Categorified trace for module tensor categories over braided tensor categories

David Penneys, UCLA
joint with André Henriques and James Tener

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Categorified traces

A trace on a k-algebra is a linear map $\mathrm{tr} : A \to k$ such that

- $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in A$.

A categorified trace on a tensor category $\mathcal{M}$ takes values in another category $\mathcal{C}$. It's a functor $\mathrm{Tr}_{\mathcal{C}} : \mathcal{M} \to \mathcal{C}$ such that

- we have isomorphisms $\tau_{x,y} : \mathrm{Tr}_{\mathcal{C}}(x \otimes y) \xrightarrow{\simeq} \mathrm{Tr}_{\mathcal{C}}(y \otimes x)$ natural in x and y called the *traciators*
- there is a hexagon compatibility axiom:

$$\begin{array}{ccccccc}
 & & \mathrm{Tr}_{\mathcal{C}}((x \otimes y) \otimes z) & \xrightarrow{\tau} & \mathrm{Tr}_{\mathcal{C}}(z \otimes (x \otimes y)) & & \\
 & \nearrow^{\mathrm{Tr}_{\mathcal{C}}(\alpha)} & & & & \searrow^{\mathrm{Tr}_{\mathcal{C}}(\alpha)} & \\
\mathrm{Tr}_{\mathcal{C}}(x \otimes (y \otimes z)) & & & & & & \mathrm{Tr}_{\mathcal{C}}((z \otimes x) \otimes y) \\
 & \searrow_{\tau} & & & & \nearrow_{\tau} & \\
 & & \mathrm{Tr}_{\mathcal{C}}((y \otimes z) \otimes x) & \xrightarrow[\mathrm{Tr}_{\mathcal{C}}(\alpha^{-1})]{} & \mathrm{Tr}_{\mathcal{C}}(y \otimes (z \otimes x)) & &
\end{array}$$

An example of a categorified trace

Let $\mathcal{M}$ be a pivotal tensor category. Define $\mathrm{Tr}_{\mathsf{Vec}} : \mathcal{M} \to \mathsf{Vec}$ by

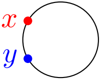
$$\mathrm{Tr}_{\mathsf{Vec}}(x) := \mathrm{Hom}_{\mathcal{M}}(1_{\mathcal{M}}, x).$$

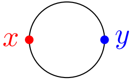
The pivotal structure gives us our natural isomorphisms:

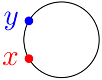
$$\begin{aligned}
\mathrm{Tr}_{\mathsf{Vec}}(x \otimes y) &= \mathrm{Hom}_{\mathcal{M}}(1_{\mathcal{M}}, x \otimes y) \\
&\cong \mathrm{Hom}_{\mathcal{M}}(y^*, x) \\
&\cong \mathrm{Hom}_{\mathcal{M}}(1_{\mathcal{M}}, y^{**} \otimes x) \\
&\cong \mathrm{Hom}_{\mathcal{M}}(1_{\mathcal{M}}, y \otimes x) \\
&= \mathrm{Tr}_{\mathsf{Vec}}(y \otimes x)
\end{aligned}$$

Diagrammatic calculus for a categorified trace

Some algebraic equations are better represented on different topological spaces. Traces on algebras can be represented on S^1:

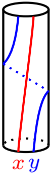
$\mathrm{tr}(xy) =$ 

$=$ 

$=$ 

$= \mathrm{tr}(yx)$

Similarly, we can denote the objects $\mathrm{Tr}_{\mathcal{C}}(x \otimes y) \cong \mathrm{Tr}_{\mathcal{C}}(y \otimes x)$ by dots on circles, but we must remember the isomorphism $\tau_{x,y}$, which we denote by a cylinder with strings:

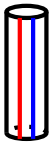
$\tau_{x,y} =$ 

$: \mathrm{Tr}_{\mathcal{C}}(x \otimes y) \xrightarrow{\simeq} \mathrm{Tr}_{\mathcal{C}}(y \otimes x)$

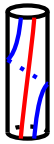
Categorified traces from central functors

When $\mathcal{M}$ is pivotal, Bezrukavnikov-Finkelberg-Ostrik showed that if $\Phi^{\mathcal{Z}} : \mathcal{C} \to \mathcal{Z}(\mathcal{M})$ is a functor such that $\Phi := F \circ \Phi^{\mathcal{Z}}$ admits a right adjoint $\mathrm{Tr}_{\mathcal{C}}$, then $\mathrm{Tr}_{\mathcal{C}}$ is a categorified trace.

The adjunction lets us construct the traciator $\tau_{x,y}$:



$$
\begin{aligned}
\in \mathcal{C}(\mathrm{Tr}_{\mathcal{C}}(x \otimes y), \mathrm{Tr}_{\mathcal{C}}(x \otimes y)) &\cong \mathcal{M}(\Phi(\mathrm{Tr}_{\mathcal{C}}(x \otimes y)), x \otimes y) \\
&\cong \mathcal{M}(\Phi(\mathrm{Tr}_{\mathcal{C}}(x \otimes y)) \otimes y^*, x) \\
(\Phi \text{ is central}) \qquad &\cong \mathcal{M}(y^* \otimes \Phi(\mathrm{Tr}_{\mathcal{C}}(x \otimes y)), x) \\
&\cong \mathcal{M}(\Phi(\mathrm{Tr}_{\mathcal{C}}(x \otimes y)), y \otimes x) \\
&\cong \mathcal{C}(\mathrm{Tr}_{\mathcal{C}}(x \otimes y), \mathrm{Tr}_{\mathcal{C}}(y \otimes x)) \ni
\end{aligned}
$$



Module categories for tensor categories

Suppose $\mathcal{C}$ is a tensor category. A module category for $\mathcal{C}$ is a category $\mathcal{M}$ together with a bifunctor $\otimes : \mathcal{C} \boxtimes \mathcal{M} \to \mathcal{M}$ satisfying certain axioms.

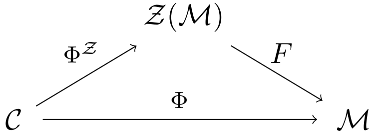
A braided tensor category $\mathcal{C}$ is a 3-category with one object and one 1-morphism, so it is one categorical dimension higher. Modules for $\mathcal{C}$ are also one categorical dimension higher.

Module tensor categories for braided tensor categories

Definition

A module tensor category for the braided tensor category $\mathcal{C}$ is a tensor category $\mathcal{M}$ and a braided functor $\Phi^{\mathcal{Z}} : \mathcal{C} \to \mathcal{Z}(\mathcal{M})$.

- We define $\Phi = F \circ \Phi^{\mathcal{Z}}$ where $F : \mathcal{Z}(\mathcal{M}) \to \mathcal{M}$ is the forgetful functor.



Example: $\mathrm{Mod}_{\mathcal{C}}(a)$

Let $\mathcal{C}$ be a braided tensor category with $a \in \mathcal{C}$ a commutative algebra object. Let $\mathcal{M} = \mathrm{Mod}_{\mathcal{C}}(a)$, the category of left a-module objects in $\mathcal{C}$.

Definition

We define the free module functor $\Phi^{\mathcal{Z}} : \mathcal{C} \to \mathcal{M}$ by $\Phi^{\mathcal{Z}}(c) = (\Phi(c), e_{\Phi(c)})$ where

- $\Phi(c) = a \otimes c$
- $e_{\Phi(c),x} : \Phi(c) \otimes_a x \cong c \otimes x \xrightarrow{\beta_{c,x}} x \otimes c \cong x \otimes_a \Phi(c)$.
- $\Phi(f : c \to d) = \mathrm{id}_a \otimes f : a \otimes c \to a \otimes d$

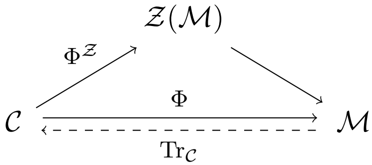
Example

Let $\mathcal{C} = SU(2)_{10}$ with objects $\mathbf{1}, \ldots, \mathbf{11}$. Let $a = \mathbf{1} \oplus \mathbf{7}$. Then $\mathcal{M} = \mathrm{Mod}_{\mathcal{C}}(a)$ is the E_6 module tensor category.

Categorified trace for module tensor categories

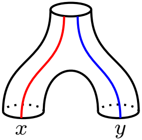
We now assume $\mathcal{C}$ is a pivotal braided tensor category, $\mathcal{M}$ is a pivotal module tensor category, and $\Phi^{\mathcal{Z}} : \mathcal{C} \to \mathcal{Z}(\mathcal{M})$ is a braided pivotal functor.

Under mild assumptions, $\Phi : \mathcal{C} \to \mathcal{M}$ has a right adjoint $\mathrm{Tr}_{\mathcal{C}}$, which is a categorified trace.



Adjoints of tensor functors are lax monoidal

It is well known that the right adjoint of a tensor functor is lax monoidal, so we get canonical natural multiplication maps

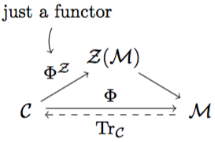
$\mu_{x,y} =$ 

$: \mathrm{Tr}_{\mathcal{C}}(x) \otimes \mathrm{Tr}_{\mathcal{C}}(y) \to \mathrm{Tr}_{\mathcal{C}}(x \otimes y)$

Theorem

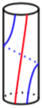
The traciators and multiplication maps are compatible, giving us a graphical caluclus of srings on tubes which can branch and braid.

Graphical calculus

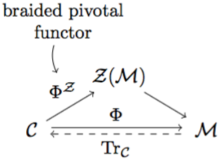


$\Downarrow$

$\mathrm{Tr}_{\mathcal{C}} : \mathcal{M} \to \mathcal{C}$ is a categorified trace [BFO09, Ost14]

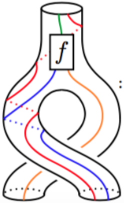


$: \mathrm{Tr}_{\mathcal{C}}(x \otimes y) \to \mathrm{Tr}_{\mathcal{C}}(y \otimes x)$

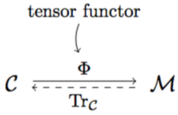


$\Downarrow$

$\mathrm{Tr}_{\mathcal{C}} : \mathcal{M} \to \mathcal{C}$ admits a full-fledged calculus of strings on tubes.

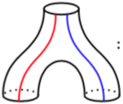


$: \mathrm{Tr}_{\mathcal{C}}(w) \otimes \mathrm{Tr}_{\mathcal{C}}(x \otimes y) \to \mathrm{Tr}_{\mathcal{C}}(z \otimes x)$



$\Downarrow$

$\mathrm{Tr}_{\mathcal{C}} : \mathcal{M} \to \mathcal{C}$ is a lax monoidal functor [Kel74].



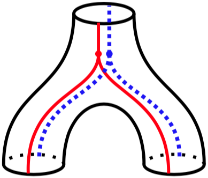
$: \mathrm{Tr}_{\mathcal{C}}(x) \otimes \mathrm{Tr}_{\mathcal{C}}(y) \to \mathrm{Tr}_{\mathcal{C}}(x \otimes y)$

Application: constructing algebras in $\mathcal{C}$

Since tensor functors are lax monoidal, given an algebra $a \in \mathcal{M}$, $\mathrm{Tr}_{\mathcal{C}}(a)$ is an algebra in $\mathcal{C}$.

Theorem

If $a, b \in \mathcal{M}$ are algebras, then $\mathrm{Tr}_{\mathcal{C}}(a \otimes_{\mathcal{M}} b)$ is an algebra in $\mathcal{C}$.



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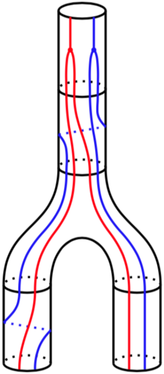


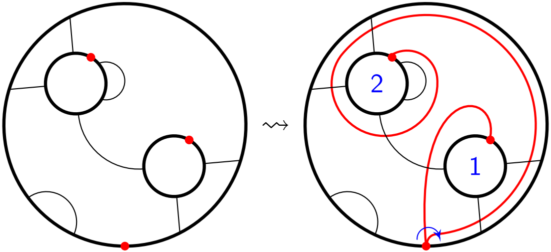
Application: anchored planar algebras

We can use the internal trace to construct planar algebras *internal* to a braided tensor category.

Definition

A planar algebra in Vec is a sequence of objects $\mathcal{P}_n \in \text{Vec}$ and an action of planar tangles.

An anchored planar algebra in $\mathcal{C}$ is a sequence of objects $\mathcal{P}_n \in \mathcal{C}$ and an action of anchored planar tangles.



$\mathcal{P}_5 \otimes_{\text{Vec}} \mathcal{P}_3 \to \mathcal{P}_6$ $\rightsquigarrow$ $\mathcal{P}_3 \otimes_{\mathcal{C}} \mathcal{P}_5 \to \mathcal{P}_6$

Thank you for listening!

Slides available at:
`http://www.math.ucla.edu/~dpenneys/PenneysAMS2016.pdf`

Article available at:
`http://arxiv.org/abs/1509.02937`