Constructing subfactors with the jellyfish algorithm

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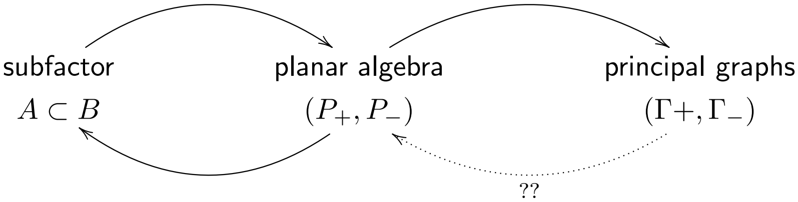
Invariants of subfactors



Subfactors

Theorem [Jon83]

For a II_1-subfactor $A \subset B$,

$$[B\colon A] \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \middle| n = 3, 4, \ldots \right\} \cup [4, \infty].$$

Definition

The Jones tower of $A = A_0 \subset A_1 = B$ (finite index) is given by

$$A_0 \subset A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \overset{e_3}{\subset} \cdots$$

where e_i is the projection in $B(L^2(A_i))$ with range $L^2(A_{i-1})$.

Two towers of centralizer algebras

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \cup & & \cup & & \\
P_{3,+} & = & A_0' \cap A_3 & \supset & A_1' \cap A_3 & = & P_{2,-} \\
 & & \cup & & \cup & & \\
P_{2,+} & = & A_0' \cap A_2 & \supset & A_1' \cap A_2 & = & P_{1,-} \\
 & & \cup & & \cup & & \\
P_{1,+} & = & A_0' \cap A_1 & \supset & A_1' \cap A_1 & = & P_{0,-} \\
 & & \cup & & & & \\
P_{0,+} & = & A_0' \cap A_0 & & & &
\end{array}$$

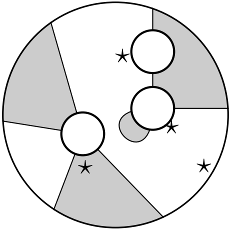
These centralizer algebras form a planar algebra.

Planar algebras [Jon99]

Definition

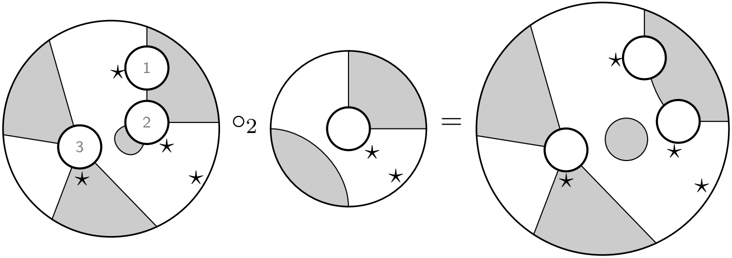
A shaded planar tangle has

- a finite number of inner boundary circles
- an outer boundary circle
- non-intersecting strings
- a marked point $\star$ on each boundary circle



Composition of tangles

We can compose planar tangles by insertion of one into another if the number of strings matches up:



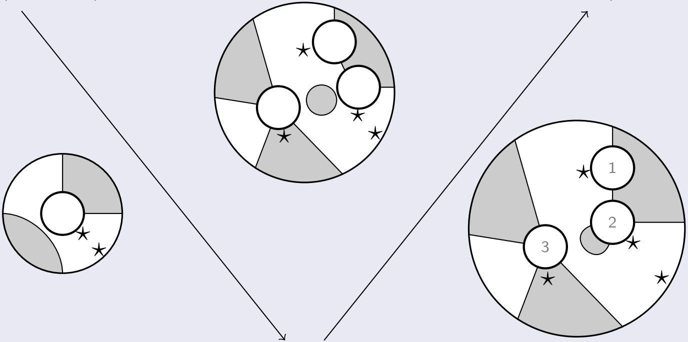
Definition

The *shaded planar operad* consists of all shaded planar tangles (up to isotopy) with the operation of composition.

Definition

A *planar algebra* is a family of vector spaces $V_{k,\pm}$, $k = 0, 1, 2, \ldots$ which are acted on by the shaded planar operad.

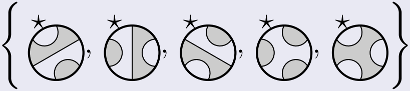
$$V_{2,-} \times V_{1,+} \times V_{1,+} \longrightarrow V_{3,+}$$



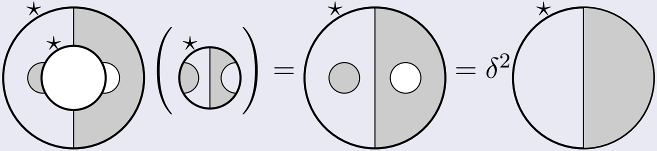
$$V_{2,-} \times V_{2,+} \times V_{1,+}$$

Example: Temperley-Lieb

$TL_{n,\pm}(\delta)$ is the span (over $\mathbb{C}$) of non-crossing pairings of $2n$ points arranged around a circle, with formal addition.

$TL_{3,+}(\delta) = \mathrm{Span}_{\mathbb{C}}$ .

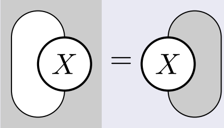
Planar tangles act on TL by inserting diagrams into empty disks, smoothing strings, and trading closed loops for factors of δ.



Subfactor planar algebras

Definition

A planar algebra $P_\bullet$ is a subfactor planar algebra if it is:

- Finite dimensional: $\dim(P_{k,\pm}) < \infty$ for all k
- Evaluable: $\dim(P_{0,\pm}) = 1$
- Sphericality: 
- Positivity: each $P_{k,\pm}$ has an adjoint $*$ such that the bilinear form $\langle x, y\rangle := \mathrm{Tr}(y^* x)$ is positive definite

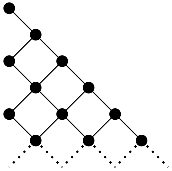
From these properties, it follows that closed circles count for a multiplicative constant δ.

Principal graphs

The complex $*$-algebras $P_{n,\pm}$ are all finite dimensional. The tower

$$P_{0,+} \subset P_{1,+} \subset P_{2,+} \subset \cdots$$

is described by its Bratteli diagram (and the trace).



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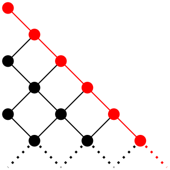


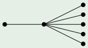
The non-reflected part is the principal graph Γ.

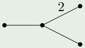
Finite depth

Definition

If the principal graph is finite, then the subfactor and planar algebra are called finite depth.

Example: $R \subset R \rtimes S_3$

Principal graph: 

Dual principal graph: 

Examples of principal graphs

Index < 4

A_n, D_{2n}, E_6, E_8

Index $= 4$

Affine Dynkin diagrams

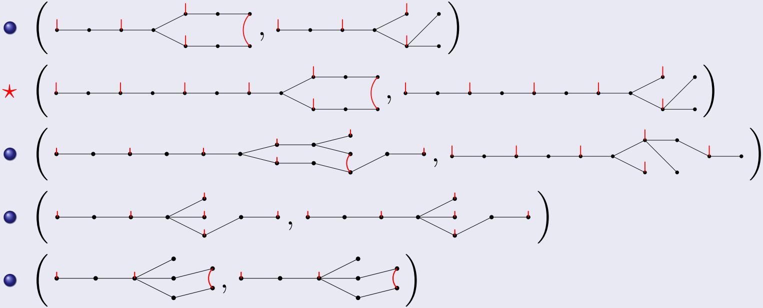
- Finite graphs $A^{(1)}_{2n-1}, D^{(1)}_{n+2}, E^{(1)}_6, E^{(1)}_7, E^{(1)}_8$.
- Infinite graphs $A_\infty, A^{(1)}_\infty, D_\infty$.

Classification to index 5

[Haa94, AH99, Bis98, AY09, BMPS09, Han11, MS11, MPPS12, IJMS11, PT12]

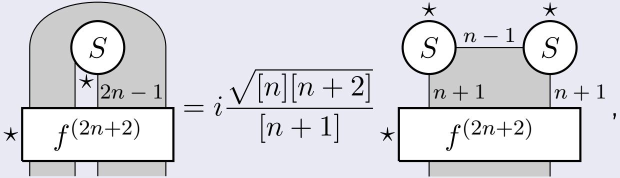
Index in $(4, 5)$

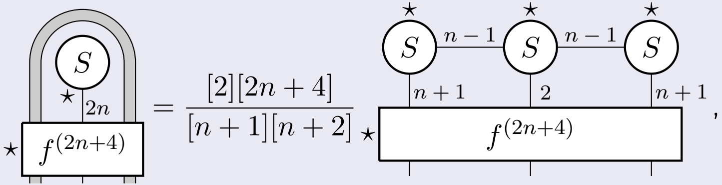
There are exactly 10 non A_∞-subfactor planar algebras in the index range $(4, 5)$:



Bigelow-Morrison-Peters-Snyder, arXiv:0909.4099

The Haagerup and extended Haagerup subfactor planar algebras have a generator $S \in P_{n,+}$ where $n = 4, 8$ respectively satisfying:

- 

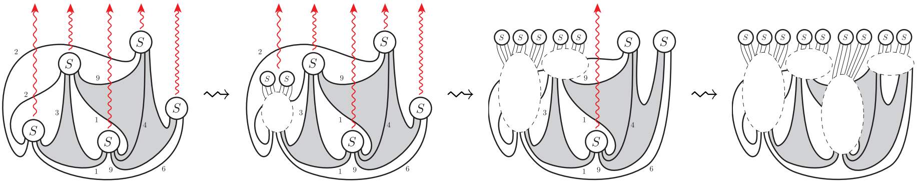
- 

- capping S gives zero, and
- (Absorption) $S^2 = f^{(n)}$.

The jellyfish algorithm

We can evaluate all closed diagrams as follows:

1. First, pull all generators to the outside using the jellyfish relations



2. Second, reduce the number of generators using the capping and absorption (multiplication) relations.

Consistency and positivity

Theorem [Jones-Penneys [JP11], Morrison-Walker]

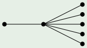
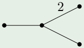
Every subfactor planar algebra embeds in the graph planar algebra of its principal graph.

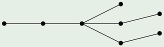
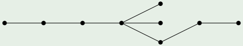
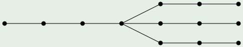
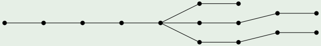
This serves two purposes:

1. To show the planar algebra is non-zero, give a representation.
2. Graph planar algebras are always finite dimensional, spherical, and positive. Only need to check evaluable.

Spoke graphs

Examples of spoke principal graphs

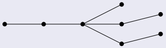
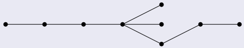
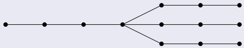
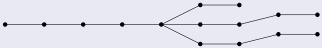
- $A_n, D_{2n}, E_6, E_8,$
- $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$
- $A_\infty, A_\infty^{(1)}, D_\infty$
- Principal graphs for $R \subset R \rtimes G$, G finite ( , 

)
- 2221 
- 3311 
- 3333 
- 4442 

Constructing spoke subfactors using jellyfish

Theorem [Morrison-Penneys `arXiv:1208.3637`]

The following subfactors can be constructed using the jellyfish algorithm. Moreover, we only need 1-strand relations.

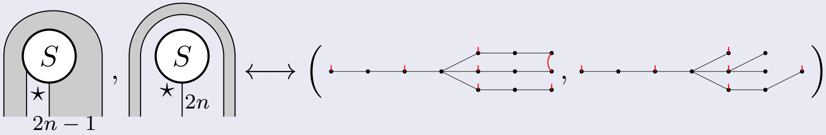
- 2221 
- 3311 
- 3333 
- 4442 

Spokes and jellyfish

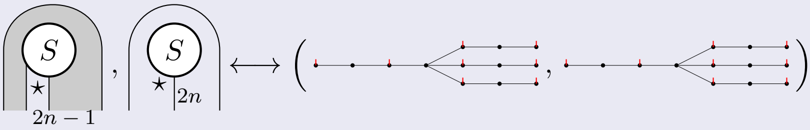
Assume all generators of $P_\bullet$ are at the same depth n.

Theorem [Bigelow-Penneys arXiv:1208.1564]

- $P_\bullet$ has 2-strand jellyfish relations $\Leftrightarrow$ one graph is a spoke.

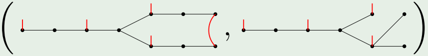
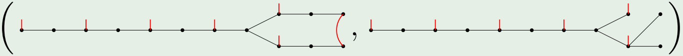
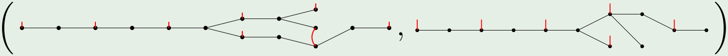
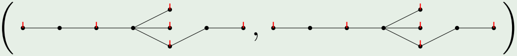
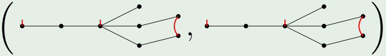


- $P_\bullet$ has 1-strand jellyfish relations $\Leftrightarrow$ both graphs are spokes.



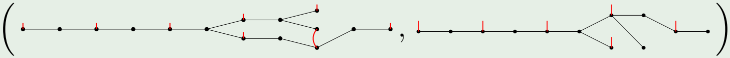
Presenting small index subfactors by jellyfish

Non-A_∞ subfactor planar algebras in $(4,5)$

- 
- 
- 
- 
- 

Presenting small index subfactors by jellyfish

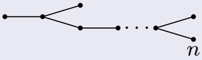
Non-A_∞ subfactor planar algebras in $(4, 5)$

- 

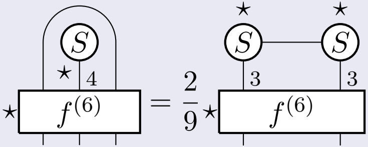
Jellyfish relations for non-spoke subfactors

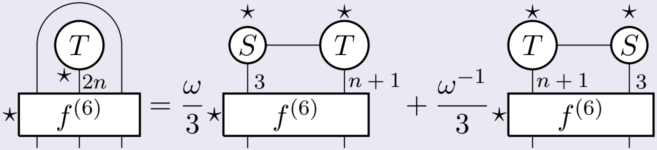
Theorem [Morrison-Penneys, in preparation]

The $D_{n+2}^{(1)}$ subfactor planar algebras



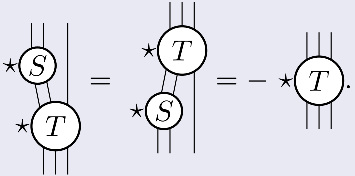
have the following presentation with a 2-box S, and an n-box T:

- 

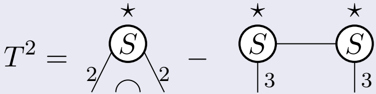
- 

$D_{n+2}^{(1)}$ relations continued

- S, T cap to zero,
- $S^2 = s + 2f^{(2)}$,
- T absorbs S (up to a scalar):



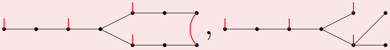
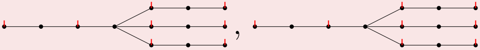
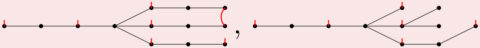
- T^2 is a diagram in jellyfish form with only S's, e.g., if $n = 3$,



Open questions

Question [Izumi [Izu01], Evans-Gannon [EG10]]

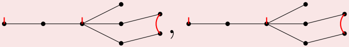
Is there an infinite family of 3^G subfactors?

- (Haagerup) $\mathbb{Z}/3$ ()
- (Izumi) $\mathbb{Z}/2 \times \mathbb{Z}/2$ ()
- (Izumi) $\mathbb{Z}/4$ ()

We can use jellyfish relations on generators at depth 4.

Question [Izumi, Evans-Gannon]

What about an infinite 2^G1 family?

- (Izumi-Xu) $\mathbb{Z}/3$ ()

Open questions 2

Question

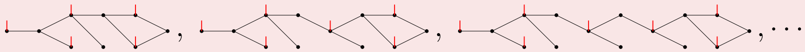
Which finite spoke graphs Γ are principal graphs of subfactors?

- $\|\Gamma\|^2$ must be a cyclotomic integer by [CG94, ENO05]!
- If you translate one spoke, eventually not cyclotomic by [CMS10].

Open questions 3

Question [Bisch-Haagerup]

Is there an infinite family of 'fish' subfactors at index $3+\sqrt{5} \approx 5.28$?



- We'd like to use techniques similar to those for the $D_{n+2}^{(1)}$'s.

Thank you for listening!

Slides available at:

`http://www.math.toronto.edu/dpenneys`

Preprints available at:

with Morrison - Constructing spokes - `arXiv:1208.3637`
with Bigelow - Spokes and jellyfish - `arXiv:1208.1564`

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