Exactly 1-ST subfactors w/ index at most 6.2.

Goals:
1. SFs + invariants
2. SF classification program – 1-ST case.
3. examples – braided + "$\mathcal{G}$-braided" categories.

0. Big picture:

$$\begin{tikzcd} \begin{matrix} B \\ \cup \\ A \end{matrix} \arrow[r, shift left] & P_\bullet \arrow[l, shift left] \arrow[r, shift left] & (\Gamma_+, \Gamma_-) \arrow[l, shift left, dashed, "?"] \end{tikzcd}$$

Subfactors — standard invariants "planar algebras" — principal graphs

1. **Subfactors:** $A \subset B$ unital inclusion of II_1-factors

- a factor $A \subset B(H)$ is a vNa ($A = A''$) w/ trivial center ($A' \cap A = Z(A) = \mathbb{C}1$)
- a factor is type II_1 if it is ∞-dim'l and has a trace.
- can think of it as a complex $*$-alg, simple, ! normalized trace, $\langle a,b\rangle = \mathrm{tr}(b^* a)$ pos. def sesq-form.

Index: finite if ${}_A B$ is a f-gen proj. module.

$$[B:A] = \mathrm{tr}\left([{}_A B] \in K_0(A)^+\right)$$

$$= n + \mathrm{tr}(p) \quad \text{where} \quad {}_A B \cong \bigoplus^n {}_A A \oplus \underbrace{{}_A A p}_{\text{fractional part.}}$$

Thm (Jones): $[B:A] \in \{4\cos^2(\frac{\pi}{n}) \mid n \geq 3\} \cup [4, \infty]$.

Standard invariant: unitary 2-category "rep theory of SF"

objects: A, B

1-mor: $\bigotimes_A^n B$, split into simple A-A, A-B, B-A, B-B

"semi-simplicity comes from analysis"

2-mor: intertwiners. $\mathrm{Hom}(P, Q)$ f.dim'l rigid structure: dual of bimodule is contragredient.

- have evaluation/coevaluation, pivotality

unitary structure: intertwiners are bimod maps

- there is an adjoint intertwiner

- also from analysis.

- a unitary 2-category + choice of 1-morphism (for us it is ${}_A B_B$) gives a planar algebra via the usual diagrammatic calculus.
- gives a generators + relations approach to constructing subfactors.

Principal graphs: "induction - restriction graphs"

$\Gamma_\pm$ bipartite graphs

Γ_+: even vertices: simple A-A bimods/$\cong$
odd vertices: simple A-B bimods/$\cong$

edges: $\dim(\mathrm{Hom}(P \underset{A}{\otimes} B, Q))$ edges from simple ${}_A P_A$ to ${}_A Q_B$. "induction"

Γ_-: restriction graph. B-B to B-A by $\underset{B}{\otimes} B_A$.

$$\begin{tikzcd} A\text{-}A \arrow[r, dash, "\Gamma_+"] \arrow[d, dash, "\Gamma_+"'] & B\text{-}A \arrow[d, dash, "\Gamma_-"] \\ A\text{-}B \arrow[r, dash, "\Gamma_-"'] & B\text{-}B \end{tikzcd}$$

(*) rigid structure gives duality
A-A $\to$ A-A
B-B $\to$ B-B
A-B $\to$ B-A

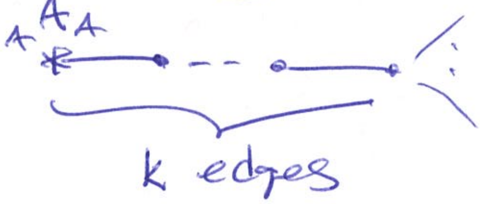
② SF classification program

i) enumerate graph pairs (Γ_+, Γ_-), apply obstructions.
ii) construct examples when graphs survive
iii) fit exotic/exceptional examples into families.

Step	index < 4	= 4	> 4
i)	ADE	affine Dynkin	hard!
ii)	no Dodd on E_7	—	
iii)	—	—	

- recent survey of Jones–Morrison–Snyder in Bull. AMS.
- complete classification of standard invariants to $\boxed{5}$
- some results above 5, not many!
- map of known subfactors

<u>Supertransitivity</u>: $\Gamma_\pm$ is k-supertransitive (ST) if



- A_n is k-ST $\forall k \geq 0$
- D_{2n} is $(2n-3)$-ST, <u>not</u> $(2n-2)$-ST

- Morrison + Peters classified 1-ST w/ index below $3+\sqrt{5}$ $\simeq 5.236\ldots$
- Liu classified at $3+\sqrt{5}$, partial proof by Izumi–Morrison–Penneys (project during last ANU visit)
- in joint work w/ Liu + Morrison, we classify up to index 6.2 [except at index 6]

<u>Intermediates</u>: exactly 1-ST if:
- ${}_B B_B$ irred.
- ${}_A B_A = {}_A A_A \oplus \underbrace{{}_A[B \ominus A]_A}_{\text{reducible}}$

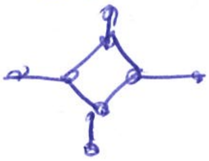
- if $\exists$ intermediate SF: $A \subset D \subset B$, then ${}_A B \ominus A_A$ is reducible. $B \cong A \oplus [D \ominus A] \oplus [B \ominus D]$
- index multiplicative. If $\exists$ nontrivial intermediate, index ≤ 6.2, $([D:A],[B:D]) = (2,2), (2,\tau^2), (2,3)$

τ smaller by taking dual SF $\qquad \tau = \frac{1+\sqrt{5}}{2}$.

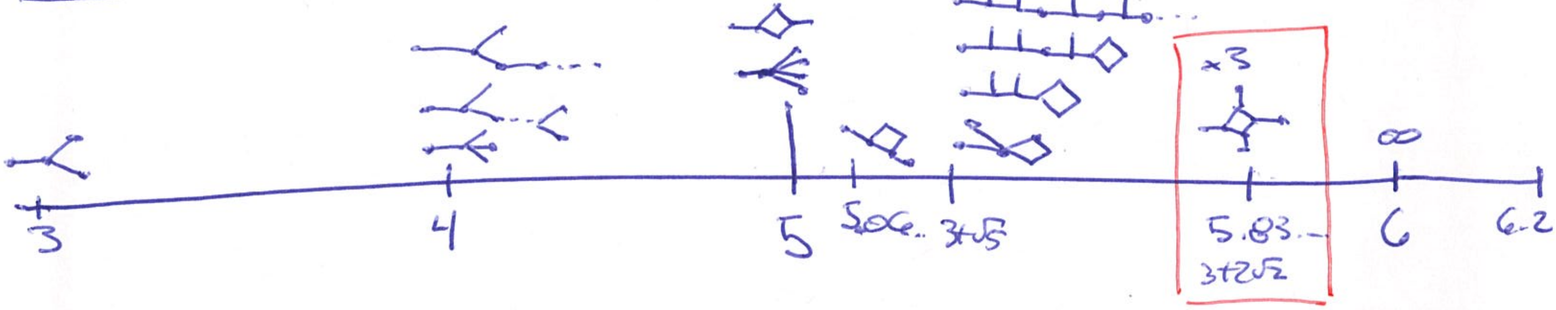
- Composite standard invariants at $2\cdot 2=4$ and $2\tau^2=3+\sqrt{5}$ are completely classified.
- at index 6, they are wild. "$N \triangleleft \mathbb{Z}/2 * \mathbb{Z}/3$"

- all 1-ST std. inv. at index 6 are composite.

Thm (Liu-Morrison-Penneys): If a standard invariant is exactly 1-ST w/o intermediate, index in $(3+\sqrt{5}, 6.2]$, then it is one of 3 at index $3+2\sqrt{2}$



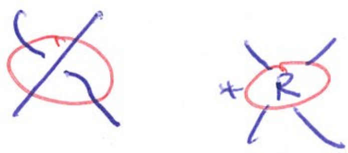
Map of 1-ST SF's:



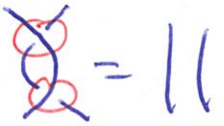
- 3 examples at $3+2\sqrt{2}$ are actually unitary fusion cats.

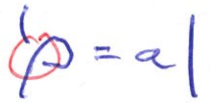
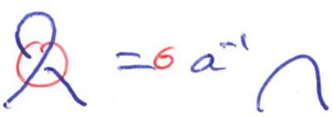
① $Rep(SO(3)_q)$ on $SU(2)_5$ ~ $(\mathfrak{g}, V, \zeta) = (\mathfrak{su}(2), V_{(2)}, e^{2\pi i/14})$

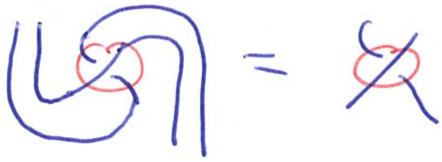
- generated by a "quadratic braiding"

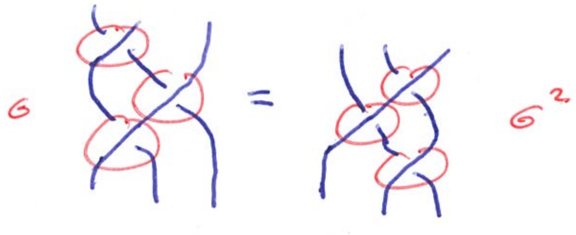


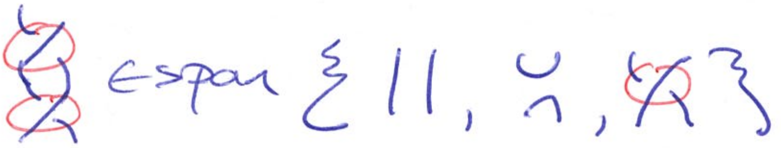
- satisfying the following relations:

(1) bi-invertible / R2  and 

(2) writhe / R1   $a \in \mathbb{C} \setminus \{0\}$

(3) rotation  "needs σ + lower order terms"

(4) braid / R3 

(5) quadratic 

- can evaluate closed diagrams via skein-template algorithm "also a Hecke alg. argument"

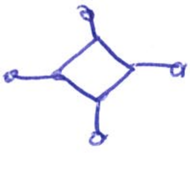
$\Rightarrow$ $\exists$ at most 1 category with these rel'ns.

(6) "σ-braided" / twisted variations. $\sigma = \pm i$

- sprinkle in some σ's in above relations.
- again, at most 1 category for $\sigma = \pm i$.

"do these also come from quantum groups? - yes, Liu's current work"

- to show existence, find representation in a graph planar algebra. (GPA)

- to show uniqueness, show that the GPA rep'n of a planar algebra w/ graph  is one of above 3 examples.