A categorical Connes' $\chi(M)$

joint w/ Quan Chen + Corey Jones

- M is a II_1 factor $!\mathrm{tr}$

Connes' $\chi(M)$ (1975):

- Abelian gp associated to M

Def: A <u>central sequence</u> in M is $(x_n) \subset M$ s.t. $\|x_n y - y x_n\|_2 \to 0$

- elts of $M' \cap M^\omega$

Examples:

- R lots
- $L\mathbb{F}_n$ trivial $M' \cap M^\omega = \mathbb{C}$

Call $\theta \in \mathrm{Aut}(M)$:

① <u>centrally trivial</u> if $\forall (x_n)$ central seq. in M, $\|\theta(x_n) - x_n\|_2 \to 0$.

$Ct(M) =$ cent. triv. aut's

② <u>approximately inner</u> if

<u>inner</u>: $\mathrm{Ad}(u)$ for $u \in \mathcal{U}(M)$

$\overline{\mathrm{Int}(M)} \subset \mathrm{Aut}(M)$

$\chi(M) :=$ image of $Ct(M) \cap \overline{\mathrm{Int}(M)}$ in $\mathrm{Out}(M) := \mathrm{Aut}(M)/\mathrm{Int}(M)$.

Jones' k-invariant (1980): a quadratic form on $\mathcal{X}(M)$

Eilenberg + Mac Lane (1954): a quad. form on an abelian gp $\longleftrightarrow$ elt of $H^3_{ab}(A, \mathbb{C}^*)$

$\rightsquigarrow$ pointed braided tensor cat!

$\hookrightarrow$ all objects invertible

$\forall x \ \exists y$ s.t. $x \otimes y = 1$

Today's Goal: extend Connes' $\mathcal{X}(M)$ to a unitary BTC of M-M bimods which are not nec. invertible.

- extend $[\mathcal{X}(M), \kappa] \rightsquigarrow \tilde{\mathcal{X}}(M)$

Why is this surprising?

- vNa's form a 2-category $\mathrm{End}(M) = \mathrm{Bim}(M)$

  obj vNa's

  1-mor bimods

  2-mor bdd intertwiners

Previous results:

Kawahigashi (1993): Relative $\chi(M,N)$ for II_1-subfactors.

Thm (K'93): $N \subset M$ AFD irred. finite index, finite depth subfactor. $|\chi(M,N)| \leq |\chi(N)/\kappa(N)| \cdot |\mathrm{Irr}(M \subset M_1)|$

- Computations for many fundamental examples.

Popa (1994): Gives a def'n of AI and CT for M-M bimods
[uses language of endo's of $M \otimes B(\ell^2)$]
$\rightsquigarrow$ full W^*-tensor subcat $\tilde{\chi}(M) \subset \mathrm{Bim}(M)$.

A first problem to solve about $\tilde{\chi}(M^\infty)$ and $\tilde{\chi}(M)$ is whether or not, like in the case of the usual $\chi(M)$, they are commutative tensor categories. Also, it is an open question whether or not $\tilde{\chi}(M^\infty)$ is selfadjoint or not, i.e., does $\rho \in \tilde{\chi}(M^\infty)$ implies $\bar{\rho} \in \tilde{\chi}(M^\infty)$?

Still open.

Masuda+Tomatsu (2009): AI+CT for endomorphisms, focus on hyperfinite type III factors.

- Lem 4.7: For ~~$\theta \in$~~ $\mathrm{Aut}(M)$ AI and $\rho \in \mathrm{End}(M)$ CT,

$$[\theta \circ \rho \circ \theta^{-1}] = [\rho] \text{ in } \mathrm{Sect}(M)$$

Theorem 1 (CJP): There is a canonical unitary braiding on $\tilde{\mathcal{X}}(M)$.

Still open: don't know if $H \in \tilde{\mathcal{X}}(M)$
$\overset{?}{\Longrightarrow} \overline{H} \in \tilde{\mathcal{X}}(M)$

Note:

① $\tilde{\mathcal{X}}(M)$ extends $\mathcal{X}(M)$:

$$\mathcal{X}(M) = \{ H \in \tilde{\mathcal{X}}(M) \mid \dim({}_M H) = \dim(H_M) = 1 \}$$

↳ paired UBTC in $\tilde{\mathcal{X}}(M)$ braiding $\to$ K.

② $\tilde{\mathcal{X}}(M)$ is a Morita invariant

- if $M \underset{ME}{\simeq} N$ $\leftarrow$ II_1-factor

$\tilde{\mathcal{X}}(M) \simeq \tilde{\mathcal{X}}(N)$

$[\Longleftrightarrow {}_M H_N$ s.t. $(N^{op})' = M]$

Question: Can we calculate $\tilde{\mathcal{X}}(M)$?

Problem: $\tilde{\mathcal{X}}(M)$ might be too large

- want dualizability to apply quantum algebra techniques.

Def: Suppose $\mathcal{C}$ is a semisimple linear monoidal cat. Call $c \in \mathcal{C}$ a fusion / finite depth object if c generates a fusion category under $\otimes, \oplus, \subseteq$.

Fact: If $c \in \mathcal{F}$ fusion cat, then $\bar{c}$ appears under $c^{\otimes n}$ for some n.

- c finite depth $\Rightarrow$ $\bar{c} \in \mathcal{C}$ $\quad \overset{1}{\cap}_c \; w_c$ $\quad c \overset{}{\underset{1}{\cup}} \bar{c}$ coev $\leadsto$ ${}^c\!\mathrm{id}_c = 1_c$

Lemma: Suppose $\mathcal{B}$ is a semisimple braided tensor cat and $a, b \in \mathcal{B}$ are finite depth.

Then $a \otimes b$ is again finite depth.

Cor: If $\mathcal{C}, \mathcal{D} \overset{\text{full}}{\subset} \mathcal{B}$ fusion, then

$\langle \mathcal{C}, \mathcal{D} \rangle \subset \mathcal{B}$ is again fusion

Def: $\tilde{\mathcal{X}}_{fus}(M) = \langle$ finite depth bimods in $\tilde{\mathcal{X}}(M) \rangle$

Examples of $\tilde{\mathcal{X}}_{fus}(M)$:

- R
- N non-Gamma
- $N \otimes R$

$\Big\}$ trivial!

Note: can also get $\tilde{\mathcal{X}}_{fus}(R)$ from Masuda + Tomatsu.

Warning: Can have $N \subset M$ finite index, finite depth w/ $\tilde{\chi}_{fus}$ radically different!

Thm (Popa 2009): Suppose $N \subset M$ finite index, finite depth, non-Gamma II_1 subfactor.

$M_0 = N \subset M_1 \subset M_2 \subset M_3 \subset \cdots \subset M_\infty$ ind. limit II_1

$$\chi(M_\infty) = \chi(S, T) \quad \text{where } T \subset S \text{ is}$$

the AFD II_1 subfactor w/ same std. inv.

Thm 2 (GJP): Suppose $N \subset M$ finite index, finite depth, non-Gamma II_1 subfactor.

Let $\mathcal{C} = \mathrm{StdInv}(N \subset M)$. Then

$$\tilde{\chi}_{fus}(M_\infty) \simeq Z(\mathcal{C}) \; [\simeq Z({}_N\mathcal{C}_N) \simeq Z({}_M\mathcal{C}_M)]$$

where $\mathcal{C}$ is the 2×2 unitary multi FC

$$\begin{bmatrix} {}_N\mathcal{C}_N & {}_N\mathcal{C}_M \\ {}_M\mathcal{C}_N & {}_M\mathcal{C}_M \end{bmatrix}$$

of bimods gen by L^2M.

Cor: Combining the results of Popa + Kawahigashi, we can calculate Connes' $\chi(M_\infty)$ and the corresponding κ-invariant.

Observe: $\underbrace{N \otimes R}_{\text{trivial } \tilde{\chi}_{tus}} \subset \underbrace{M_\infty}_{\tilde{\chi}_{tus} = \mathbb{Z}(\mathcal{C})}$ fin. fin depth.

essentially the assymptotic inclusion for $\underbrace{N_1 \subset N}_{\text{tunnel}} \subset M \subset \cdots \subset \underset{N_\infty}{\overset{\shortparallel}{M_\infty}}$

We'll now give defn of $\tilde{\chi}(M)$.

Recall: for a bimod ${}_M H_M$,

$H^\circ := \{ \xi \in H \mid \underset{\text{bdd}}{L_\xi : M_{L^2} \to H \text{ by } m \mapsto \xi m} \}$

- on H°, $\langle \xi \mid \eta \rangle_M := L_\xi^* L_\eta \in (M^{op})' \cap \mathcal{B}(L^2 M) = M$

$\mathrm{Bim}(M) \ni H \longmapsto H^\circ \subset W^*\mathrm{Corr}(M \to M)$

Def of $\mathcal{X}(M)$: Call ${}_M H_M$: essentially Popa's def's. from 1994

- Centrally trivial (CT) if

$\forall$ central seq. $(x_n) \subset M$, $\forall \xi \in H^\circ$,

$$\| x_n \xi - \xi x_n \|_H \to 0$$

- approximately inner (AI) if

① $\exists$ approximate H_M-basis $(\{\beta_i^{(n)}\}_{i=1}^{m(n)})$

where $m(n) < L$ $\forall n$ s.t.

- $\| \langle \beta_i^{(n)} | \beta_i^{(n)} \rangle_M \|_\infty < K$ $\forall i, \forall n$

basis for H^ω M^ω

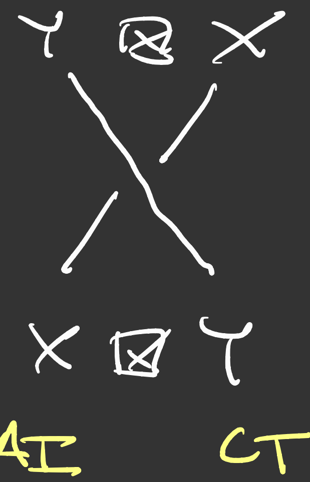
- $\| \xi - \sum_i \beta_i^{(n)} \langle \beta_i^{(n)} | \xi \rangle_M \|_H \to 0$

② approx H_M-basis satisfies

$\forall x \in M$, $\| x \beta_i^{(n)} - \beta_i^{(n)} x \| \to 0$.

basis M-central

$\tilde{\mathcal{X}}(M) := CT \cap AI$



braiding on $\tilde{\mathcal{X}}(M)$:

<u>X</u>: $(\{\beta_i^{(n)}\}_{i=1}^{n})_n$ satisfying ① + ②

<u>Y</u>: pick any Y_M-basis $\{\gamma_j\}$

$$\times := \lim_n \sum_{i,j} \underbrace{|\gamma_j \boxtimes \beta_i^{(n)}\rangle}_{Y \boxtimes X}\underbrace{\langle \beta_i^{(n)} \boxtimes \gamma_j|}_{X \boxtimes Y}$$

Must verify:

- limit exists pointwise / sot
- indep of choices
- left M-linear
- unitary
- braiding coherences

We do this in greater generality:

$\mathcal{C}$ C^*/W^*-Cat w/ "metric structure" $\Big\}$ AI CT for cts endo's.

$\mathcal{C} = \mathrm{Mod}_{fgp}(M)$

$\|\cdot\|_2$ on homs

Steps of pf of Thm 2 [$\tilde{\mathcal{X}}_{fus}(M_\infty) = Z(\mathcal{C})$]

Step 1: Construct a braided $\otimes$-functor

$$Z(\mathcal{C}) \longrightarrow \tilde{\mathcal{X}}_{fus}(M_\infty).$$

nondeg $\rightsquigarrow$ fully faithful.

Step 2:

Recall: by Longo (1984, 89, ...) there is a correspondence between

$$\left\{ \begin{array}{c} \text{com. Q-systems} \\ A \in \mathrm{Bim}(M) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irred. finite} \\ \text{index extensions} \\ M \subset |A| \end{array} \right\}$$

In the braided setting, have correspondence

$$\left\{ \begin{array}{c} \text{com. commutative} \\ \text{Q-systems } B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irred. finite} \\ \text{index local} \\ \text{extensions} \\ M \subset |B| \end{array} \right\}$$

Thm (CJP): For $B \in \tilde{\mathcal{X}}_{fus}(M)$ a connected, commutative Q-system,

$$\tilde{\mathcal{X}}(|B|) = \tilde{\mathcal{X}}(M)_B^{loc}, \quad \tilde{\mathcal{X}}_{fus}(|B|) \supseteq \tilde{\mathcal{X}}_{fus}(M)_B^{loc}$$

<u>Step 3:</u> Show $\mathcal{Z}(\mathcal{C}) \hookrightarrow \tilde{\mathcal{X}}_{fus}(M_\infty)$ is essentially surjective.

- If $X \in \mathcal{Z}(\mathcal{C})' \subset \tilde{\mathcal{X}}_{fus}(M_\infty)$,

  $\langle \mathcal{Z}(\mathcal{C}), X \rangle$ is a BFC.

  $L :=$ canonical Lagrangian in $\mathcal{Z}(\mathcal{C})$.

$$\tilde{\mathcal{X}}_{fus}(M_\infty)^{loc}_{L} \subset \tilde{\mathcal{X}}_{fus}(|L|) \simeq \underbrace{\tilde{\mathcal{X}}_{fus}(R \otimes \Omega)}_{\text{trivial!}}$$

$$R \otimes \Omega \subset M_\infty \subset |L|$$

Morita equiv!

$\Rightarrow X$ is trivial!