Classifying small index subfactors

Great Plains Operator Theory Symposium

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In celebration of the 60th birthday of Vaughan Jones
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Where do subfactors come from?

Some examples include:

- Groups – from $G \curvearrowright R$, we get $R^G \subset R$ and $R \subset R \rtimes_\alpha G$.
- finite dimensional unitary Hopf/Kac algebras
- Quantum groups – Rep$(\mathcal{U}_q(\mathfrak{g}))$
- Conformal field theory
- endomorphisms of Cuntz C*-algebras
- composites of known subfactors

However, there are certain possible infinite families without uniform constructions.

Remark

Just as von Neumann algebras come in pairs (M, M'), subfactors come in pairs $(A \subset B, B' \subset A')$.

Index for subfactors

Theorem [Jon83]

For a II_1-subfactor $A \subset B$,

$$[B \colon A] \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \middle| n = 3, 4, \ldots \right\} \cup [4, \infty].$$

Moreover, there exists a subfactor at each index.

Definition

The Jones tower of $A = A_0 \subset A_1 = B$ (finite index) is given by

$$A_0 \subset A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \overset{e_3}{\subset} \cdots$$

where e_i is the projection in $B(L^2(A_i))$ with range $L^2(A_{i-1})$.

Two towers of centralizer algebras

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \cup & & \cup & & \\ P_{3,+} & = & A_0' \cap A_3 & \supset & A_1' \cap A_3 & = & P_{2,-} \\ & & \cup & & \cup & & \\ P_{2,+} & = & A_0' \cap A_2 & \supset & A_1' \cap A_2 & = & P_{1,-} \\ & & \cup & & \cup & & \\ P_{1,+} & = & A_0' \cap A_1 & \supset & A_1' \cap A_1 & = & P_{0,-} \\ & & \cup & & & & \\ P_{0,+} & = & A_0' \cap A_0 & & & & \end{array}$$

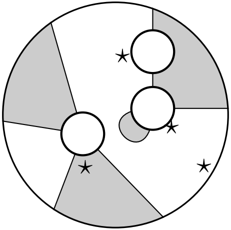
These centralizer algebras are finite dimensional [Jon83], and they form a planar algebra [Jon99].

Planar algebras [Jon99]

Definition

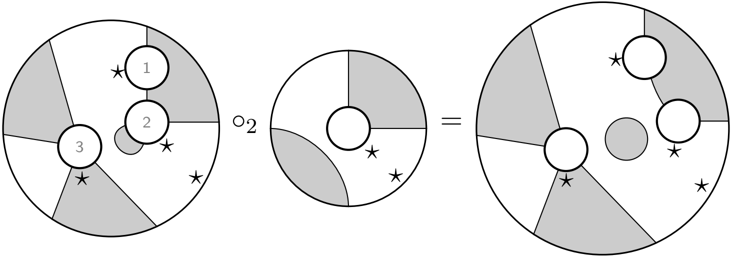
A shaded planar tangle has

- a finite number of inner boundary disks
- an outer boundary disk
- non-intersecting strings
- a marked interval $\star$ on each boundary disk



Composition of tangles

We can compose planar tangles by insertion of one into another if the number of strings matches up:

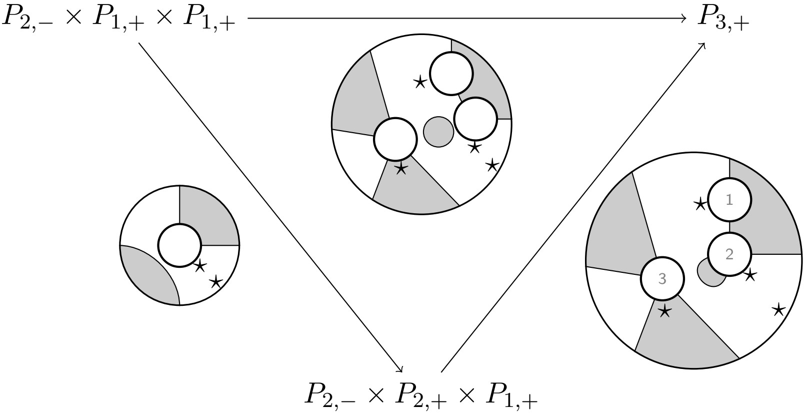


Definition

The *shaded planar operad* consists of all shaded planar tangles (up to isotopy) with the operation of composition.

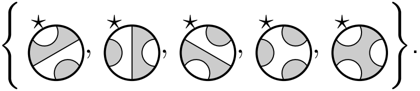
Definition

A *planar algebra* is a family of vector spaces $P_{k,\pm}$, $k = 0, 1, 2, \ldots$ and an action of the shaded planar operad.

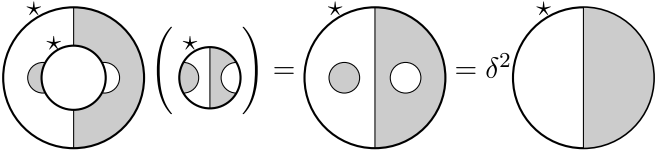


Example: Temperley-Lieb

$TL_{n,\pm}(\delta)$ is the complex span of non-crossing pairings of $2n$ points arranged around a circle, with formal addition and scalar multiplication.

$TL_{3,+}(\delta) = \mathrm{Span}_{\mathbb{C}}$ 

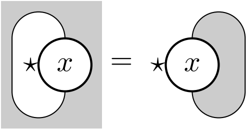
Planar tangles act on TL by inserting diagrams into empty disks, smoothing strings, and trading closed loops for factors of δ.



Subfactor planar algebras

Definition

A planar $*$-algebra $P_\bullet$ is a subfactor planar algebra if it is:

- Finite dimensional: $\dim(P_{k,\pm}) < \infty$ for all k
- Evaluable: $P_{0,\pm} \cong \mathbb{C}$ by sending the empty diagram to $1_\mathbb{C}$
- Sphericality: $\mathrm{Tr}(x) =$ 

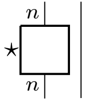
- Positivity: each $P_{k,\pm}$ has an adjoint $*$ such that the sesquilinear form $\langle x, y\rangle := \mathrm{Tr}(y^* x)$ is positive definite

From these properties, it follows that closed circles count for a multiplicative constant $\delta \in \{2\cos(\pi/n) | n \geq 3\} \cup [2, \infty)$.

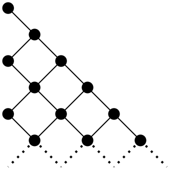
Principal graphs

The complex $*$-algebras $P_{n,\pm}$ are all finite dimensional. The tower

$$P_{0,+} \subset P_{1,+} \subset P_{2,+} \subset \cdots$$

where the inclusion is given by 

is described by its Bratteli diagram (and the trace).



Principal graphs

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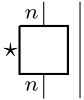
where the inclusion is given by 

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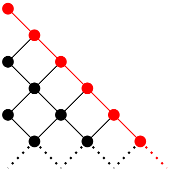
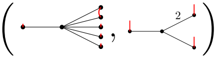
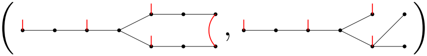
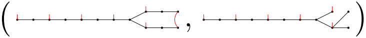


- The non-reflected part is the principal graph Γ_+.
- Get the dual principal graph Γ_- by looking at the Bratteli diagram for the tower $(P_{n,-})$.

Examples of principal graphs

- index < 4: A_n, D_{2n}, E_6, E_8. No D_{odd} or E_7.
- index $= 4$: $A^{(1)}_{2n-1}, D^{(1)}_{n+2}, E^{(1)}_6, E^{(1)}_7, E^{(1)}_8, A_\infty, A^{(1)}_\infty, D_\infty$
- Graphs for $R \subset R \rtimes G$ obtained from G and Rep(G).



- Haagerup 333 
- extended Haagerup 733 

- First graph is principal, second is dual principal.
- Leftmost vertex corresponds to $P_{0,\pm} \cong \mathbb{C}$.
- Red tags for duality of even vertices.
- Duality of odd vertices by depth and height

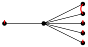
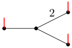
Finite depth

Definition

If the principal graph is finite, then the subfactor and standard invariant/planar algebra are called finite depth.

Example: $R \subset R \rtimes G$ for finite G

For $G = S_3$:

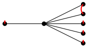
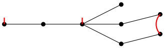
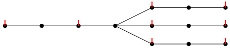
- Principal graph: 
- Dual principal graph: 

Supertransitivity

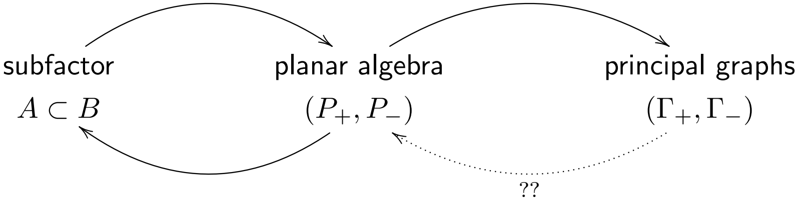
Definition

We say a principal graph is n-supertransitive if it begins with an initial segment consisting of the Coxeter-Dynkin diagram A_{n+1}, i.e., an initial segment with n edges.

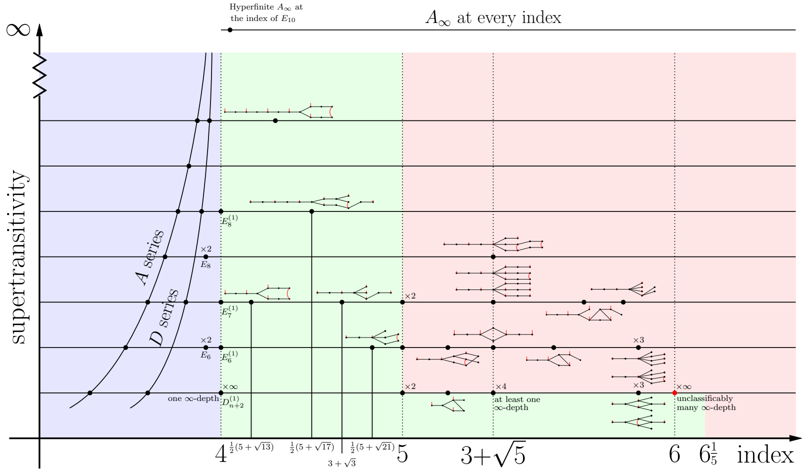
Examples

-  is 1-supertransitive
-  is 2-supertransitive
-  is 3-supertransitive

Invariants of subfactors



Known small index subfactors

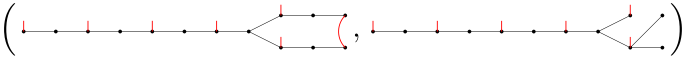


- Map of known small index subfactors modified from Jones-Morrison-Snyder Bulletin AMS survey [JMS14].

The extended Haagerup subfactor

[Bigelow-Morrison-Peters-Snyder [BMPS12]]

The extended Haagerup subfactor is the unique subfactor with principal graphs

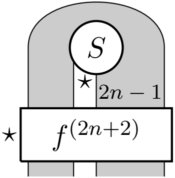


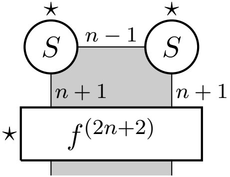
- Last remaining possible graph in Haagerup's classification to $3 + \sqrt{3}$ [Haa94] by work of Asaeda-Yasuda [AY09].
- Largest known supertransitivity outside the A and D series. High supertransitivity is exceedingly rare!
- Planar algebra constructed using Bigelow's jellyfish algorithm.

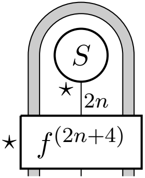
Jellyfish relations

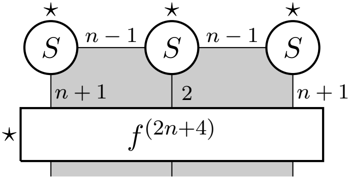
Theorem [Bigelow-Morrison-Peters-Snyder [BMPS12]]

The Haagerup and extended Haagerup subfactor planar algebras have a generator $S \in P_{n,+}$ where $n = 4, 8$ respectively satisfying:

- 

$= i\frac{\sqrt{[n][n+2]}}{[n+1]}$


- 

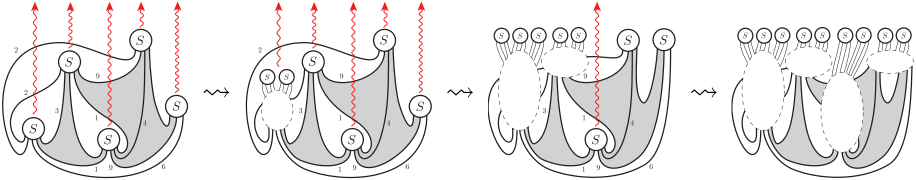
$= \frac{[2][2n+4]}{[n+1][n+2]}$


- (Absorption) capping S gives zero and $S^2 = f^{(n)} \in TL_{n,+}$.

The jellyfish algorithm

We can evaluate all closed diagrams as follows:

1. First, pull all generators to the outside using the jellyfish relations



2. Second, reduce the number of generators using the capping and absorption (multiplication) relations.

Consistency and positivity

Theorem [Jones-Penneys [JP11], Morrison-Walker]

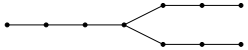
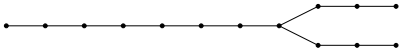
Every subfactor planar algebra embeds in the graph planar algebra of its principal graph.

This serves two purposes:

1. To show the planar algebra is non-zero, give a representation.
2. Graph planar algebras are always finite dimensional, spherical, and positive. Only need to check evaluable.

Spoke graphs

Examples of spoke principal graphs

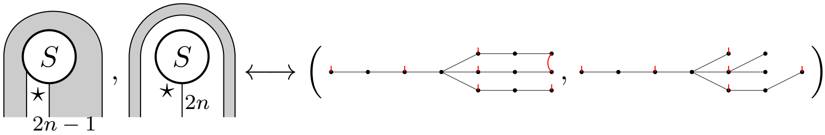
- $A_n, D_{2n}, E_6, E_8,$
- $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$
- $A_\infty, A_\infty^{(1)}, D_\infty$
- Principal graphs for $R \subset R \rtimes G$, G finite ( , )
- 2221 
- Haagerup 333 
- 3311 
- 3333 
- 4442 
- extended Haagerup 733 

Spokes and jellyfish

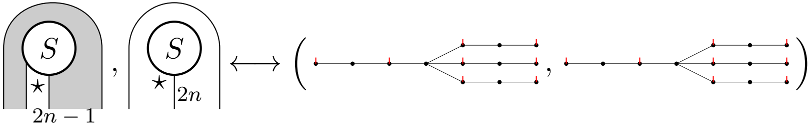
Assume all generators of $P_\bullet$ are at the same depth n.

Theorem [Bigelow-Penneys [BP14]]

- $P_\bullet$ has 2-strand jellyfish relations $\Leftrightarrow$ one graph is a spoke.



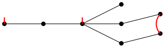
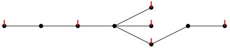
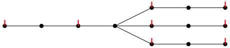
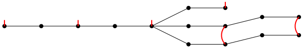
- $P_\bullet$ has 1-strand jellyfish relations $\Leftrightarrow$ both graphs are spokes.



Constructing spoke subfactors with jellyfish

Theorem [Morrison-Penneys [MP12a]]

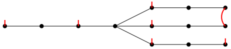
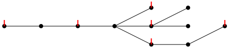
We automate finding 1-strand relations for these subfactors:

- Izumi-Xu 2221  [Han10]
- [GdlHJ89] 3311 
- Izumi $3^{\mathbb{Z}/2\times\mathbb{Z}/2}$  (index $3+\sqrt{5}$)
- 4442  (index $3+\sqrt{5}$)

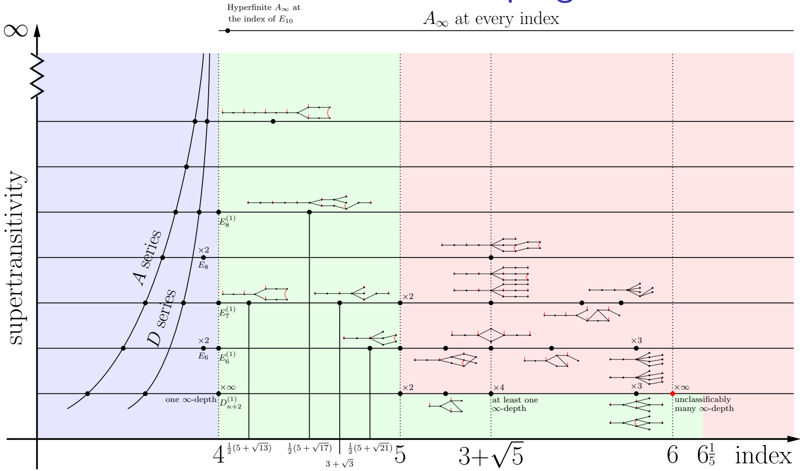
For the above, both principal graphs are the same spoke graph.

Theorem [Penneys-Peters [PP13]]

We give explicit 2-strand relations for Izumi's $3^{\mathbb{Z}/4}$ subfactor

- ( , ) (index $3+\sqrt{5}$)

Small index subfactor classification program



Focuses of the classification program:

- Enumerate graph pairs and apply obstructions.
- Construct examples when graphs survive.
- Place exotic examples into families.

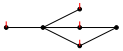
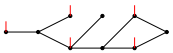
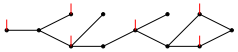
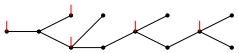
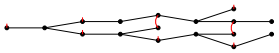
Why do we care about index $3 + \sqrt{5}$?

- Standard invariants at index 4 are completely classified.
  - $\mathbb{Z}/2 * \mathbb{Z}/2 = D_\infty$ is amenable
- Standard invariants at index 6 are wild.
  - There is (at least) one standard invariant for every normal subgroup of the modular group $\mathbb{Z}/2 * \mathbb{Z}/3 = PSL(2, \mathbb{Z})$
  - There are unclassifiably many distinct hyperfinite subfactors with the same standard invariant [BNP07, BV13]
- $4 = 2 \times 2$ and $6 = 2 \times 3$ are composite indices, as is $3 + \sqrt{5} = 2\tau^2$ where $\tau = \frac{1+\sqrt{5}}{2}$.

1-supertransitive subfactors at index $3+\sqrt{5}$

Theorem [Liu [Liu13a]], partial proof by [IMP13]

There are exactly seven 1-supertransitive subfactor planar algebras with index $3+\sqrt{5}$:

- ( , ) self-dual
- ( , ) and its dual
- ( , ) and its dual
- ( $\cdots$,  $\cdots$) and its dual ($A_3 * A_4$)

These are all the standard invariants of composed inclusions of A_3 and A_4 subfactors.

Open question

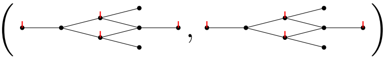
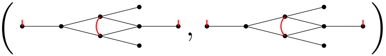
How many hyperfinite subfactors have Bisch-Jones' Fuss-Catalan $A_3 * A_4$ standard invariant at index $3+\sqrt{5}$?

- $A_3 * A_4$ and $A_2 * T_2$ are not amenable [Pop94, HI98].

1-supertransitive with index at most $6\frac{1}{5}$

Theorem [Liu-Morrison-Penneys [LMP13]]

An exactly 1-supertransitive subfactor planar algebra with index at most $6\frac{1}{5}$ either comes from a composed inclusion (and has index $3+\sqrt{5}$ or 6), or is one of 3 self-dual planar algebras at index $3+2\sqrt{2}$:

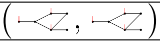
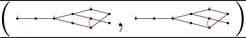
- 
-  two complex conjugate

- Can push classification results above index 6!
- Could hope that the only wildness at index 6 is "group-like"

Index $(5, 3+\sqrt{5})$

Conjecture [Morrison-Peters [MP12b]]

There are exactly two non Temperley-Lieb subfactor planar algebras in the index range $(5, 3+\sqrt{5})$:

name	Principal graphs	Index	Constructed
$SU(2)_5$		5.04892	[Wen90], [MP12b]
$SU(3)_4$		5.04892	[Wen88], [MP12b]

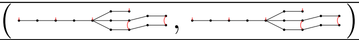
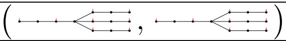
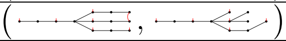
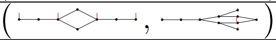
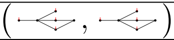
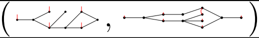
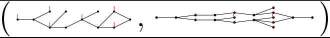
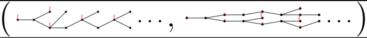
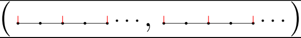
Theorem [Morrison-Peters [MP12b]]

There is exactly one 1-supertransitive subfactor in the index range $(5, 3+\sqrt{5})$

Subfactor planar algebras at index $3+\sqrt{5}$

Conjecture [Morrison-Penneys]

At $3+\sqrt{5}$, we have only the following subfactor planar algebras:

name	Principal graphs	#	Constructed
4442		1	[MP12a], Izumi
$3^{\mathbb{Z}/2\times\mathbb{Z}/2}$		1	Izumi, [MP12a]
$3^{\mathbb{Z}/4}$		2	Izumi, [PP13]
$2D2$		2	Izumi, [MPP]
$A_3 \otimes A_4$		1	$\otimes$
fish 2		2	BH
fish 3		2	[IMP13]
$A_3 * A_4$		2	[BJ97]
A_∞		1	[Pop93]

- 1-supertransitive case known by [Liu13a, IMP13, LMP13]

Methods to push classification results further

- The non-initial triple point obstruction
- Popa's principal graph stability [Pop95, BP14]
- Liu's virtual normalizers for 1-supertransitive subfactors [Liu13b] (pushed 1-supertransitive classification to $6\frac{1}{5}$ [LMP13])
- Afzaly's principal graph enumerator, based on Brendan McKay's isomorph free enumeration by canonical construction paths
- New general initial triple point obstruction [Pen13]

Theorem [Afzaly-Morrison-Penneys]

The conjectures of Morrison-Peters and Morrison-Penneys hold with at most finitely many exceptions.

Thank you for listening!

Recent articles:

- with Liu and Morrison - 1-supertransitive below $6\frac{1}{5}$ - to appear **Comm. Math. Phys.** - `arXiv:1310.8566`
- with Bigelow - Spokes and jellyfish - **Math. Ann.** - `MR3157990`
- with Morrison - Constructing spokes with 1-strand jellyfish - to appear **Trans. AMS** - `arXiv:1208.3637`
- with Peters - Constructing spokes with 2-strand jellyfish - Submitted - `arXiv:1308.5197`
- with Izumi and Morrison - 1-supertransitive at $3+\sqrt{5}$ - Submitted - `arXiv:1308.5723`
- new obstruction - Submitted - `arXiv:1307.5890`

Marta Asaeda and Seidai Yasuda, *On Haagerup's list of potential principal graphs of subfactors*, Comm. Math. Phys. **286** (2009), no. 3, 1141–1157, arXiv:0711.4144 MR2472028 DOI:10.1007/s00220-008-0588-0. MR MR2472028

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Masaki Izumi, Scott Morrison, and David Penneys, *Fusion categories between* $\mathcal{C} \boxtimes \mathcal{D}$ *and* $\mathcal{C} * \mathcal{D}$, 2013, arXiv:1308.5723.

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