j \circ \iota_j^\dagger = \mathrm{id}_{\bigoplus_{i=1}^n c_i}$, i.e., the $\iota_j \circ \iota_j^\dagger$ are mutually orthogonal projections in $\mathrm{End}(\bigoplus_{i=1}^n c_i)$ which sum to $\mathrm{id}_{\bigoplus_{i=1}^n c_i}$.

Exercise 5.4.15. Suppose $\mathcal{C}$ is either a W* category or a C* category with finite dimensional morphism spaces. Suppose $(\bigoplus_{i=1}^n c_i, (\iota_j)_{j=1}^n, (\pi_j)_{j=1}^n)$ is the direct sum of $c_1, \ldots, c_n$. Find isometries $v_j : c_j \to \bigoplus_{i=1}^n c_i$ such that $(\bigoplus_{i=1}^n c_i, (v_j)_{j=1}^n)$ is the orthogonal direct sum of $c_1, \ldots, c_n$.

Exercise 5.4.16. Prove that any two orthogonal direct sums of $c_1, \ldots, c_n$ are uniquely unitarily isomorphic.

Exercise 5.4.17. Suppose that $(\bigoplus_{i=1}^n c_i, (\iota_j)_{j=1}^n)$ is the orthogonal direct sum of $c_1, \ldots, c_n$. Find a canonical unital †-isomorphism

$$\mathcal{C}\left(\bigoplus_{i=1}^n c_i \to \bigoplus_{j=1}^n c_j\right) \cong \bigoplus_{i,j=1}^n \mathcal{C}(c_i \to c_j). \tag{5.8}$$

Exercise 5.4.18. Suppose $\mathcal{C}, \mathcal{D}$ are C* categories and $c, a_1, \dots, a_n \in \mathcal{C}$. Show that the property that c is *unitarily* isomorphic to $\bigoplus_{i=1}^n a_i$ is preserved by all †-functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$.

Definition 5.4.19. Given a C* category $(\mathcal{C}, \dagger)$, we define $\mathsf{Add}^\dagger(\mathcal{C})$ to be the dagger category whose objects are finite orthogonal direct sums, whose morphism spaces and compostion are defined analogously to (5.6), and with dagger structure $(x_{ij})^\dagger := (x_{ji}^\dagger)$. There is an analogous obvious †-functor $\mathcal{C} \hookrightarrow \mathsf{Add}^\dagger(\mathcal{C})$.

Exercise 5.4.20 (Roberts' 2×2 trick [GLR85])**.** Show that if $(\mathcal{C}, \dagger)$ is a dagger category with orthogonal direct sums, then $(\mathcal{C}, \dagger)$ is a C* category if and only if for all $a \in \mathcal{C}$, $\mathcal{C}(a \to a)$ is a unital C* algebra with involution †.

Exercise 5.4.21. Suppose $(\mathcal{C}, \dagger)$ is C*. Use Exercise 5.2.4 and (5.8) to show that $\mathsf{Add}^\dagger(\mathcal{C})$ is C*.

Exercise 5.4.22. Formulate and prove the universal property for $\mathsf{Add}^\dagger(\mathcal{C})$ analogous to (5.7) for †-functors from $\mathcal{C}$ to a C* category $\mathcal{D}$ which admits finite orthogonal direct sums.

Exercise 5.4.23. Use Exercise 5.4.22 to show that if $\mathcal{C}$ admits finite orthogonal direct sums, then $\mathcal{C}$ is dagger equivalent to $\mathsf{Add}^\dagger(\mathcal{C})$.

Exercise 5.4.24 ($**$)**.** Define the notion of an infinite orthogonal direct sum in a W* category.

5.4.3 Idempotent and Cauchy completions

Let $\mathcal{C}$ be a linear category.

Definition 5.4.25. An *idempotent* in $\mathcal{C}$ is a pair (c, e) where $c \in \mathcal{C}$ and $e \in \mathcal{C}(c \to c)$ such that $e \circ e = e$. A *splitting* for an idempotent (c, e) is an triple (a, r, s) where $a \in \mathcal{C}$, $r \in \mathcal{C}(c \to a)$ called a *retract*, and $s \in \mathcal{C}(a \to c)$ such that $s \circ r = e$ and $r \circ s = \mathrm{id}_a$. A linear category $\mathcal{C}$ is called *idempotent complete* if every idempotent admits a splitting.

Exercise 5.4.26. Suppose $(a, r_a, s_a), (b, r_b, s_b)$ are two splittings of (c, e). Show that there is a unique isomorphism $f : a \to b$ which is compatible with (r_a, s_a) and (r_b, s_b).

Exercise 5.4.27. Suppose $\mathcal{C}, \mathcal{D}$ are linear categories.

- Show that the property that the idempotent (c, e) admits a splitting is preserved by all functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$.
- Show that the property of being idempotent complete is preserved by all functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$.

Definition 5.4.28. The *idempotent completion* $\mathsf{Idem}(\mathcal{C})$ is the linear category whose objects are pairs (c, e) where $c \in \mathcal{C}$ and $e \in \mathcal{C}(c \to c)$ is an idempotent. The morphism spaces are given by

$$\mathcal{C}((a, e) \to (b, f)) := \{x \in \mathcal{C}(a \to b) | x = f \circ x \circ e\}.$$

Observe that $\mathcal{C}((a, e) \to (b, f)) \subseteq \mathcal{C}(a \to b)$ is a linear subspace, and if $x \in \mathcal{C}((a, e) \to (b, f))$, then $x = x \circ e = f \circ x$. Composition of morphisms is exactly composition in $\mathcal{C}$, i.e., if $x \in \mathcal{C}((a, e) \to (b, f))$ and $y \in \mathcal{C}((b, f) \to (c, g))$, then $y \circ x \in \mathcal{C}((a, e) \to (c, g))$.

There is an obvious faithful inclusion functor $\mathcal{C} \hookrightarrow \mathsf{Idem}(\mathcal{C})$ given by $c \mapsto (c, \mathrm{id}_c)$.

Exercise 5.4.29. Show that $\mathsf{Idem}(\mathcal{C})$ is idempotent complete.

Exercise 5.4.30. Suppose $\mathcal{C}$ is a linear category and $\mathcal{D}$ is an idempotent complete linear category. Show that any linear functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ factors uniquely through $\mathsf{Idem}(\mathcal{C})$. That is, show there is a linear functor $\mathsf{Idem}(\mathcal{F}) : \mathsf{Idem}(\mathcal{C}) \to \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathsf{Add}(\mathcal{C}) \arrow[dr, "\exists!\mathsf{Idem}(\mathcal{F})"] & \\ \mathcal{C} \arrow[u, hook] \arrow[r, "\mathcal{F}"] & \mathcal{D} \end{tikzcd} \tag{5.9}$$

and prove the functor $\mathsf{Idem}(\mathcal{F})$ is unique up to unique natural isomorphism (using Exercise 5.4.26).

Exercise 5.4.31. Use Exercise 5.4.30 to show that if $\mathcal{C}$ is idempotent complete, then $\mathcal{C}$ is equivalent to $\mathsf{Idem}(\mathcal{C})$.

Definition 5.4.32. A linear category $\mathcal{C}$ is called *Cauchy complete* if it admits all finite direct sums and it is idempotent complete. The *Cauchy completion* of a linear category $\mathcal{C}$ is $\overline{\mathcal{C}} := \mathsf{Idem}(\mathsf{Add}(\mathcal{C}))$. Observe that $\overline{\mathcal{C}}$ is Cauchy complete by Exercise 5.4.29, and $c \mapsto (c, \mathrm{id}_c)$ gives a faithful linear functor $\mathcal{C} \hookrightarrow \overline{\mathcal{C}}$.

Exercise 5.4.33. Suppose $\mathcal{C}$ is a linear category and $\mathcal{D}$ is a Cauchy complete linear category. Show that any linear functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ factors uniquely through $\overline{\mathcal{C}}$, i.e., there is a unique linear functor $\overline{\mathcal{F}} : \overline{\mathcal{C}} \to \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd} \overline{\mathcal{C}} \arrow[dr, "\exists!\overline{\mathcal{F}}"] & \\ \mathcal{C} \arrow[u, hook] \arrow[r, "\mathcal{F}"] & \mathcal{D} \end{tikzcd} \tag{5.10}$$

Exercise 5.4.34. Use Exercise 5.4.33 to show that if $\mathcal{C}$ is Cauchy complete, then $\mathcal{C}$ is equivalent to $\overline{\mathcal{C}}$.

Exercise 5.4.35. Find an example of a linear category $\mathcal{C}$ such that $\mathsf{Add}(\mathsf{Idem}(\mathcal{C}))$ is not equivalent to $\mathsf{Idem}(\mathsf{Add}(\mathcal{C}))$.
Hint: Use an algebra without non-trivial idempotents with projective modules which are not free, e.g., $C(S^2)$.

Example 5.4.36. The Temperley-Lieb category $\mathcal{TL}(d)$ is not idempotent complete. For example, $f^{(2)}$ is an idempotent which does not split unless $d = 1$.

Proposition 5.4.37. *Suppose* $[1], \ldots, [k+1] \neq 0$*, so that* $f^{(k)}$ *exists and has non-zero trace. For every idempotent* $e \in TL_k(d)$ *such that* $e(TL_k(d))e = \mathbb{C}e$*, there is an* $n \in \{1 \ldots, k\}$ *such that* $(k, e) \cong (n, f^{(n)})$ *in* $\mathsf{Idem}(\mathcal{TL}(d))$.

Proof. [[todo]] $\square$

Exercise 5.4.38 (**)**.** A colimit in a (linear) category is called *absolute* if it is preserved by every (linear) functor.

- Show that the absolute colimits in an ordinary category are split idempotents.
- Show that the absolute colimits in a linear category are split idempotents and direct sums.

5.4.4 Projection and C* Cauchy completions

Suppose now that $\mathcal{C}$ is a C* category.

Definition 5.4.39. A *projection* in $\mathcal{C}$ is a pair (c,p) where $c \in \mathcal{C}$ and $p \in \mathcal{C}(c \to c)$ such that $p \circ p = p = p^\dagger$. An *orthogonal splitting* for a projection (c,p) is a pair (a,v) where $a \in \mathcal{C}$, $v \in \mathcal{C}(a \to c)$ is an isometry such that $v^\dagger \circ v = \mathrm{id}_a$ and $v \circ v^\dagger = p$. A C* category $\mathcal{C}$ is called *projection complete* if every projection admits an orthogonal splitting.

Exercise 5.4.40. Suppose $(a, b_a), (b, v_b)$ are two orthogonal splittings of (c,p). Show that there is a unique unitary isomorphism $u : a \to b$ which is compatible with v_a, v_b.

Exercise 5.4.41. Suppose $\mathcal{C}, \mathcal{D}$ are C* categories.

- Show that the property that the projection (c,p) admits an orthogonal splitting is preserved by all †-functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$.
- Show that the property of being projection complete is preserved by all †-functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$.

Definition 5.4.42. The *projection completion* $\mathsf{Proj}(\mathcal{C})$ is the †-category whose objects are projections (c,p) in $\mathcal{C}$, and whose morphism spaces are

$$\mathcal{C}((a,p) \to (b,q)) := \{x \in \mathcal{C}(a \to b) | x = q \circ x \circ p\}.$$

Composition and † are just composition and † in $\mathcal{C}$. There is an obvious faithful dagger functor $\mathcal{C} \hookrightarrow \mathsf{Proj}(\mathcal{C})$.

Exercise 5.4.43. Prove that $\mathsf{Proj}(\mathcal{C})$ is C* and projection complete.

Exercise 5.4.44. Formulate and prove the universal property for $\mathsf{Proj}(\mathcal{C})$ analogous to (5.9) for †-functors from $\mathcal{C}$ to a projection complete C* category $\mathcal{D}$. (Use Exercise 5.4.40 for the uniqueness.)

Exercise 5.4.45. Use Exercise 5.4.44 to show that if $\mathcal{C}$ is projection complete, then $\mathcal{C}$ is dagger equivalent to $\mathsf{Proj}(\mathcal{C})$.

Definition 5.4.46. A C* category is called C* *Cauchy complete* if it admits finite orthogonal direct sums and is projection complete. The C* *Cauchy completion* of $\mathcal{C}$ is $\overline{\mathcal{C}} := \mathsf{Proj}(\mathsf{Add}^\dagger(\mathcal{C}))$. By Exercise 5.4.43, $\overline{\mathcal{C}}$ is C* Cauchy complete. Again, there is an obvious faithful dagger functor $\mathcal{C} \hookrightarrow \overline{\mathcal{C}}^\dagger$.

Exercise 5.4.47. Formulate and prove the universal property for $\overline{\mathcal{C}}^\dagger$ analogous to (5.10) for †-functors from $\mathcal{C}$ to a C* Cauchy complete category $\mathcal{D}$.

Exercise 5.4.48. Use Exercise 5.4.47 to show that if $\mathcal{C}$ is C* Cauchy complete, then $\mathcal{C}$ is dagger equivalent to $\overline{\mathcal{C}}^\dagger$.

Again, $\mathcal{TL}^\dagger(d)$ is not idempotent complete, but we have the following proposition.

Proposition 5.4.49. *Suppose d is as in (4.1) and $[1], \ldots, [k+1] \neq 0$, so that $f^{(k)}$ exists and has strictly positive trace. For every minimal projection $p \in TL_k^\dagger(d)$ such that $p(TL_k^\dagger(d))p = \mathbb{C}p$, there is an $n \in \{1 \ldots, k\}$ such that $(k,e) \cong (n, f^{(n)})$ in $\mathsf{Proj}(\mathcal{TL}(d))$.*

Proof. [[todo: use Exercise 4.4.2]] □

5.5 Semisimplicity

In this section, we give a functional definition of semisimplicity for linear categories following [Müg03]. We also give a criterion in the spirit of Roberts' 2×2 trick [GLR85] which ensures semisimplicity of the (C*) Cauchy completion.

5.5.1 Semisimplicity for linear categories

Let $\mathcal{C}$ be a linear category.

Definition 5.5.1. An object $c \in \mathcal{C}$ is called *simple* if $\operatorname{End}_{\mathcal{C}}(c) = \mathbb{C}\operatorname{id}_c$. Two simple objects $a, b \in \mathcal{C}$ are called *distinct* if $\mathcal{C}(a \to b) = (0)$.

Definition 5.5.2 ([Müg03])**.** We call $\mathcal{C}$ *semisimple* if

(SS1) $\mathcal{C}$ admits (finite) direct sums,

(SS2) $\mathcal{C}$ is idempotent complete, and

(SS3) there is a set of pairwise distinct simple objects $\{c_i\}_{i \in I}$ where I is some index set such that for any $a, b \in \mathcal{C}$, the composition map

$$\bigoplus_{i \in I} \mathcal{C}(a \to c_i) \otimes_{\mathbb{C}} \mathcal{C}(c_i \to b) \longrightarrow \mathcal{C}(a \to b) \tag{5.11}$$

is an isomorphism. (The direct sum in (5.11) is the direct sum in the category of finite dimensional complex vector spaces.)

We call $\mathcal{C}$ *finitely semisimple* if in addition $\mathcal{C}$ has finitely many isomorphism classes of simple objects.

Definition 5.5.3. Suppose $\mathcal{C}$ is semisimple. We call $a \in \mathcal{C}$ *isotypic* if there is a single distinct simple object c such that the composition map

$$\mathcal{C}(a \to c) \otimes_{\mathbb{C}} \mathcal{C}(c \to a) \to \mathcal{C}(a \to a) \tag{5.12}$$

is an isomorphism.

Lemma 5.5.4. *Every isotypic object is isomorphic to a finite direct sum of the same simple object.*

Proof. Suppose $a \in \mathcal{C}$ is isotypic. By (SS3) and (5.12), there are non-zero morphisms $\iota_j : c \to a$ and $\pi_j : a \to c$ for $j = 1, \ldots, n$ such that $\sum_{j=1}^{n} \iota_j \circ \pi_j = \operatorname{id}_a$, so ($\oplus$2) holds. We claim that ($\oplus$1) also holds. For all $x \in \mathcal{C}(a \to a)$,

$$x = \left(\sum_{i=1}^{n} \iota_i \circ \pi_i\right) \circ x \circ \left(\sum_{j=1}^{n} \iota_j \circ \pi_j\right) = \sum_{i,j=1}^{n} x_{ij}(\iota_i \circ \pi_j)$$

where each $x_{ij} \in \mathbb{C}$ is defined by $x_{ij}\operatorname{id}_c = \pi_i \circ x \circ \iota_j \in \mathcal{C}(c \to c) = \mathbb{C}\operatorname{id}_c$. We calculate

$$(x \circ y)_{ik}\operatorname{id}_c = \pi_i \circ x \circ y \circ \iota_k = \sum_{j=1}^{n} \pi_i \circ x \circ \iota_j \circ \pi_j \circ y \circ \iota_k = \sum_{j=1}^{n} x_{ij} y_{jk}\operatorname{id}_c,$$

so the map $\mathcal{C}(a \to a) \to M_n(\mathbb{C})$ by $x \mapsto (x_{ij})_{i,j=1}^{n}$ is a unital algebra isomorphism. Hence $\pi_i \circ \iota_j = \delta_{i=j}\operatorname{id}_c$, and ($\oplus$1) holds. $\square$

Theorem 5.5.5. *Suppose $\mathcal{C}$ is semisimple. Every object $c \in \mathcal{C}$ is isomorphic to a direct sum of simple objects.*

Proof. By (SS3), there are $c_1, \ldots, c_n$ pairwise distinct simples and non-zero morphisms $(\iota_j^i : c_j \to c)_{i=1}^{k_j}, (\pi_j^i : c \to c_j)_{i=1}^{k_j}$ such that

$$\mathrm{id}_c = \sum_{j=1}^{n} \sum_{i=1}^{k_j} \iota_j^i \circ \pi_j^i,$$

i.e., ($\oplus$2) holds. Since the only map between pairwise distinct simples is the zero map, we have $\pi_j^i \circ \iota_{j'}^{i'} = 0$ unless $j = j'$. Notice that for each $j = 1, \ldots, n$, $e_j := \sum_{i=1}^{k_j} \iota_j^i \circ \pi_j^i$ is an idempotent in $\mathcal{C}(c \to c)$. By (SS2), (c, e_j) splits, so there is an isotypic object $a_j \in \mathcal{C}$ with $a_j \cong \bigoplus_{i=1}^{k_j} c_j$ by Lemma 5.5.4, $v_j : a_j \to c$, and $w_j : c \to a_j$ such that $w_i \circ v_j = \delta_{i=j} \mathrm{id}_{a_j}$ and $\sum_{j=1}^{n} v_j \circ w_j = \mathrm{id}_c$. Hence

$$c \cong \bigoplus_{j=1}^{n} a_j \cong \bigoplus_{j=1}^{n} \bigoplus_{i=1}^{k_j} c_j,$$

and ($\oplus$1) holds. □

Definition 5.5.6. Suppose $\mathcal{C}$ is semisimple and $\{c_i\}_{i \in I}$ is a list of pairwise distinct simples from the factorization axiom (SS3). By Theorem 5.5.5, every object $c \in \mathcal{C}$ can be expressed as a direct sum of isotypic objects

$$c \cong \bigoplus_{j=1}^{n} a_j$$

where $a_j \cong \bigoplus_{i=1}^{k_j} c_j$. We call a_j the j-th *isotypic component* of c.

Exercise 5.5.7. Suppose $\mathcal{C}$ is a Cauchy complete linear category whose isomorphism classes of objects form a set. Show that the following conditions are equivalent.

(1) $\mathcal{C}$ is semisimple.

(2) For all $a, b \in \mathcal{C}$, the *linking algebra*

$$\mathcal{L}(a,b) := \mathrm{End}_{\mathcal{C}}(a \oplus b) = \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(b \to a) \\ \mathcal{C}(a \to b) & \mathcal{C}(b \to b) \end{pmatrix}$$

with composition as in (C*) is a finite dimensional complex semisimple algebra.

(3) For all $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in \mathcal{C}$, the *linking algebra*

$$\mathcal{L}(a_1, \ldots, a_n) := \mathrm{End}_{\mathcal{C}}\left(\bigoplus_{i=1}^{n} a_i\right)$$

with the obvious algebra structure is a finite dimensional complex semisimple algebra.

(4) Every object in $\mathcal{C}$ is a isomorphic to a finite direct sum of simple objects.

Example 5.5.8. The category $\mathsf{Vec_{fd}}$ of finite dimensional vector spaces is finitely semisimple. Indeed there is only one isomorphism class of simple objects.

Example 5.5.9. Let S be a set. The category $\mathsf{Vec_{fd}}(S)$ of finite dimensional S-graded vector spaces is semisimple. Here, objects are of the form

$$V = \bigoplus_{s \in S} V_s$$

where $V_s \in \mathsf{Vec_{fd}}$ for each $s \in S$, and $V_s = (0)$ for all but finitely many elements of S. The morphisms are S-graded linear maps, i.e., if $f : V \to W$, then

$$f = \{f_s : V_s \to W_s\}_{s \in S}.$$

Example 5.5.10. The Temperley-Lieb category $\mathcal{TL}(d)$ is not semisimple. [[todo: finish]]

Exercise 5.5.11 $(*)$**.** Show that a linear category $\mathcal{C}$ is finitely semisimple if and only if it is equivalent to $\mathsf{Vec_{fd}}^n$ for some $n \in \mathbb{N}$.

Exercise 5.5.12 $(**)$**.** Show that a linear category $\mathcal{C}$ is semisimple and the isomorphism classes of simple objects of $\mathcal{C}$ form a set if and only if it is equivalent to $\mathsf{Vec_{fd}}(S)$ for some set S.

Exercise 5.5.13 $(**)$**.** Show that a linear category $\mathcal{C}$ is semisimple if and only if it is abelian, has finite dimensional hom spaces, and every exact sequence in $\mathcal{C}$ splits.

5.5.2 Unitary categories

The following exercises explore semisimplicity for C* categories.

Exercise 5.5.14. Suppose $\mathcal{C}$ is a semisimple C* category. Show that $\mathcal{C}$ admits finite orthogonal direct sums and $\mathcal{C}$ is projection complete.

Exercise 5.5.15. Suppose $\mathcal{C}$ is a semisimple C* category. Prove that every object $c \in \mathcal{C}$ is *unitarily* isomorphic to a finite orthogonal direct sum of simple objects.

Exercise 5.5.16. Suppose $\mathcal{C}$ is a C* category with finite dimensional morphism spaces. Show that the C* Cauchy completion of $\mathcal{C}$ is semisimple.

Definition 5.5.17. A dagger category $(\mathcal{C}, \dagger)$ is called a *unitary category* if $(\mathcal{C}, \dagger)$ is C* and the underlying linear category $\mathcal{C}$ is semisimple.

Chapter 6

Tensor categories

6.1 Tensor categories

We now define tensor categories as linear monoidal categories. We do not use the terminology from [EGNO15].

Definition 6.1.1. A *tensor category* is a linear category $\mathcal{C}$ together with the following additional data:

- A linear bifunctor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- A distinguished object $1_{\mathcal{C}} \in \mathcal{C}$,
- *assocaitor* isomorphisms $\alpha_{a,b,c} : a \otimes (b \otimes c) \xrightarrow{\cong} (a \otimes b) \otimes c$ for all $a, b, c \in \mathcal{C}$, separately natural in all components, and
- *unitor* natural isomorphisms $\lambda_a : 1_{\mathcal{C}} \otimes a \xrightarrow{\cong} a$ and $\rho_a : a \otimes 1_{\mathcal{C}} \xrightarrow{\cong} a$ for all $a \in \mathcal{C}$,

and this data must satisfy the following axioms:

- (pentagon) for all $a, b, c, d \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
a \otimes (b \otimes (c \otimes d)) \arrow[r, "\alpha_{a,b,c\otimes d}"] \arrow[dd, "\mathrm{id}_a \otimes \alpha_{b,c,d}"] & (a \otimes b) \otimes (c \otimes d) \arrow[dr, "\alpha_{a\otimes b,c,d}"] & \\
 & & ((a \otimes b) \otimes c) \otimes d \\
a \otimes ((b \otimes c) \otimes d) \arrow[r, "\alpha_{a,b\otimes c,d}"] & (a \otimes (b \otimes c)) \otimes d \arrow[ur, "\alpha_{a,b,c}\otimes \mathrm{id}_d"'] &
\end{tikzcd} \tag{$\pentagon$}$$

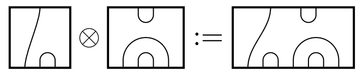
- (triangle) for all $a, b \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
a \otimes (1_{\mathcal{C}} \otimes b) \arrow[rr, "\mathrm{id}_a \otimes \lambda_b"] \arrow[dr, "\alpha_{a,1_{\mathcal{C}},b}"'] & & a \otimes b \\
 & (a \otimes 1_{\mathcal{C}}) \otimes b \arrow[ur, "\rho_a \otimes \mathrm{id}_b"'] &
\end{tikzcd} \tag{$\triangle$}$$

A tensor category $(\mathcal{C}, \otimes, 1_{\mathcal{C}}, \alpha, \lambda, \rho)$ is called *strict* if for every $a, b, c \in \mathcal{C}$, $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ and $1_{\mathcal{C}} \otimes a = a \otimes 1_{\mathcal{C}} = a$, and the natural isomorphisms $\alpha_{a,b,c}$, λ_a, and ρ_a are all identity morphisms.

Remark 6.1.2. Removing linearlity from above definition *mutatis mutandis*, we obtain the definition of a *monoidal category*.

Definition 6.1.3. We endow $\mathcal{TL}(d)$ with the structure of a strict tensor category as follows. On objects, we define $m \otimes n := m + n$. For string diagrams $x \in \mathcal{TL}(d)(m \to n)$ and $y \in \mathcal{TL}(d)(p \to q)$, we define $x \otimes y \in \mathcal{TL}(d)(m + p \to n + q)$ to be the horizontal concatenation of x and y. For an explicit example,



We then extend $- \otimes - : \mathcal{TL}(d) \times \mathcal{TL}(d) \to \mathcal{TL}(d)$ bilinearly in each argument.

Exercise 6.1.4. Verify that this tensor product endows $\mathcal{TL}(d)$ with the structure of a struct tensor category. That is, verify (⬠) and (△) hold using only identity morphisms.

Exercise 6.1.5. Suppose $\mathcal{C}$ is a tensor category. Endow the following linear categories with tensor product structures which are compatible with the fully faithful inclusion of $\mathcal{C}$:

- the additive envelope $\mathsf{Add}(\mathcal{C})$,
- the idempotent completion $\mathsf{Idem}(\mathcal{C})$, and
- the Cauchy completion $\overline{\mathcal{C}}$.

6.1.1 Tensor functors and monoidal natural transformations

[[todo.]]

6.1.2 Semisimple multitensor categories

Definition 6.1.6. A *semisimple multitensor category* is a tensor category whose underlying linear category is semisimple. If in addition $1_{\mathcal{C}}$ is simple, we call $\mathcal{C}$ a *semisimple tensor category*

Definition 6.1.7. Let $\mathrm{Irr}(\mathcal{C})$ denote a set of representatives for the isomorphism classes of $\mathcal{C}$. For $a, b, c \in \mathrm{Irr}(\mathcal{C})$, we define the *fusion coefficeint*

$$N_{a,b}^{c} := \dim(\mathrm{Hom}(a \otimes b \to c)).$$

We call $\mathcal{C}$ *multiplicity free* if $N_{a,b}^{c} \in \{0, 1\}$ for all $a, b, c \in \mathrm{Irr}(\mathcal{C})$.

Exercise 6.1.8. Show that for all $a, b, c, d \in \mathrm{Irr}(\mathcal{C})$,

$$\sum_{e \in \mathrm{Irr}(\mathcal{C})} N_{a,b}^{e} N_{e,c}^{d} = \sum_{f \in \mathrm{Irr}(\mathcal{C})} N_{a,f}^{d} N_{b,c}^{f}.$$

Definition 6.1.9. Suppose $\mathcal{C}$ is a semisimple tensor category. The *fusion graph* of $c \in \mathcal{C}$ has vertices the set $\mathrm{Irr}(\mathcal{C})$ and $\dim(\mathrm{Hom}(a \otimes c \to b))$ oriented edges between the vertices $a, b \in \mathrm{Irr}(\mathcal{C})$.

Exercise 6.1.10. Suppose $d = \exp(2\pi i/(2k))$ for $k \geq 3$. Show that the fusion graph for the strand $X \in \mathcal{TL}^{\dagger}(d)$ is the A_{k-1} Coxeter-Dynkin diagram.

Exercise 6.1.11. Suppose G is a finite group. Show that the fusion graph of $\mathbb{C}[G] \in \mathsf{Vec}(G)$ is the graph with vertices labelled by $g \in G$ and one edge from g to h for every $g, h \in G$.

[[todo.]]

6.2 Tensor C* categories

For this section, let $(\mathcal{C}, \dagger)$ be a (linear) dagger category.

Definition 6.2.1. A morphism $u \in \mathcal{C}(a \to b)$ is called *unitary* if $u^\dagger \circ u = \mathrm{id}_a$ and $u \circ u^\dagger = \mathrm{id}_b$.

Definition 6.2.2. A *tensor dagger category* is a tensor category $(\mathcal{C}, \otimes 1_\mathcal{C}, \alpha, \lambda, \rho)$ with a dagger structure $\dagger$ such that $(\mathcal{C}, \dagger)$ is a dagger category, $-\otimes- : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is a dagger bifunctor ($(x\otimes y)^d ag = x^\dagger \otimes y^\dagger$ for all morphisms $x \in \mathcal{C}(a \to b)$ and $y \in \mathcal{C}(c \to d)$), and all assocaitor and unitor natural isomorphisms are unitary.

A tensor dagger category is called a *tensor* C* *category* if the underlying dagger category $(\mathcal{C}, \dagger)$ is a C* category.

Exercise 6.2.3. Show that the tensor structure for $\mathcal{TL}(d)$ from Definition 6.1.3 makes $\mathcal{TL}(d)$ a tensor dagger category.

Exercise 6.2.4. Suppose $x \in \mathcal{TL}(d)(m \to n)$ and $y \in \mathcal{TL}(d)(p \to q)$. Show that if x or y is negligible, then so is $x \otimes y$.

Exercise 6.2.5. Show that $(x + \mathcal{N}_{m\to n}) \otimes (y + \mathcal{N}_{p\to q}) := x \otimes y + \mathcal{N}_{m+p\to n+q}$ endows $\mathcal{TL}^\dagger(d)$ with the structure of a strict tensor C* category.

6.2.1 Unitary tensor categories

Definition 6.2.6. A tensor C* category whose underlying C* category is unitary (semisimple) is called a *unitary multitensor category*. If in addition $1_\mathcal{C}$ is simple, we call $\mathcal{C}$ a *unitary tensor category*.

[[todo: expand]]

6.3 Examples from groups and cohomology

We now present more examples of tensor (C*) categories coming from groups. We begin with some basics on cocycles.

Definition 6.3.1. Let G be a group, A an abelian group, and $\pi : G \to \mathrm{Aut}(A)$ a group homomorphism (a.k.a. an action of G on A). We define the space of *n-cochains* as $C^n(G; A) := \{f : G^n \to A\}$. Given an n-cochain $\omega \in C^n(G; A)$, for $0 \le i \le n + 1$, we define $d_i(\omega) \in C^{n+1}(G; A)$ by

$$\begin{aligned} d_0(\omega)(g_0, \ldots, g_n) &:= \pi_{g_0} \cdot \omega(g_1, \ldots, g_n) \\ d_i(\omega)(g_0, \ldots, g_n) &:= \omega(g_0, \ldots, g_{i-2}, g_{i-1}g_i, g_{i+1}, \ldots, g_n) \qquad \forall 1 \le i \le n \\ d_{n+1}(\omega)(g_0, \ldots, g_n) &:= \omega(g_0, \ldots, g_{n-1}). \end{aligned}$$

We define $d := \sum_{i=0}^n (-1)^i d_i$. An n-cochain ω is called:

- an *n-cocycle* if $d(\omega) = 0$, and
- an n-cochain is called an *n-coboundary* if it is in the image of $d : C^{n-1}(G; A) \to C^n(G; A)$.

We denote the space of n-cocycles and n-coboundaries by $Z^n(G; A)$ and $B^n(G; A)$ repsectively.

Exercise 6.3.2. Prove that $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$. Deduce that $d \circ d = 0$.

Exercise 6.3.3. Show that $C^n(G;A)$, $Z^n(G,A)$, and $B^n(G;A)$ are abelian groups under pointwise addition of functions valued in A.

Definition 6.3.4. The n-th *cohomology* of G with *values* in (A,π) is the abelian group

$$H^n(G;A) := Z^n(G;A)/B^n(G;A).$$

Two cocycles $\omega, \upsilon \in Z^n(G;A)$ are called *cohomologous* if $\omega + B^n(g;A) = \upsilon + B^n(g;A)$, i.e., they induce the same cohomology class in $H^n(g;A)$.

Exercise 6.3.5. Suppose $A = \mathbb{C}^\times$ and $\pi : G \to \operatorname{Aut}(\mathbb{C}^\times)$ is the trivial action. Show that if G is finite, any $\omega \in Z^n(G;\mathbb{C}^\times)$ is cohomologous to a cocycle which only takes values in $U(1)$, the unit circle.

6.3.1 Categorical 1-groups

Example 6.3.6. Let G be a group. The category $\underline{G}$ has objects the elements of G and only identity morphisms, i.e., $\operatorname{Hom}(g,h) = \delta_{g=h}\operatorname{id}_g$. Hence composition is trivial. We define a strict tensor structure by $g \otimes h := gh$.

Example 6.3.7. Let G be a group and $\omega \in Z^3(G;\mathbb{C}^\times)$, where the action of $G \to \operatorname{Aut}(\mathbb{C}^\times)$ is trivial. The category $\underline{G}(\mathbb{C}^\times,\omega)$ has objects the elements of G and $\operatorname{Hom}(g,h) = \delta_{g=h}\mathbb{C}^\times \operatorname{id}_g$, where composition is given by multiplication. We define a tensor structure by $g \otimes h := gh$, tensor product of morphisms is just multiplication, $\alpha_{g,h,k} := \omega(g,h,k)\operatorname{id}_{ghk}$, $\lambda_g := \omega(g,1,1)\operatorname{id}_g$, and $\rho_g := \omega(1,1,g)^{-1}$.

Exercise 6.3.8. Show that the pentagon axiom ($\pentagon$) for Example 6.3.7 is exactly the 3-cocycle condition for $\omega \in Z^3(G;\mathbb{C}^\times)$. Then show that the triangle axiom ($\triangle$) holds for Example 6.3.7 by analyzing $d\omega(g,1,1,h)$.

Exercise 6.3.9. Show that if $\omega, \upsilon \in Z^3(G;\mathbb{C}^\times)$ are cohomologous, then $\underline{G}(\mathbb{C}^\times,\omega)$ is monoidally equivalent to $\underline{G}(\mathbb{C}^\times,\upsilon)$.

The previous examples are special cases of the following.

Example 6.3.10. Suppose G is a group, A is an abelian group, $\pi : G \to \operatorname{Aut}(A)$ is a group homomorphism , and $\omega \in Z^3(G;A)$. Let $\underline{G}(A,\pi,\omega)$ be the category whose objects are the group elements $g \in G$, whose morphism sets are $\operatorname{Hom}(g,h) := \delta_{g=h}A$, and whose composition law is the group law in A. We define a tensor structure on $\underline{G}(A,\pi,\omega)$ similar to Exercise 6.3.7, i.e., $g \otimes h := gh$, $\alpha_{g,h,k} := \omega(g,h,k) \in \operatorname{End}(ghk)$, $\lambda_g := \omega(g,1,1) \in \operatorname{End}(g)$, $\rho_g := -\omega(1,1,g) \in \operatorname{End}(g)$.[1] However, the tensor product of morphisms is more subtle. For $a \in \operatorname{End}(g)$ and $b \in \operatorname{End}(h)$, we define $a \otimes b := a + \pi_g(b)$.

Exercise 6.3.11. Repeat Exercise 6.3.8 in the context of Example 6.3.10.

Exercise 6.3.12 (Sinh, [BS10, §4.2])**.** Show that every monoidal category $\mathcal{C}$ whose objects are all *invertible* (for every $a \in \mathcal{C}$, there is a $b \in \mathcal{C}$ such that $a \otimes b \cong 1_{\mathcal{C}} \cong b \otimes a$) and whose morphisms are all invertible is monoidally equivalent to a monoidal category of the form $\underline{G}(A,\pi,\omega)$.

[1] We write the group law multiplicatively in $\mathbb{C}^\times$ and additively in A.

6.3.2 G-graded spaces

Exercise 6.3.13. Describe the associators and unitors in the tensor category Vec of complex vector spaces.

Definition 6.3.14. The category $\mathsf{Vec}(G)$ of *G-graded vector spaces* has objects complex vector spaces graded by G

$$V = \bigoplus_{g \in G} V_g$$

and morphisms G-graded linear transformations. For $U, V \in \mathsf{Vec}(G)$, we define the tensor product by defining its g-graded component by

$$(U \otimes V)_g := \bigoplus_{\substack{h,k \in G \\ g = hk}} U_h \otimes V_k,$$

and the associator and unitor isomorphisms are the same as in the category of vector spaces.

We define $\mathsf{Vec}_{\mathsf{fd}}(G)$ to be the full tensor subcategory of finite dimensional complex vector spaces. Similarly, $\mathsf{Hilb}(G)$ denotes the tensor C* category of *G-graded Hilbert spaces* with G-graded bounded linear operators and adjoint of bounded operators. We denote by $\mathsf{Hilb}_{\mathsf{fd}}(G)$ the full tensor C* subcategory of finite dimensional G-graded Hilbert spaces.

Exercise 6.3.15. Suppose $\omega \in Z^3(G; \mathbb{C}^\times)$ as in Example 6.3.7. Define $\mathsf{Vec}(G, \omega)$ and show it is a tensor category. Then show that as in Exercise (6.3.9), cohomologous cocycles give tensor equivalent categories.

Exercise 6.3.16. Repeat Exercise 6.3.15 for $\mathsf{Hilb}(G, \omega)$ where $\omega \in Z^3(G; U(1))$.

Chapter 7

Graphical calculus, part I

In this section, we develop the graphical calculus for tensor categories. We begin with an unbiased version of an ordinary unital complex algebra called a *linear algebra* before proceeding to an unbiased version of a tensor category called a *tensor algebras*.

7.1 Linear algebras and unbiased multiplication

7.1.1 Unbiased definition of a monoid

Recall that a *monoid* $(M, \cdot)$ is a set together with an associative binary operation $\cdot : M \times M \to M$ and a unit $e \in M$. Consider the following standard introductory exercise.

Exercise 7.1.1. Show that for any n elements $x_1, \ldots, x_n \in M$, the product of $x_1 \cdots x_n$ does not depend on the parenthesization of $x_1, \ldots, x_n$.

Here, we see that the definition of monoid is *biased* toward multiplying 2 elements at a time. We can define an unbiased version as follows.

Definition 7.1.2. A *monoid* is a set M together with n-ary operations $\mu_n : M^n \to M$ for every $n \in \mathbb{N}$ such that

- $\mu_1 : M \to M$ is the identity, and
- Inserting μ_k into the $i+1$-th spot of μ_{n+1} gives the map $\mu_{n+k} : M^{n+k} \to M$.

$$\mu_{n+1} \circ (\underbrace{\mathrm{id}_M \times \cdots \times \mathrm{id}_M}_{i \text{ factors}} \times \mu_k \times \underbrace{\mathrm{id}_M \times \cdots \times \mathrm{id}_M}_{n-i \text{ factors}}) = \mu_{n+k}.$$

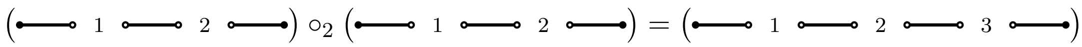
Exercise 7.1.3. Add linearity to the above discussion to get the unbiased definition of a unital associative complex lgebra.

7.1.2 Linear algebras

We give the following diagrammatic calculus for the unbiased definition of a unital associative complex algebra by adapting [GMP+18, §3.2].

Definition 7.1.4. The *linear operad* consists of the 1D *Swiss cheese diagrams* [Vor99] consisting of a large interval, several removed subintervals called *holes*, all considered up to diffeomorphism.

These interval diagrams can be composed by plugging some new big intervals into the holes to get a new diagram.

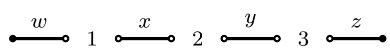


A *linear algebra* in vector spaces is an algebra for the linear operad, which means it consists of a vector space A together with a linear map $A^{\otimes n} \to A$ attached to each linear Swiss cheese diagram with n holes. These maps must be compatible with the operad structure (i.e., plugging elements of A into holes, and plugging diagrams into larger diagrams, associates). Unpacking this definition, a linear algebra in vector spaces consists of multiplication maps $\mu_n : A^{\otimes n} \to A$ for every natural number n ($n = 0$ gives the unit) which satisfy the appropriate associativity relations. This is exactly the unbiased definition of a unital associative algebra.

7.1.3 Linear categories and algebras

We now add string labels to get an unbiased version of a linear category which allows us to compose n morphisms at a time.

Definition 7.1.5. Suppose S is a set of *string labels*. The *S-linear operad* consists of 1D Swiss cheese diagrams whose connected components are labelled by elements of S. To agree with future examples, it helps to think of these components as *strings* connecting each hole to the next hole or to the outside interval.



Again the operadic structure comes from gluing linear tangles into the holes, but since substitution only makes sense when the labels match, this is a *colored operad*. An algebra ${}_\bullet V_\bullet$ for this operad consists of a family of vector spaces $\{{}_xV_y\}_{x,y\in S}$ together with an action of linear tangles with holes. That is, to each linear tangle T with components labelled by $x_1, \ldots, x_n$, we get a linear map $Z(T) : {}_{x_1}V_{x_2} \otimes \cdots \otimes {}_{x_{n-1}}V_{x_n} \to {}_{x_1}V_{x_n}$ which is compatible with composition of linear tangles with holes.

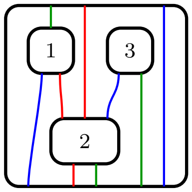
Exercise 7.1.6. Show that an algebra for the S-linear operad gives an unbiased definition of a linear category whose set of objects is S, whose hom spaces $\mathrm{Hom}(x \to y)$ are the vector spaces ${}_xV_y$, and whose composition of morphisms is the action of linear tangles.
Hint: The identity morphisms come from the tangles with no holes.

7.2 Tensor algebras and unbiased tensor categories

Sometimes there is a good analogy between going one categorical dimension higher and going one topological dimension higher. This is the case for a *tensor algebra*, which is an unbiased version of a linear category. Again, we adapt [GMP+18, §3.2].

Definition 7.2.1. An *S-tensor tangle* is a 2D Swiss cheese diagram consisting of a rectangle, with several smaller rectangles (with edges parallel to those of the big one) removed, and some non-crossing smooth strings labelled by elements of S which are oriented upward, have no minima or maxima, and begin and end on the tops or bottoms of the rectangles. We say a monoidal tangle T has *type* $((s_0, t_0); (s_1, t_1), \ldots, (s_k, t_k))$ where $s_0, \ldots, s_k, t_0, \ldots, t_k$ are finite words on S if the tangle T has k input rectangles, and there are $|s_i|, |t_i|$ strings attached to the bottom and top respectively of the i-th rectangle (the zeroth rectangle is the output rectangle and $1 \leq i \leq k$ corresponds to the

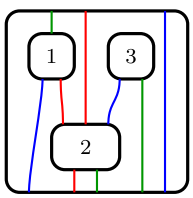
i-th input rectangle), which are labelled by the characters in the words s_i, t_i respectively. Here is an example of a tangle with $S = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}\}$, where we color the strings instead of labelling them:



has type $$((\underbrace{{\color{blue}\bullet}{\color{red}\bullet}{\color{green}\bullet}{\color{green}\bullet}{\color{blue}\bullet}}_{s_0}, \underbrace{{\color{green}\bullet}{\color{red}\bullet}{\color{blue}\bullet}}_{t_0}); (\underbrace{{\color{blue}\bullet}{\color{red}\bullet}}_{s_1}, \underbrace{{\color{green}\bullet}}_{t_1}), (\underbrace{{\color{red}\bullet}{\color{green}\bullet}}_{s_2}, \underbrace{{\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet}}_{t_2}), (\underbrace{{\color{blue}\bullet}{\color{green}\bullet}}_{s_3}, \underbrace{\emptyset}_{t_3})).$$

Tensor tangles are considered up to isotopy (through diagrams that again have no minima or maxima). Tensor tangles form a colored operad, because you can insert tensor tangles into the rectangles of a large tensor tangle to get a new tensor tangle.

Definition 7.2.2. An *S-tensor algebra* is an algebra for the operad of S-tensor tangles. Unpacking this definition, a tensor algebra $\mathcal{P}_{\bullet\to\bullet}$ consists of a family of finite dimensional vector spaces $\mathcal{P}_{s\to t}$ where s, t are finite words in S, together with an action of tensor tangles. To each tensor tangle T of type $((s_0, t_0); (s_1, t_1, \ldots, (s_k, t_k)))$, we associate a multilinear map $Z(T) : \prod_{j=1}^k \mathcal{P}_{s_j\to t_j} \to \mathcal{P}_{s_0\to t_0}$, and composition of tensor tangles corresponds to composition of multilinear maps. Here is an an example:

$Z\Big($  $\Big)$

$$: \mathcal{P}_{{\color{blue}\bullet}{\color{red}\bullet}\to{\color{green}\bullet}} \times \mathcal{P}_{{\color{red}\bullet}{\color{green}\bullet}\to{\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet}} \times \mathcal{P}_{{\color{blue}\bullet}{\color{green}\bullet}\to\emptyset} \to \mathcal{P}_{{\color{blue}\bullet}{\color{red}\bullet}{\color{green}\bullet}{\color{green}\bullet}{\color{blue}\bullet}\to{\color{green}\bullet}{\color{red}\bullet}{\color{blue}\bullet}}$$

A tensor algebra is called *semisimple* if for every pair of words s, t on S, the 2×2 *linking algebra*

$$\mathcal{L}(s,t) := \begin{pmatrix} \mathcal{P}_{s\to s} & \mathcal{P}_{t\to s} \\ \mathcal{P}_{s\to t} & \mathcal{P}_{t\to t} \end{pmatrix} \tag{7.1}$$

whose multiplication given by matrix multiplication together with the appropriate 'stacking' multiplication tangles is a finite dimensional semisimple algebra.

Example 7.2.3. Suppose $\mathcal{C}$ is a tensor category with a set of objects $\mathcal{S} := \{X_s\}_{s\in S}$ which *Cauchy tensor generates* $\mathcal{C}$, i.e., every object in $\mathcal{C}$ is isomorphic to a summand of a direct sum of tensor products of objects in $\mathcal{S}$. We define an S-tensor algebra $\mathcal{P}(\mathcal{C}, \mathcal{S})_{\bullet\to\bullet}$ as follows. For $s_1, \ldots, s_k, t_1, \ldots, t_\ell \in S$, we define

$$\mathcal{P}(\mathcal{C}, \mathcal{S})_{s_1\cdots s_k \to t_1\cdots t_\ell} := \mathcal{C}(X_{s_1} \otimes \cdots \otimes X_{s_k} \to X_{t_1} \otimes \cdots \otimes X_{t_\ell}).$$

We use the convention that if $\emptyset$ is the empty word on S, then the empty tensor product of objects is $1_\mathcal{C}$. The action of tangles is given as follows:

[[todo: explain]]

Theorem 7.2.4. *There is an equivalence of categories*[1]

$$\left\{\begin{array}{l} S\text{-tensor algebras } \mathcal{P}_{\bullet\to\bullet} \text{ with finite} \\ \text{dimensional box spaces } \mathcal{P}_{m\to n} \end{array}\right\} \cong \left\{\begin{array}{l} \text{Pairs } (\mathcal{C}, \{X_s\}_{s\in S}) \text{ with } \mathcal{C} \text{ a tensor category} \\ \text{with Cauchy tensor generators } X_s \in \mathcal{C} \text{ for} \\ s \in S \end{array}\right\}.$$

Under this equivalence, semsimple S-tensor algebras correspond to semisimple tensor categories.

[1] Pairs $(\mathcal{C}, \{X_s\}_{s\in S})$ form a 2-category where between any two 1-morphisms, there is at most one 2-morphism, which is necessarily invertible when it exists [HPT16b, Lem. 3.5]. Hence this 2-category is equivalent to its truncation to a 1-category.

7.2.1 Shadings is tensor algebras

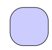
Now suppose $\mathcal{C}$ is a semisimple tensor category.

Exercise 7.2.5. Show that $\mathcal{C}(1_{\mathcal{C}} \to 1_{\mathcal{C}})$ is abelian, i.e., there is an $r \in \mathbb{N}$ such that $\mathcal{C}(1_{\mathcal{C}} \to 1_{\mathcal{C}}) \cong \mathbb{C}^r$. *Hint: Mimic the proof that $\pi_2(X)$ is abelian for a topological space X.*

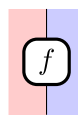
By Exercise 7.2.5, we may decompose $1_{\mathcal{C}} \cong \bigoplus_{i=1}^r 1_i$ as a direct sum of r distinct simples. We call such a $\mathcal{C}$ an *r-shaded semisimple tensor category.* We write

$$\mathcal{C}_{ij} := 1_i \otimes \mathcal{C} \otimes 1_j, \tag{7.2}$$

and we note that $\mathcal{C} = \bigoplus_{i,j=1}^r \mathcal{C}_{ij}$. We also have distinguished idempotents $p_i \in \mathcal{C}(1_{\mathcal{C}} \to 1_{\mathcal{C}})$ corresponding to each summand 1_i for $1 \leq i \leq r$. In the graphical calculus, we represent these projections, which freely float about in their regions, as a single shading. For example, we could denote

 $= p_i$  $= p_j$.

Then for objects $a, b \in \mathcal{C}_{ij}$, we would denote a morphism $f \in \mathcal{C}(a \to b)$ by

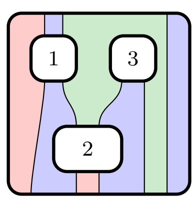


.

This motivates the following definition.

Definition 7.2.6. An *R-shaded S-tensor tangle* with label set S is a tensor tangle with string label set S whose regions are shaded by the elements of R such that each element $x \in S$ has a left *source* shading $s_x \in R$ and a right *target* shading $t_y \in R$.

Example 7.2.7. For the shading set $R = \{ , , \}$, and the label set $S = \{ , , , \}$, we have the following R-shaded monoidal tangle with label set S:



.

Definition 7.2.8. An *R-shaded S-tensor algebra* is an algebra over the operad of R-shaded S-tensor tangles. Notice this means that the spaces $\mathcal{P}_{x \to y}$ are only well-defined when consecutive characters in the words x and y have compatible target and source shadings, and the source and target shadings of the words x and y agree.

We have the following shaded version Theorem 7.2.4.

Corollary 7.2.9. *There is an equivalence of categories (see Footnote 1)*

$$\left\{ \begin{matrix} \text{Semisimple } \{1,\ldots,r\}\text{-shaded} \\ S\text{-tensor algebras } \mathcal{P}_{\bullet\to\bullet} \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{Pairs } (\mathcal{C}, \{X_y\}_{y\in S}) \text{ with } \mathcal{C} \text{ an } r\text{-shaded semisimple} \\ \text{tensor category with decomposition } 1 = \bigoplus_{i=1}^r 1_i \\ \text{with Cauchy tensor generators } X_y \in \mathcal{C}_{s_y,t_y} \text{ for } y \in S \end{matrix} \right\}.$$

Chapter 8

The synoptic chart of tensor categories

We now provide a brief interlude to provide the synoptic chart of properties and structures for tensor categories. We then provide the modified chart for unitary tensor categories. In the upcoming chapters, we will discuss in detail dualizability and pivotal structures and braidings on tensor categories.

In the charts below,

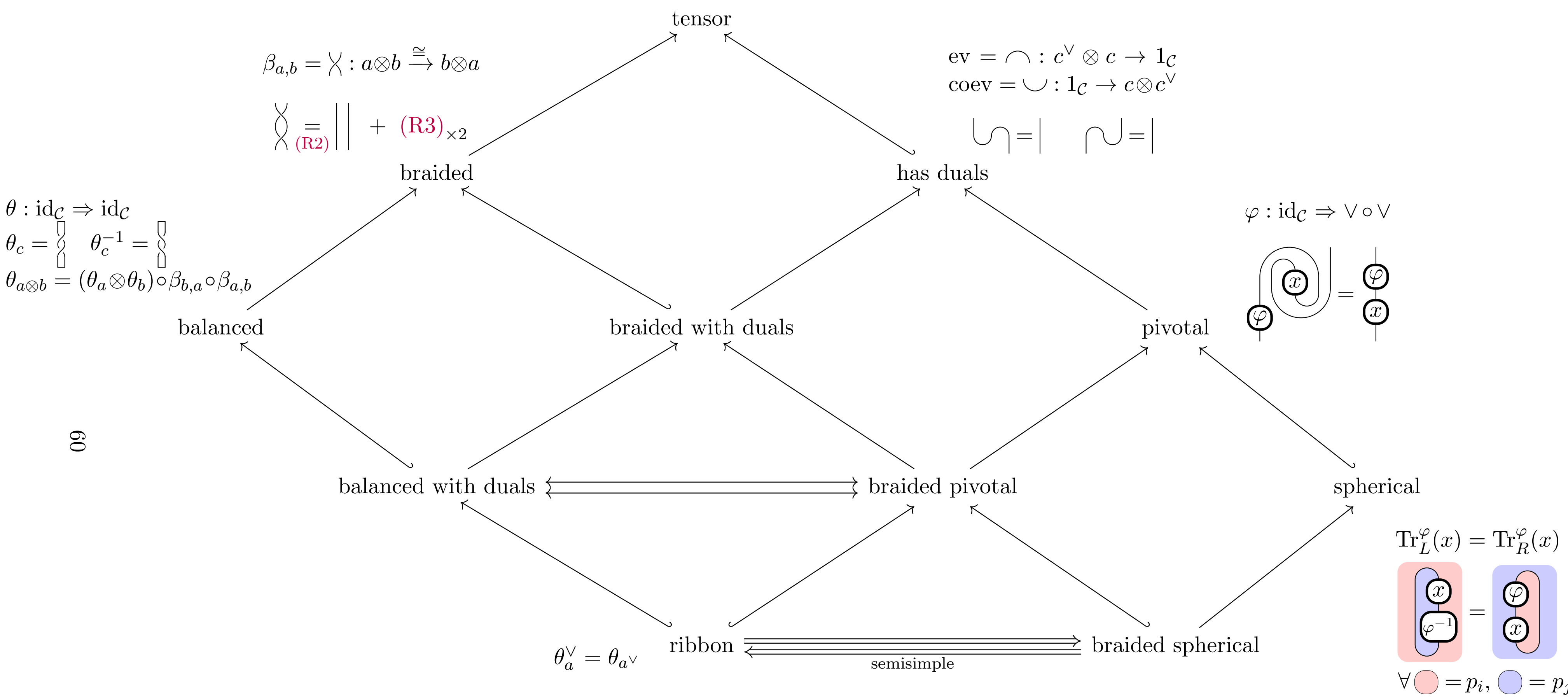
- An arrow $A \longrightarrow B$ indicates that notion B can be obtained from notion A by forgetting part of the data. This also means that notion A can be obtained from B by adding extra structure.

- An arrow $A \hookrightarrow B$ indicates that notion A can be obtained from notion B by imposing extra axioms. That is, notion A should be considered as a *property* of notion of B, and not extra structure.

- A dashed arrow $A \overset{\mathcal{Z}}{\dashrightarrow} B$ indicates that the Drinfeld center construction goes from notion A to notion B.

- An arrow with two heads $A \longleftrightarrow B$ indicates an equivalence between notions A and B.

- A double arrow $A \overset{P}{\Longrightarrow} B$ indicates that notion A implies notion B assuming property P.


tensor
$\beta_{a,b} = \times : a \otimes b \xrightarrow{\cong} b \otimes a$
$=_{(R2)}$
$+$ (R3)$_{\times 2}$
ev $= \frown : c^\vee \otimes c \to 1_\mathcal{C}$
coev $= \smile : 1_\mathcal{C} \to c \otimes c^\vee$
braided
has duals
$\theta : \mathrm{id}_\mathcal{C} \Rightarrow \mathrm{id}_\mathcal{C}$
$\theta_c =$
$\theta_c^{-1} =$
$\theta_{a\otimes b} = (\theta_a \otimes \theta_b) \circ \beta_{b,a} \circ \beta_{a,b}$
$\varphi : \mathrm{id}_\mathcal{C} \Rightarrow \vee \circ \vee$
balanced
braided with duals
pivotal
balanced with duals
braided pivotal
spherical
$\mathrm{Tr}_L^\varphi(x) = \mathrm{Tr}_R^\varphi(x)$
ribbon
braided spherical
semisimple
$\theta_a^\vee = \theta_{a^\vee}$
$\forall$ $= p_i$, $= p_j$

Chapter 9

Dualizability in tensor categories

In this section, we investigate common properties and structures of tensor categories. We keep in mind the example of finite dimensional G-graded vector/Hilbert spaces. For this section, $\mathcal{C}$ will denote a linear tensor category which is not necessarily semisimple. Throughout, we suppress all associators and unitors whenever possible.

9.1 Dualizability

Consider the category of complex vector spaces Vec. The *dual space* of $V \in \mathsf{Vec}$ is the space $V^\vee := \operatorname{Hom}(V \to \mathbb{C})$ of linear functionals on V. Notice that the dual space comes with a canonical *evaluation map* $V^\vee \otimes V \to \mathbb{C}$ given by the linear extension of $f \otimes v \mapsto f(v)$. When V is *finite dimensional*, we have a canonical isomorphism $V \otimes V^\vee \cong \operatorname{End}(V)$ given by the linear extension of the $v \otimes f \mapsto f(\,\cdot\,)v \in \operatorname{End}(V)$. This gives us a canonical *coevaluation map* $\mathbb{C} \to V \otimes V^\vee$ given by $\lambda \mapsto \lambda \sum_b b \otimes b^\vee$ where $\{b\}$ is a basis for B and $\{b^\vee\}$ denotes the dual basis of $V^\vee$.

Exercise 9.1.1. Show that $\sum_b b^\vee(\,\cdot\,)b = \mathrm{id}_V$ and is thus independent of the choice of basis. Then use the canonical isomorphism $V \otimes V^\vee \cong \operatorname{End}(V)$ to show $\sum_b b \otimes b^\vee$ is independent of the choice of basis.

The situation is analogous in the category $\mathsf{Rep}(G)$ of finite dimensional complex representations of G, and in the category $\mathsf{Vec}(G)$ of G-graded vector spaces, perhaps twisted by a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$.

Exercise 9.1.2. Show that when $V \in \mathsf{Rep}(G)$, $V^\vee$ also carries a canonical linear representation of G given by $(g \cdot f)(v) := f(g^{-1} \cdot v)$. Then show that the evaluation and coevaluation maps are both G-equivariant and thus define maps in $\mathsf{Rep}(G)$.

9.1.1 Duals and preduals

Definition 9.1.3. An object $c \in \mathcal{C}$ is called *dualizable* if there is an object $c^\vee \in \mathcal{C}$ together with morphisms $\mathrm{ev}_c \in \mathcal{C}(c^\vee \otimes c \to 1_\mathcal{C})$ and $\mathrm{coev}_c \in \mathcal{C}(1_\mathcal{C} \to c \otimes c^\vee)$ which satisfy the *zig-zag axioms*

$$
\begin{tikzcd}
c \arrow[r, "\mathrm{coev}_c \otimes \mathrm{id}_c"] \arrow[dr, "\mathrm{id}_c"'] & c \otimes c^\vee \otimes c \arrow[d, "\mathrm{id}_c \otimes \mathrm{ev}_c"] \\
& c
\end{tikzcd}
\qquad
\begin{tikzcd}
c^\vee \arrow[r, "\mathrm{id}_c \otimes \mathrm{coev}_c"] \arrow[dr, "\mathrm{id}_{c^\vee}"'] & c^\vee \otimes c \otimes c^\vee \arrow[d, "\mathrm{ev}_c \otimes \mathrm{id}_c"] \\
& c^\vee
\end{tikzcd}
\tag{9.1}
$$

The triple $(c^\vee, \mathrm{ev}_c, \mathrm{coev}_c)$ is called a *dual* of $\mathcal{C}$. We say that $\mathcal{C}$ *has duals* if every $c \in \mathcal{C}$ has a dual in $\mathcal{C}$.

In the tensor algebra graphical calculus, the evaluation and coevaluation are best represented by caps and cups respectively:

$$\mathrm{ev}_c = [[\textit{todo}]]$$

We then see that the zig-zag axioms are exactly Morse cancellation of critical points:

[[*todo*]]

Exercise 9.1.4. Show that a vector space $V \in \mathsf{Vec}$ is dualizable if and only if it is finite dimensional. Repeat this exercise for G-graded vector spaces.

Exercise 9.1.5. Suppose $a, b \in \mathcal{C}$ both have duals. Find a dual for $a \otimes b$.

Exercise 9.1.6. Show that given objects $c, c^\vee \in \mathcal{C}$ and a map $\mathrm{ev}_c \in \mathcal{C}(c^\vee \otimes c \to 1_{\mathcal{C}})$, there is at most one $\mathrm{coev}_c \in \mathcal{C}(1_{\mathcal{C}} \to c \otimes c^\vee)$ satisfying (9.1). In this sense, we say that evaluation *determines* coevaluation.

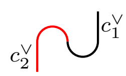
Exercise 9.1.7. Show that coevaluation also determines evaluation, i.e., given a map $\mathrm{coev} \in \mathcal{C}(1_{\mathcal{C}} \to c \otimes c^\vee)$, there is at most one $\mathrm{ev}_c \in \mathcal{C}(c^\vee \otimes c \to 1_{\mathcal{C}})$ satisfying (9.1).

Exercise 9.1.8. Show that given two duals $(c_1^\vee, \mathrm{ev}_1, \mathrm{coev}_1)$ and $(c_2^\vee, \mathrm{ev}_2, \mathrm{coev}_2)$ for $c \in \mathcal{C}$, the following two conditions are equivalent for a map $\zeta_c \in \mathcal{C}(c_2^\vee \to c_1^\vee)$:

(1) $\mathrm{ev}_2 \circ (\zeta_c \otimes \mathrm{id}_c) = \mathrm{ev}_1$, and

(2) $(\mathrm{id}_c \otimes \zeta_c) \circ \mathrm{coev}_1 = \mathrm{coev}_2$.

Show that the above conditions *uniquely determine* ζ_c, which is necessarily an isomorphism. Deduce that given two duals of $c \in \mathcal{C}$, there is a unique isomorphism $\zeta_c \in \mathcal{C}(c_2^\vee \to c_1^\vee)$ satisfying the above conditions.
For later use, we record the following formula for ζ_c:

$\zeta_c :=$ 

$= (\mathrm{ev}_c^2 \otimes \mathrm{id}_{c_1^\vee}) \circ (\mathrm{id}_{c_2^\vee} \otimes \mathrm{coev}_c^1). \quad (9.2)$

Definition 9.1.9. An object $c_\vee \in \mathcal{C}$ is called a *predual* of $c \in \mathcal{C}$ if there exists an isomorphism $(c_\vee)^\vee \cong c$. We say that $\mathcal{C}$ *has preduals* if every object in $\mathcal{C}$ admits a predual.

Exercise 9.1.10. Find an object in a tensor category which:

- admits a predual but does not admit a dual, and
- admits a dual but does not admit a predual.

Remark 9.1.11. [[comment about left/right rigidity]]

Definition 9.1.12. Suppose that $\mathcal{C}$ has duals. A *dual functor* consists of a choice of dual $(c^\vee, \mathrm{ev}_c, \mathrm{coev}_c)$ for each $c \in \mathcal{C}$, which may be amalgamated into a linear anti-tensor functor $(\vee, \nu) : \mathcal{C} \to \mathcal{C}$ as follows. On objects, we define $\vee(c) := c^\vee$, and on morphisms $f \in \mathcal{C}(a \to b)$, we define

$$f^\vee := (\mathrm{ev}_b \otimes \mathrm{id}_{a^\vee}) \circ (\mathrm{id}_{b^\vee} \otimes f \otimes \mathrm{id}_{a^\vee}) \circ (\mathrm{id}_{b^\vee} \otimes \mathrm{coev}_a).$$

The canonical anti-tensorator $\nu_{a,b} \in \mathcal{C}(a^\vee \otimes b^\vee \to (b \otimes a)^\vee)$ is defined by taking the unique isomorphism $\zeta_{b \otimes a}$ for the two duals $(b \otimes a)^\vee$ and $a^\vee \otimes b^\vee$ for $b \otimes a$:

$$\nu_{a,b} := \zeta_{b \otimes a} = (\mathrm{ev}_a \otimes \mathrm{id}_{(b \otimes a)^\vee}) \circ (\mathrm{id}_{a^\vee \otimes \mathrm{ev}_b \otimes \mathrm{id}_a \otimes \mathrm{id}_{(b \otimes a)^\vee}}) \circ (\mathrm{id}_{a^\vee \otimes b^\vee} \otimes \mathrm{coev}_{b \otimes a}).$$

Observe that ν is not part of the data of $\vee$ as ν is completely determined by $\vee$.

Exercise 9.1.13. Show that given any two dual functors $\vee_1, \vee_2 : \mathcal{C} \to \mathcal{C}$, the unique isomorphisms from Exercise 9.1.8 give a unique monoidal natural isomorphism $\zeta : \vee_1 \Rightarrow \vee_2$ which is compatible with the evaluation and coevaluation maps.

9.1.2 Duality and semisimplicity

For this section, we assume $\mathcal{C}$ is a semisimple tensor category which has duals.

Exercise 9.1.14 ([HPT16a, Lem. A.5])**.** Suppose $c \in \mathcal{C}$ and $(c^\vee, \mathrm{ev}_c, \mathrm{coev}_c)$ is a dual of c. Suppose that $\varepsilon : c^\vee \otimes c \to 1_\mathcal{C}$ and $\eta : 1_\mathcal{C} \to c^\vee \otimes c$ are non-zero morphisms. Prove that $\varepsilon \circ \eta \neq 0$.

Definition 9.1.15. A *multifusion category* is a finitely semisimple tensor category with duals and preduals. If in addition $1_\mathcal{C}$ is simple, we call $\mathcal{C}$ a *fusion category*.

Example 9.1.16. For G a finite group, the category $\mathsf{Rep}(G)$ of finite dimensional complex representations of G is a fusion category, as is the category $\mathsf{Vec}(G, \omega)$ of G-graded vector spaces with associator twisted by $\omega \in Z^3(G, \mathbb{C}^\times)$.

9.1.3 Duality in unitary tensor categories

Definition 9.1.17.

[[canonical iso $c \to c^{\vee\vee}$ in the presence of a unitary dual functor.]]

9.1.4 Unitary fusion categories

Exercise 9.1.18. Show that a unitary fusion category has a *unique* unitary dual functor up to unique unitary monoidal natural isomorphism.

9.2 Pivotal structures

9.2.1 Trace and dimension

9.2.2 Semisimplicity and nondegeneracy of the trace

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