Subfactors of index $3+\sqrt{5}$

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Invariants of subfactors

$$\begin{tikzcd}[column sep=huge]
\text{subfactor} & \text{planar algebra} & \text{principal graphs} \\[-2em]
A \subset B \arrow[r, bend left=40] & (P_+, P_-) \arrow[l, bend left=40] \arrow[r, bend left=40] & (\Gamma+, \Gamma_-) \arrow[l, dotted, bend left=40, "??"]
\end{tikzcd}$$

Subfactors

Theorem [Jon83]

For a II_1-subfactor $A \subset B$,

$$[B\colon A] \in \left\{4\cos^2\left(\frac{\pi}{n}\right)\middle|\, n = 3, 4, \ldots\right\} \cup [4, \infty].$$

Definition

The Jones tower of $A = A_0 \subset A_1 = B$ (finite index) is given by

$$A_0 \subset A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \overset{e_3}{\subset} \cdots$$

where e_i is the projection in $B(L^2(A_i))$ with range $L^2(A_{i-1})$.

Two towers of centralizer algebras

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \cup & & \cup & & \\
P_{3,+} & = & A_0'\cap A_3 & \supset & A_1'\cap A_3 & = & P_{2,-}\\
 & & \cup & & \cup & & \\
P_{2,+} & = & A_0'\cap A_2 & \supset & A_1'\cap A_2 & = & P_{1,-}\\
 & & \cup & & \cup & & \\
P_{1,+} & = & A_0'\cap A_1 & \supset & A_1'\cap A_1 & = & P_{0,-}\\
 & & \cup & & & & \\
P_{0,+} & = & A_0'\cap A_0 & & & &
\end{array}$$

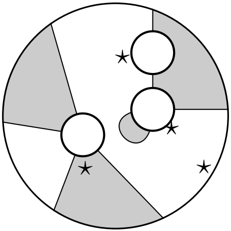
These centralizer algebras form a planar algebra.

Planar algebras [Jon99]

Definition

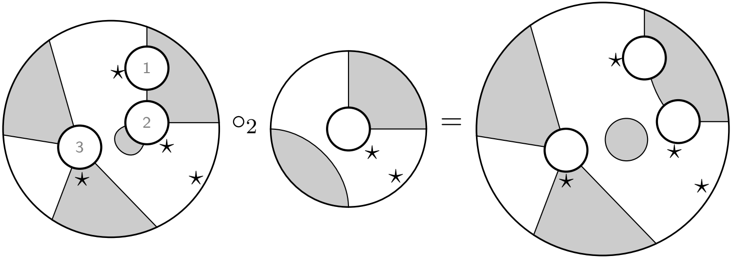
A shaded planar tangle has

- a finite number of inner boundary disks
- an outer boundary disk
- non-intersecting strings
- a marked interval $\star$ on each boundary disk



Composition of tangles

We can compose planar tangles by insertion of one into another if the number of strings matches up:



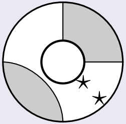
Definition

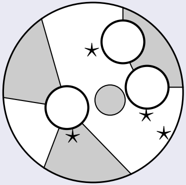
The *shaded planar operad* consists of all shaded planar tangles (up to isotopy) with the operation of composition.

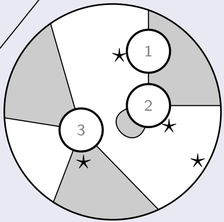
Definition

A *planar algebra* is a family of vector spaces $P_{k,\pm}$, $k = 0, 1, 2, \ldots$ and an action of the shaded planar operad.

$$\begin{tikzcd} P_{2,-} \times P_{1,+} \times P_{1,+} \arrow[rr] \arrow[dr] & & P_{3,+} \\ & P_{2,-} \times P_{2,+} \times P_{1,+} \arrow[ur] & \end{tikzcd}$$





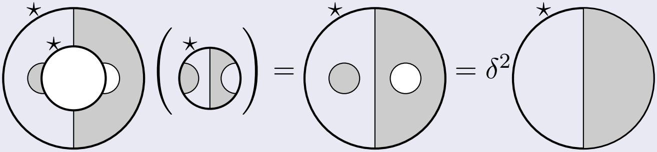


Example: Temperley-Lieb

$TL_{n,\pm}(\delta)$ is the complex span of non-crossing pairings of $2n$ points arranged around a circle, with formal addition and scalar multiplication.

$TL_{3,+}(\delta) = \mathrm{Span}_{\mathbb{C}} \Big\{$  $\Big\}$.

Planar tangles act on TL by inserting diagrams into empty disks, smoothing strings, and trading closed loops for factors of δ.

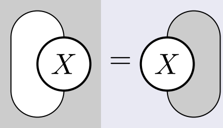


Subfactor planar algebras

Definition

A planar algebra $P_\bullet$ is a subfactor planar algebra if it is:

- Finite dimensional: $\dim(P_{k,\pm}) < \infty$ for all k
- Evaluable: $\dim(P_{0,\pm}) = 1$
- Sphericality:



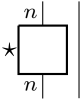
- Positivity: each $P_{k,\pm}$ has an adjoint $*$ such that the sesquilinear form $\langle x, y\rangle := \mathrm{Tr}(y^* x)$ is positive definite

From these properties, it follows that closed circles count for a multiplicative constant δ.

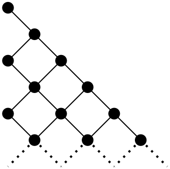
Principal graphs

The complex $*$-algebras $P_{n,\pm}$ are all finite dimensional. The tower

$$P_{0,+} \subset P_{1,+} \subset P_{2,+} \subset \cdots$$

where the inclusion is given by 

is described by its Bratteli diagram (and the trace).



Principal graphs

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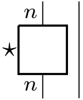
where the inclusion is given by 

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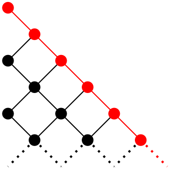


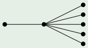
The non-reflected part is the principal graph Γ.

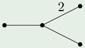
Finite depth

Definition

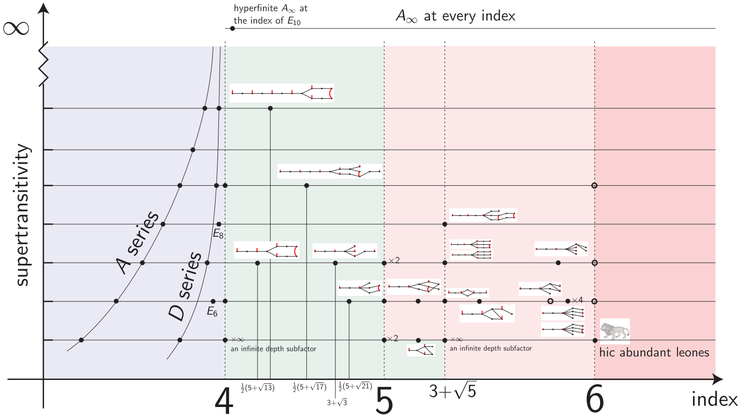
If the principal graph is finite, then the subfactor and planar algebra are called finite depth.

Example: $R \subset R \rtimes S_3$

Principal graph: 

Dual principal graph: 

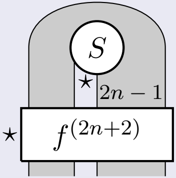
Known subfactors

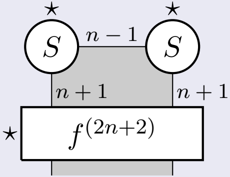


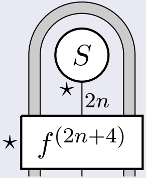
- See Jones-Morrison-Snyder survey [JMS13]

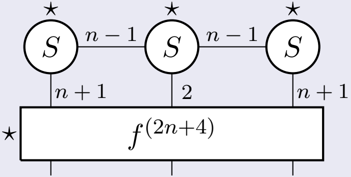
Bigelow-Morrison-Peters-Snyder, [BMPS12]

The Haagerup and extended Haagerup subfactor planar algebras have a generator $S \in P_{n,+}$ where $n = 4, 8$ respectively satisfying:

- 

$= i\dfrac{\sqrt{[n][n+2]}}{[n+1]}$


- 

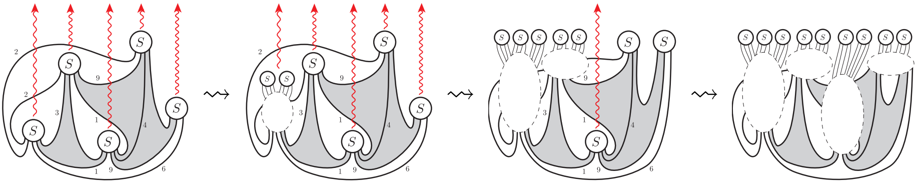
$= \dfrac{[2][2n+4]}{[n+1][n+2]}$


- (Absorption) capping S gives zero and $S^2 = f^{(n)} \in TL_{n,+}$.

The jellyfish algorithm

We can evaluate all closed diagrams as follows:

1. First, pull all generators to the outside using the jellyfish relations



2. Second, reduce the number of generators using the capping and absorption (multiplication) relations.

Consistency and positivity

Theorem [Jones-Penneys [JP11], Morrison-Walker]

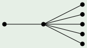
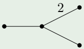
Every subfactor planar algebra embeds in the graph planar algebra of its principal graph.

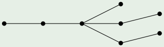
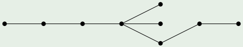
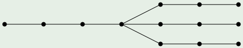
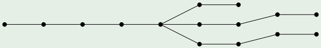
This serves two purposes:

1. To show the planar algebra is non-zero, give a representation.
2. Graph planar algebras are always finite dimensional, spherical, and positive. Only need to check evaluable.

Spoke graphs

Examples of spoke principal graphs

- $A_n, D_{2n}, E_6, E_8,$
- $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$
- $A_\infty, A_\infty^{(1)}, D_\infty$
- Principal graphs for $R \subset R \rtimes G$, G finite ( , 

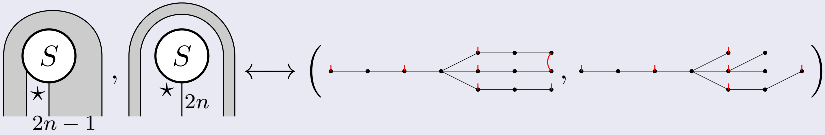
)
- 2221 
- 3311 
- 3333 
- 4442 

Spokes and jellyfish

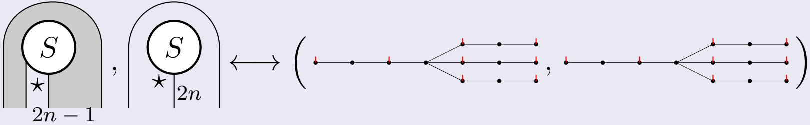
Assume all generators of $P_\bullet$ are at the same depth n.

Theorem [Bigelow-Penneys [BP13]]

- $P_\bullet$ has 2-strand jellyfish relations $\Leftrightarrow$ one graph is a spoke.



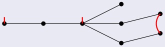
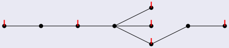
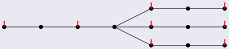
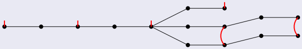
- $P_\bullet$ has 1-strand jellyfish relations $\Leftrightarrow$ both graphs are spokes.



Constructing spoke subfactors with jellyfish

Theorem [Morrison-Penneys [MP13]]

We automate finding 1-strand relations for these subfactors:

- Izumi-Xu 2221  [Han11]
- [GdlHJ89] 3311 
- Izumi $3^{\mathbb{Z}/2\times\mathbb{Z}/2}$  $(3+\sqrt{5})$
- 4442  $(3+\sqrt{5})$

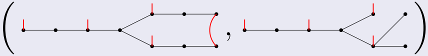
For the above, both principal graphs are the same spoke graph.

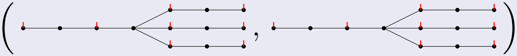
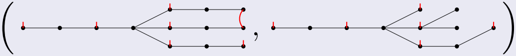
Constructing spoke subfactors with jellyfish, part 2

Theorem [Penneys-Peters (in preparation)]

We give explicit 2-strand relations for the following subfactors:

- 333 $(\mathbb{Z}/3)$, AKA Haagerup

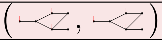
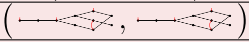


- 3333 $(\mathbb{Z}/2 \times \mathbb{Z}/2)$ 
- 3333 $(\mathbb{Z}/4)$ 

Index $(5, 3+\sqrt{5})$

Conjecture [Morrison-Peters] [MP12]

There are exactly two non Temperley-Lieb subfactor planar algebras in the index range $(5, 3+\sqrt{5})$:

name	Principal graphs	Index	Constructed
$SU(2)_5$		5.04892	[Wen90], [MP12]
$SU(3)_4$		5.04892	[Wen88], [MP12]

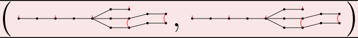
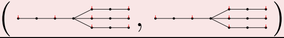
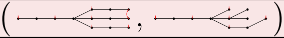
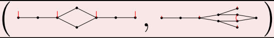
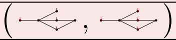
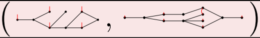
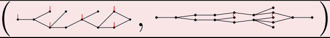
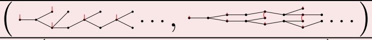
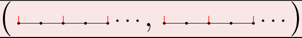
Theorem [Morrison-Peters] [MP12]

There is exactly one 1-supertransitive subfactor in the index range $(5, 3+\sqrt{5})$

Subfactor planar algebras at $3+\sqrt{5}$

Conjecture [Morrison-Penneys]

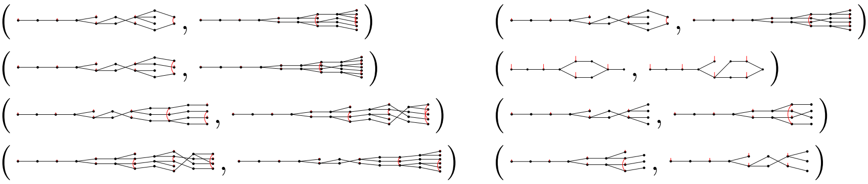
At $3+\sqrt{5}$, we have the following subfactor planar algebras:

name	Principal graphs	#	Constructed
4442		2	[MP13], Izumi
$3^{\mathbb{Z}/2\times\mathbb{Z}/2}$		2	Izumi, [MP13]
$3^{\mathbb{Z}/4}$		2	Izumi, [PP13]
$2D2$		2	Izumi, [MPP13]
$A_3 \otimes A_4$		1	$\otimes$
fish 2		2	BH
fish 3		2	[IMP13]
$A_3 * A_4$		2	[BJ97]
A_∞		1	[Pop93]

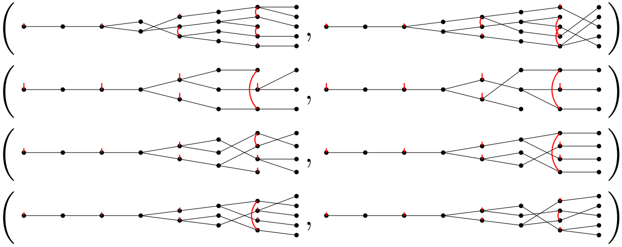
How can we prove these conjectures?

Biggest hurdle: need to eliminate certain weeds.

$*10$ weeds:

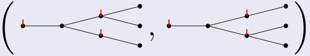


$*11$ weeds:

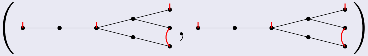


New obstruction

Theorem [Penneys, last week]

(1) If (Γ_+, Γ_-) is a translated extension of
, then

$$(\check{r}-1)\frac{r}{\check{r}} + \frac{(\sigma_S + \sigma_S^{-1})}{[n]}\frac{\sqrt{r}}{\sqrt{\check{r}}} = \frac{r[n]-[n+2]}{[n]}.$$

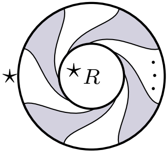
(2) If (Γ_+, Γ_-) is a translated extension of
, then

$$(r-1) + \frac{(\sigma_S + \sigma_S^{-1})}{[n]} = \frac{[n+2]-r[n]}{r[n]}.$$

σ_S is the chirality (σ_S^2 is rotational eigenvalue)
$r, \check{r}$ are the branch factors (ratio of dimensions past branch)

Remaks on the new obstruction

- The obstruction is far more general. Recovers Jones' quadratic tangles obstruction and Snyder's single valent obstruction.
- Proven by analyzing rotation after using Liu's relation, which is a clever manipulation of Wenzl's relation.



- Can obtain rotational eigenvalues for most small index subfactors.

Thank you for listening!

arXiv preprints available at:

with Bigelow - Spokes and jellyfish - to appear Math. Ann. - `arXiv:1208.1564`
with Morrison - Constructing spokes - to appear Trans. AMS - `arXiv:1208.3637`
with Peters - coming soon!
new obstruction - coming soon!

Dietmar Bisch and Vaughan F. R. Jones, *Algebras associated to intermediate subfactors*, Invent. Math. **128** (1997), no. 1, 89–157, MR1437496.

Stephen Bigelow, Scott Morrison, Emily Peters, and Noah Snyder, *Constructing the extended Haagerup planar algebra*, Acta Math. **209** (2012), no. 1, 29–82, MR2979509, arXiv:0909.4099, DOI:10.1007/s11511-012-0081-7. MR 2979509

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