Q-system completion for C^* 2-categories

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Brazos Analysis Seminar

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Overview

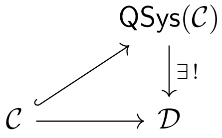
- Unitary tensor categories (UTCs) encode quantum symmetry and act on operator algebras via unitary tensor functors

$$\mathbf{H} : \mathcal{C} \to \mathsf{Bim}(A) = \mathrm{End}(\mathsf{Mod}(A))$$

- A (...) subfactor $N \subset M$ can be viewed a triple $(\mathcal{C}, \mathbf{H}, \mathbf{A})$ where $\mathcal{C}$ is a UTC, $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}(N)$ is an action, and $A \in \mathcal{C}$ is an (...) algebra object.

$$N \subset N \rtimes_{\mathbf{H}} A = M$$

- Q-systems in UTCs are particularly nice algebra objects where the above construction is easy. They are *higher idempotents*, and we can take a *higher idempotent completion*.



Unitary (multi)tensor categories

A monoidal category $\mathcal{C}$ is called a *unitary multitensor category* if:

- (linear) hom spaces $\mathcal{C}(a \to b)$ finite dimensional vector spaces
- (Karoubi complete) admits finite direct sums, and all idempotents split
- (C^*) For all $a, b \in \mathcal{C}$, $\dagger : \mathcal{C}(a \to b) \to \mathcal{C}(b \to a)$ such that $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ and $f^{\dagger\dagger} = f$, and all endomorphism algebras are C^* algebras under $\dagger$.
- (tensor) $\dagger$-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ $((f \otimes g)^\dagger = f^\dagger \otimes g^\dagger)$ with *unitary* $(u^{-1} = u^\dagger)$ coherence isomorphisms α, λ, ρ
- (rigid) every object admits left and right duals

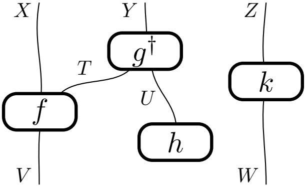
We call $\mathcal{C}$ a *unitary tensor category* if the unit is simple.
We call $\mathcal{C}$ a *unitary (multi)fusion category* if there are only finitely many isomorphism classes of simple objects.

Fact

Every UMC is *semisimple*, i.e., every object is a finite direct sum of simples $(\mathrm{End}_{\mathcal{C}}(c) = \mathbb{C})$

2D graphical calculus for UMCs

0. Objects denoted by labelled strands, oriented bottom to top.
1. 1-morphisms denoted by coupons



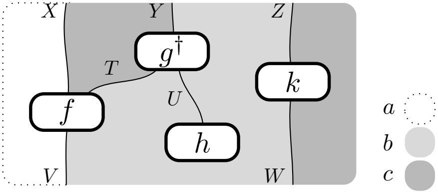
$f : V \to X \otimes T$
$g : Y \to T \otimes U$
$h : 1_{\mathcal{C}} \to U$
$k : W \to Z$

- vertical stacking is composition
- horizontal juxtaposition is $\otimes$
- vertical reflection is $\dagger$
- suppress unit $1_{\mathcal{C}}$ and all coheretors α, λ, ρ

2D graphical calculus for $\mathrm{C}^*/\mathrm{W}^*$ 2-categories

A tensor category is a 2-category with one object. For 2-categories, we have a dimension shift.

0. shadings for regions to denote objects
1. 1-morphisms denoted by strands
2. 2-morphisms denoted by coupons



$T : c \to b$
$U : b \to b$
$V : a \to b$
$W : b \to c$
$X : a \to c$
$Y : c \to b$
$Z : b \to c$

$f : V \Rightarrow X \otimes T$
$g : Y \Rightarrow T \otimes U$
$h : 1_c \Rightarrow U$
$k : W \Rightarrow Z$

Where do UTCs come from?

1. Subfactor standard invariants $A \subset B \rightsquigarrow \mathcal{C}(A \subset B)$
2. Compact groups $G \rightsquigarrow \mathsf{Rep}(G)$
3. Discrete/compact quantum groups (Tannaka-Krein duality)

$$\mathbb{G} \qquad \rightsquigarrow \qquad (\mathsf{Rep}(\mathbb{G}), \mathbf{F} : \mathsf{Rep}(\mathbb{G}) \to \mathsf{Hilb})$$

4. Generators and relations [VV19]
5. Constructions of new UTCs from existing UTCs

Many people care about UTCs because of physics

- conformal field theory ($\mathsf{Rep}(\mathcal{A})$ of a conformal net)
- unitary fusion categories give Turaev-Viro TQFTs
- unitary modular categories give Reshetikhin-Turaev TQFTs
- topological phases of matter (UMTCs)

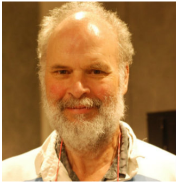
Subfactors

- A II_1 *factor* is an infinite dimensional von Neumann algebra with trivial center and a trace. (Eg: $L\Gamma := \mathbb{C}[\Gamma]'' \subset B(\ell^2\Gamma)$)
- A II_1 *subfactor* is a unital inclusion of type II_1 factors.

Jones' Index Rigidity Theorem [Jon83]

The index $[B : A] := \dim({}_A L^2 B)$ of a II_1 subfactor $A \subset B$ takes values in:

$$[B : A] \in \left\{4\cos^2(\pi/n)\middle|n \geq 3\right\} \cup [4, \infty].$$



Example

Given a finite index II_1 subfactor $A \subset B$, the UTC ${}_A\mathcal{C}_A$ is the category of $A - A$ bimodules generated by L^2B under

- $\bigoplus$ direct sum
- $\boxtimes$ Connes' fusion relative tensor product over A
- $\subseteq$ sub-bimodules
- $\overline{\cdot}$ conjugates

The standard invariant

Definition

The *standard invariant* of $A \subset B$ is the collection of all $A-A$, $A-B$, $B-B$, and $B-A$ bimodules generated by L^2B under

- $\oplus$ direct sum
- $\boxtimes$ Connes' fusion relative tensor product (over A or B)
- $\subseteq$ sub-bimodules
- $\overline{\cdot}$ conjugates.

We can think of this as a 2×2 UMC of bimodules of $A \oplus B$

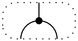
$$\mathcal{C} = \mathcal{C}(A \subset B) := \begin{pmatrix} {}_A\mathcal{C}_A & {}_A\mathcal{C}_B \\ {}_B\mathcal{C}_A & {}_B\mathcal{C}_B \end{pmatrix} \subset \mathsf{Bim}(A \oplus B)$$

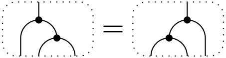
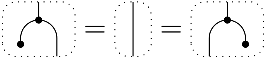
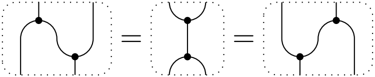
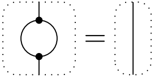
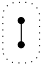
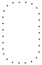
with the *generator* ${}_AL^2B_B$.

- If there are only finitely many isomorphism classes of simple bimodules, we call $A \subset B$ and $\mathcal{C}$ *finite depth*.

Alternate definition via Q-systems

Alternatively, we can define the standard invariant as the UTC ${}_A\mathcal{C}_A$ of $A - A$ bimodules generated by L^2B with the *Q-system* ${}_A L^2 B_A$.

 $: L^2B \boxtimes_A L^2B \xrightarrow{\text{multiplication}} L^2B$  $: L^2A \xrightarrow{\text{unit}} L^2B$

- (associative) 
- (unital) 
- (Frobenius) 
- (separable) 
- (minimal/standard)  $= \dim_{\min}(Q)$ 

Classification of subfactors/UTCs

Example

The subfactor $R \subset R \rtimes G$ for a finite group G 'remembers' G. So classifying hyperfinite subfactors is hopeless. We must restrict to some notion of 'smallness.'

Strategy for small index classification:

1. Classify possible standard invariants with $\dim({}_A L^2 B_B)$ small
2. Determine how many subfactors give each standard invariant.

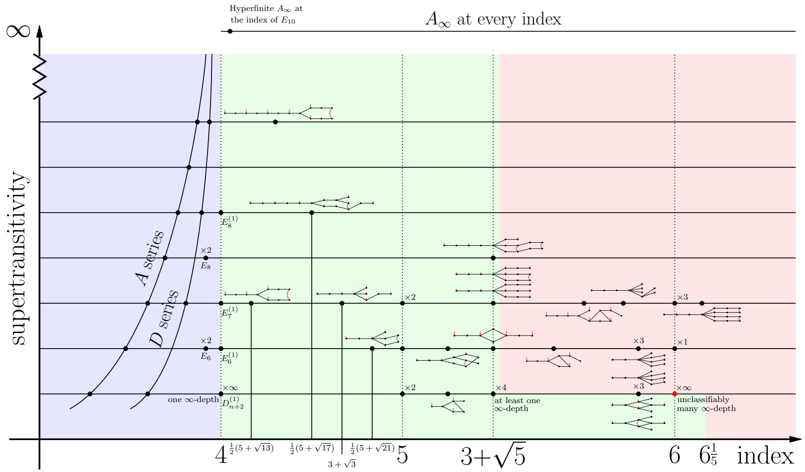
Popa's Subfactor Reconstruction Theorem [Pop90, Pop95]

Every standard invariant comes from a subfactor. If the standard invariant is *strongly amenable* (eg: finite depth), the subfactor can be taken to be hyperfinite.

Theorem [BHP12], cf. [PS03]

Every UTC admits a fully faithful embedding into $\mathsf{Bim}_{\mathsf{ext}}(L\mathbb{F}_\infty)$.

Known small index standard invariants



Theorem [AMP15, Liu15]

We know all standard invariants up to index $5\frac{1}{4} > 3+\sqrt{5}$, the first interesting composite index.

Amenability

Amenability arises in *two places* when subfactors can be classified:

1. We restrict to subfactors of the amenable II_1 factor R
2. We embed amenable unitary tensor categories into Bim(R).

Question

How many ways can $\mathrm{Ad}(A_3 * A_4)$ embed into Bim(R)?

Question

How many ways can $TLJ(d)$ embed into Bim(R) for $d > 2$?

Beyond small index classification

What new directions can we go in?

- Infinite index
- Horizontal categorification
- Vertical categorification
- Ask higher categorical questions in this context
- Actions of unitary tensor categories on C^*-algebras

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- Ask higher categorical questions in this context
  - Q-system completion is a higher idempotent completion
- Actions of unitary tensor categories on C^*-algebras
  - Use Q-system completion to induce new actions from existing actions

Discrete subfactors

With Corey Jones [JP19], we characterize the class of extremal irreducible discrete subfactors $(N \subset M, E)$ with N type II_1 with trace τ and $E : M \to N$ a f.n. conditional expectation.

- (discrete) Setting $\phi := \tau \circ E$, ${}_N L^2(M,\phi)_N$ decomposes as a direct sum of dualizable $N - N$ bimodules (generates a UTC!)
- (irreducible) $N' \cap M = \mathrm{End}_{N-M}(L^2(M,\phi)) = \mathbb{C}$
- (extremal) For every $N - N$ sub-bimodule ${}_N K_N \subset {}_N L^2(M,\phi)_N$, $\dim({}_N K) = \dim(K_N)$.

Examples

- Any finite depth, finite index irreducible II_1 subfactor is automatically extremal and discrete.
- If $\alpha : \Gamma \curvearrowright N$ is an outer action of a discrete countable group, then $N \subset N \rtimes_\alpha \Gamma$ is an extremal irreducible discrete subfactor.

Characterization of discrete subfactors

Such a subfactor $(N \subset M, E)$ can be viewed as a triple $(\mathcal{C}, \mathbf{A}, \mathbf{H})$:

1. Unitary tensor category $\mathcal{C}$,
2. Connected W^* algebra object $\mathbf{A} \in \mathsf{Vec}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}^{\mathsf{op}} \to \mathsf{Vec})$ ($\mathsf{Vec}(\mathcal{C})$ is a model for $\mathrm{ind}(\mathcal{C}^{\natural})$, where $\natural$ means forget $\dagger$),
3. Fully faithful unitary tensor functor $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}_{\mathsf{ext}}(N)$ which lands in extremal $N - N$ bimodules.

The *standard invariant* of $(N \subset M, E)$ is the pair $(\mathcal{C}, \mathbf{A})$.

W^* algebra objects

Definition

A connected W^* algebra object $\mathbf{A} = \underline{\mathrm{End}}_{\mathcal{C}}(m)$ for some simple object m in some $\mathcal{C}$-module $\mathrm{C}^*/\mathrm{W}^*$-category ${}_{\mathcal{C}}\mathcal{M}$.

$$\mathbf{A}(c) := \mathcal{M}(c \triangleright m \to m) \in \mathsf{Vec}$$

Example

For an irreducible extremal discrete subfactor $(N \subset M, E)$ and $K \in \mathcal{C} = \langle {}_N L^2(M, \phi)_N \rangle$,

$$\begin{aligned}\mathbf{A}(K) : &= \mathrm{Hom}_{N-N}(K \to L^2(M, \phi)) \\ &\cong \mathrm{Hom}_{N-M}(K \boxtimes_N L^2(M, \phi) \to L^2(M, \phi)).\end{aligned}$$

Theorem [JP19]

Fix a unitary tensor category $\mathcal{C}$ and a fully faithful unitary tensor functor $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}_{\mathsf{ext}}(N)$ where N is a II_1 factor. There is an equivalence of categories

$$\left\{ \begin{array}{l} \text{Connected } \mathrm{W}^* \text{ algebra} \\ \text{objects } \mathbf{A} \in \mathsf{Vec}(\mathcal{C}) \\ \text{with ucp morphisms} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Extremal irreducible discrete inclu-} \\ \text{sions } (N \subseteq M, E) \text{ supported on} \\ \mathbf{H}(\mathcal{C}) \text{ with normal } N-N \text{ bilinear} \\ \text{ucp maps preserving } \tau \circ E \end{array} \right\}$$

- This effectively splits subfactor classification into 2 parts:
  1. Classify embeddings of unitary tensor categories $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}(N)$
  2. Classify connected W^* algebra objects in $\mathsf{Vec}(\mathcal{C})$.

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- Generalizes all known Galois correspondences for intermediate subfactors. (finite groups: [NT60], discrete groups: [ILP98], compact quantum groups: [Tom09])

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- Generalizes all known Galois correspondences for intermediate subfactors. (finite groups: [NT60], discrete groups: [ILP98], compact quantum groups: [Tom09])
- Gives well-behaved notion of standard invariant for a large class of infinite index subfactors.

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- Gives new examples of subfactors from an embedding of $\mathcal{C}$, a $\mathcal{C}$-module $\mathrm{C}^*/\mathrm{W}^*$-category $\mathcal{M}$, and a simple object $m \in \mathcal{M}$.

Example

$\mathbf{F} : \mathcal{C} \to \mathsf{Hilb}$ a fiber functor (discrete quantum group) and $m = \mathbb{C}$. M is type II_1 iff $(\mathcal{C}, \mathbf{F})$ is Kac-type; otherwise M is type III!

Realization [JP19, CPJP]

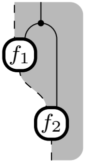
The main tool we provide is *realization*. Given $(\mathcal{C}, \mathbf{A}, \mathbf{H})$, we reconstruct a subfactor

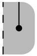
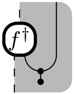
$$N = \underbrace{A(1_{\mathcal{C}})}_{\mathbb{C}} \otimes \underbrace{\mathbf{H}^\circ(1_{\mathcal{C}})}_{N} \subset \overline{\bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} \mathbf{A}(c) \otimes \underbrace{\mathbf{H}^\circ(c)}_{\text{bdd. vects.}}}^{\mathrm{W}^*} =: \begin{cases} N \rtimes_{\mathbf{H}} \mathbf{A} \\ |\mathbf{A}|_{\mathbf{H}} \end{cases}$$

This is much easier when $\mathbf{A}$ is a Q-system in $\mathcal{C}$ rather than a W^*-algebra object in $\mathsf{Vec}(\mathcal{C})$. In this case,

$$|\mathbf{A}|_{\mathbf{H}} = \mathbf{H}(\mathbf{A})^\circ := \mathrm{Hom}_{-N}(L^2 N \to \mathbf{H}(\mathbf{A}))$$

is easily equipped with the structure of a unital C^*-algebra which has a predual and is thus a von Neumann algebra:

$f_1 \cdot f_2 :=$ 

, $\quad 1_{|\mathbf{A}|_{\mathbf{H}}} :=$  , and $\quad f^* :=$ 

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Realization is a †-2-functor

With Quan Chen, Roberto Hernandez Palomares, and Corey Jones,



we extend realization to a †-2-functor in the C^* setting (proof also works in W^* setting).

- Given a $\mathrm{C}^*/\mathrm{W}^*$ 2-category $\mathcal{C}$, Q-systems, separable bimodules, and intertwiners in $\mathcal{C}$ form a $\mathrm{C}^*/\mathrm{W}^*$ 2-category $\mathsf{QSys}(\mathcal{C})$.
- Have canonical inclusion $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{QSys}(\mathcal{C})$. $\mathcal{C}$ is *Q-system complete* if $\iota_{\mathcal{C}}$ is a †-2-equivalence.
- Realization inverse †-2-functor $|\cdot| : \mathsf{QSys}(\mathsf{C^*Corr}) \to \mathsf{C^*Corr}$. $\mathsf{C^*Corr}$ is Q-system complete (as is $\mathsf{W^*Corr} \simeq \mathsf{vNA}$).

Idempotent completion example: K-theory

Recall the definition of $K_0(A)$ for a untial C^*-algebra.

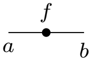
1. Look at the C^*-category $\mathsf{Mod}_{\mathsf{fgp}}(A)$ of finitely generated projective A-modules.
2. $\mathsf{Mod}_{\mathsf{fgp}}(A)$ admits all finite direct sums.
3. $K_0(A) := K_0(\mathsf{Mod}_{\mathsf{fgp}}(A))$, the Grothendieck group of $\mathsf{Mod}_{\mathsf{fgp}}(A)$.

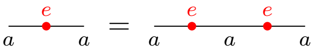
- $\mathsf{Mod}_{\mathsf{fgp}}(A)$ is *Karoubi complete*: it has finite direct sums and all projections *split*: given a projection $p \in \mathrm{End}(M_A)$, $pM_A \in \mathsf{Mod}_{\mathsf{fgp}}(A)$.

$$M_A \underset{i}{\overset{p}{\rightleftarrows}} pM_A \qquad\qquad p \circ i = \mathrm{id}_{pM} \qquad i \circ p = p$$

Here, p is a *retract* and i is a *section*.

Idempotents and condensation

- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

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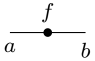
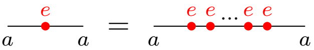
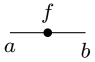
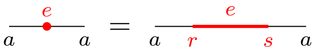
- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

Idempotents and condensation

- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

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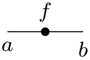
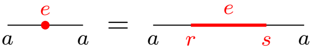
- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

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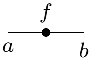
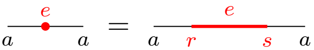
- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

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- We can do this process again; starting with $\mathsf{Vec}_{\mathsf{fd}}$, we can deloop to get $\mathsf{Vec}_{\mathsf{fd}}$ as a *tensor* category. We then 'higher' Karoubi complete (unital separable algebra object completion) to obtain $2\mathsf{Vec}_{\mathsf{fd}}$, the 2-category of multimatrix algebras, finite dim bimodules, and intertwiners.

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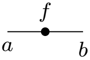
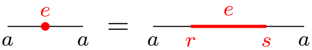
- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

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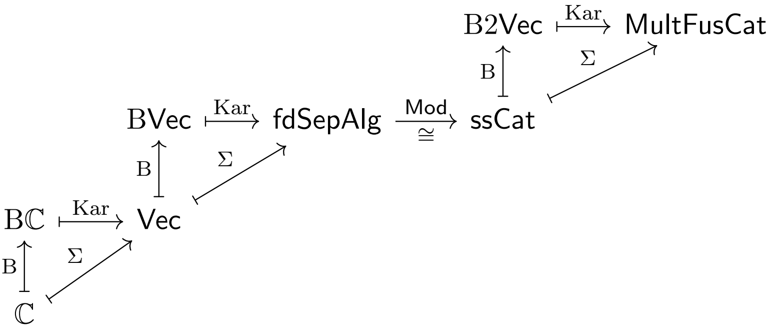
- The next step yields $3\mathsf{Vec}$, the 3-category of multifusion categories! [GJF19, JF20]

Higher Karoubi completion

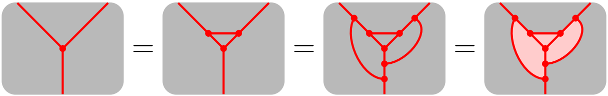
Multifusion categories arise *freely** from higher Karoubi completion.



- B means take the *delooping* [BS10, §5.6], i.e., consider the monoidal k-category as a $(k+1)$-category with one object.
- Kar means take higher Karoubi completion [GJF19].
- Σ is the composite $\mathrm{Kar} \circ \mathrm{B}$, called the *suspension*.

Q-systems are higher categorical idempotents

A Q-system is a unitary (co)unital *higher categorical idempotent*.



Now strands and tri/univalent vertices can replicate freely.

Warning: Unitary condensation (in progress with Reutter and Steinebrunner) is extremely nuanced, and one may not want to use Q-systems!

Definition based on
[Yam04, EGNO15, BKLR15, CR16, NY16, DR18, GY20]

The *Q-system completion* $\mathsf{QSys}(\mathcal{C})$ of a $\mathrm{C}^*/\mathrm{W}^*$ 2-category $\mathcal{C}$ has

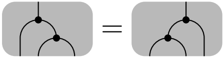
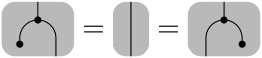
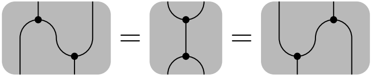
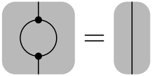
- objects are Q-systems,
- 1-morphisms are unitarily separable bimodules, and
- 2-morphisms are intertwiners.

Q-systems

Recall that a Q-system in a $\mathrm{C}^*/\mathrm{W}^*$ 2-category $\mathcal{C}$ is a 1-morphism $Q \in \mathcal{C}(b \to b)$ together with

 $: Q \otimes_b Q \xrightarrow{\text{multiplication}} Q$  $: 1_b \xrightarrow{\text{unit}} Q$

such that the following relations hold:

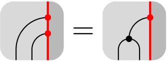
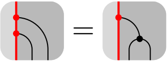
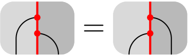
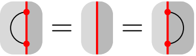
- (associative) 
- (unital) 
- (Frobenius) 
- (unitarily separable) 
- (non-degenerate)  $\in \mathrm{End}_{\mathcal{C}}(1_b)^\times$

Frobenius actually follows from associative, unital, and unitarily separable by [LR97]; see [BKLR15, Lem. 3.7].

Unitarily separable bimodules

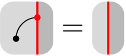
Suppose $P \in \mathcal{C}(a \to a)$, $Q \in \mathcal{C}(b \to b)$ are Q-systems and $X \in \mathcal{C}(a \to b)$.

 $: P \otimes_a X \xrightarrow{\text{left action}} X$  $: X \otimes_b Q \xrightarrow{\text{right action}} X$

- (bimod) , , and 
- (unitarily separable) 

[BKLR15, Lem. 3.23]

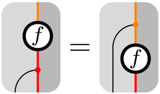
A unitarily separable $P - Q$ bimodule ${}_P X_Q$ over Q-systems P, Q is automatically unital and Frobenius:

- (unital)  and 
- (Frobenius)  and .

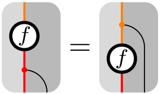
Intertwiners

Definition

If $P \in \mathcal{C}(a \to a)$ and $Q \in \mathcal{C}(b \to b)$ are Q-systems and ${}_aX_b, {}_aY_b \in \mathcal{C}(a \to b)$ are $P - Q$ bimodules, we define $\mathrm{QSys}(\mathcal{C})({}_PX_Q \Rightarrow {}_PY_Q)$ as the set of $f \in \mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$ such that



and

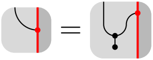


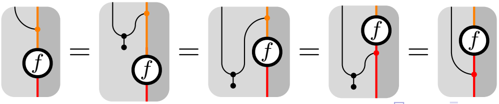
.

Lemma

$f^\dagger \in \mathcal{C}({}_aY_b \Rightarrow {}_aX_b)$ is also a $P - Q$ bimodule map.

Proof.

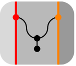
Step 1: 

Step 2: Apply † to 

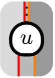
. □

Composition of 1-morphisms

To compose the $P - Q$ bimodule ${}_aX_b$ and the $Q - R$ bimodule ${}_bY_c$, we unitarily split the separability projector

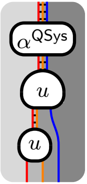
$p_{X.Y}$:=  :=  =  $= u^\dagger_{X,Y} u_{X,Y}$

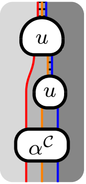
for a coisometry $u_{X,Y}$, unique up to unique unitary.

 $= X \otimes_Q Y$ 

$= u_{X,Y}$.

As in [NY16, Rem. 2.6], associator $\alpha^{\mathsf{QSys}(\mathcal{C})}$ uniquely determined by



$=$ 

$: (X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z \to X \otimes_Q (Y \otimes_R Z)$.

Theorem [CPJP] cf. [GY20]

C*Corr, W*Corr, vNA are Q-system complete (Q-systems split).

Corollary [CPJP] cf. [GY20]

Can induce action $\mathcal{C} \to \mathsf{Bim}(A) \subset \mathsf{R} \in \{\mathsf{C^*Corr}, \mathsf{W^*Corr}, \mathsf{vNA}\}$

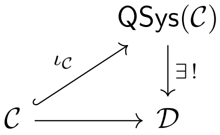
$$\mathsf{QSys}(\mathcal{C}) \to \mathsf{QSys}(\mathsf{Bim}(A)) \to \mathsf{QSys}(\mathsf{R}) \xrightarrow{\cong} \mathsf{R}$$

Followup results with Quan Chen:

Theorem [CP] cf. [DR18]

QSys is a 3-functor on $\mathrm{C}^*/\mathrm{W}^*$ 2-categories.

Universal property for Q-system completion [CP] cf. [DR18]



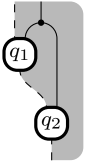
for every †-2-functor from $\mathcal{C}$ to a Q-system complete $\mathcal{D}$.

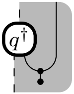
Main idea for C*Corr Q-system complete

Realization $|\cdot| : \mathsf{QSys}(\mathsf{C^*Corr}) \to \mathsf{C^*Corr}$ is inverse †-2-functor to natural inclusion $\iota : \mathsf{C^*Corr} \to \mathsf{QSys}(\mathsf{C^*Corr})$.

Definition

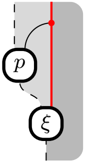
Q-system $Q \in \mathsf{C^*Corr}(B \to B)$ maps to $|Q| := \mathrm{Hom}_{\mathbb{C}-B}(B \to Q)$

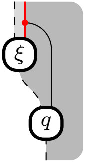
$q_1 \cdot q_2 :=$ 

, $1_{|Q|} :=$ 
, and $q^* :=$ 

.

For $P - Q$ bimod ${}_AX_B$, define $|X| := \mathrm{Hom}_{\mathbb{C}-B}(B \to A \boxtimes_A X)$.

$p \rhd \xi :=$ 

$\xi \lhd q :=$ 

$\forall f \in |P|$,
$\forall \eta \in |M|$, and
$\forall g \in |Q|$.

Induced actions on C^*-algebras

Theorem [Jon20]

Every pointed unitary fusion category $\mathsf{Hilb}_{\mathsf{fd}}(G, \omega)$ admits an action on $C(X)$ where X is some 'nice' compact Hausdorff space (e.g. closed connected n-manifold for $n \geq 2$).

- Can use our results to induce actions of group-theoretical unitary fusion categories on unital C^*-algebras with connected spectrum.
- Unlike actions on II_1 factors, there are K-theoretic obstructions to unitary fusion category actions on C^*-algebras.

Thank you for listening!

Slides available at:

`https://people.math.osu.edu/penneys.2/PenneysBrazos2021.pdf`

Articles in preparation, expected Summer 2021:

- Q-system completion for C^* 2-categories (with Quan Chen, Roberto Hernandez Palomares, and Corey Jones)
- Q-system completion is a 3-functor (with Quan Chen)

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