The 2D2 subfactor planar algebra

Great Plains Operator Theory Symposium

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Where do subfactors come from?

Some examples include:

- Groups – from $G \curvearrowright R$, we get $R^G \subset R$ and $R \subset R \rtimes_\alpha G$.
- Finite dimensional unitary Hopf/Kac algebras
- Quantum groups – $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$
- Conformal field theory

Operations to obtain more subfactors from known subfactors:

- Composites
- Intermediates
- Morita equivalence

However, there are possible infinite families for which finitely many have only been constructed by brute force methods:

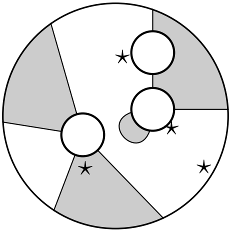
- Endomorphisms of Cuntz C*-algebras
- Planar subalgebras of graph planar algebras

Planar algebras [Jon99]

Definition

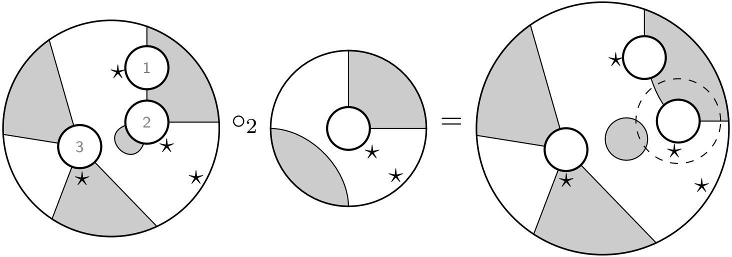
A shaded planar tangle has

- a finite number of inner boundary disks
- an outer boundary disk
- non-intersecting strings
- a marked interval $\star$ on each boundary disk
- a checkerboard shading



Composition of tangles

We can compose planar tangles by insertion of one into another if the number of strings matches up:



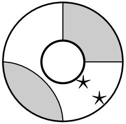
Definition

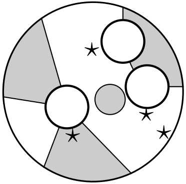
The *shaded planar operad* consists of all shaded planar tangles (up to isotopy) with the operation of composition.

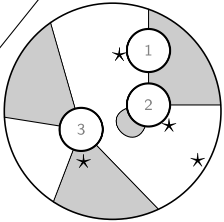
Definition

A *planar algebra* is a family of vector spaces $P_{k,\pm}$, $k = 0, 1, 2, \ldots$ and an action of the shaded planar operad.

$$\begin{tikzcd}
P_{1,+} \times P_{1,+} \times P_{2,-} \arrow[rr] \arrow[dr] & & P_{3,+} \\
& P_{1,+} \times P_{2,+} \times P_{2,-} \arrow[ur] &
\end{tikzcd}$$





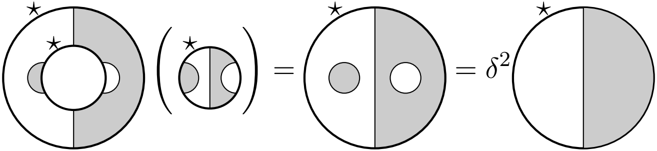


Example: Temperley-Lieb

$TL_{n,\pm}(\delta)$ is the complex span of non-crossing pairings of $2n$ points on a circle, with formal addition and scalar multiplication.

$TL_{3,+}(\delta) = \mathrm{Span}_{\mathbb{C}} \Big\{$  $\Big\}$

Planar tangles act on TL by inserting diagrams into empty disks, smoothing strings, and trading closed loops for factors of δ.

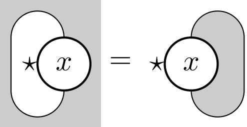


This is a planar $*$-algebra where the involution is the conjugate linear extension of reflection.

Subfactor planar algebras

Definition

A planar $*$-algebra $P_\bullet$ is a *subfactor planar algebra* if it is:

- Finite dimensional: $\dim(P_{k,\pm}) < \infty$ for all k
- Evaluable: $P_{0,\pm} \cong \mathbb{C}$ by sending the empty diagram to $1_\mathbb{C}$
- Sphericality: $\mathrm{Tr}(x) =$ 

- Positivity: each $P_{k,\pm}$ has an adjoint $*$ such that the sesquilinear form $\langle x, y\rangle := \mathrm{Tr}(y^* x)$ is positive definite

From these properties, we get:

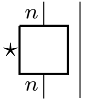
Jones' index rigidity theorem [Jon83]

The value of a closed loop is a multiplicative constant $\delta \in \{2\cos(\pi/n) | n \geq 3\} \cup [2, \infty)$. We call δ^2 the index.

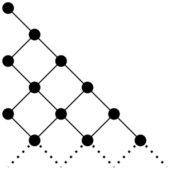
Principal graphs

The C*-algebras $P_{n,\pm}$ are all finite dimensional. The tower

$$P_{0,+} \subset P_{1,+} \subset P_{2,+} \subset \cdots$$

where the inclusion is given by 

is described by its Bratteli diagram (and the trace).



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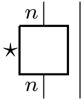
where the inclusion is given by 

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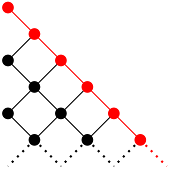
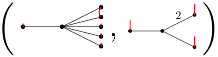


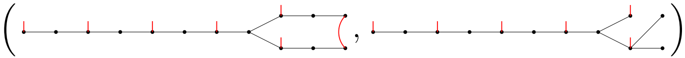
- The non-reflected part is the principal graph Γ_+.

Examples of principal graphs

- index < 4: A_n, D_{2n}, E_6, E_8. No D_{odd} or E_7.
- index $= 4$: $A_{2n-1}^{(1)}, D_{n+2}^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, A_\infty, A_\infty^{(1)}, D_\infty$
- Graphs for $R \subset R \rtimes G$ obtained from G and $\mathrm{Rep}(G)$.



- extended Haagerup [BMPS12]

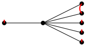
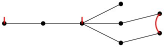
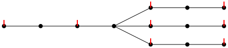


Supertransitivity

Definition

We say a principal graph is n-supertransitive if it begins with an initial segment consisting of the Coxeter-Dynkin diagram A_{n+1}, i.e., an initial segment with n edges.

Examples

-  is 1-supertransitive
-  is 2-supertransitive
-  is 3-supertransitive

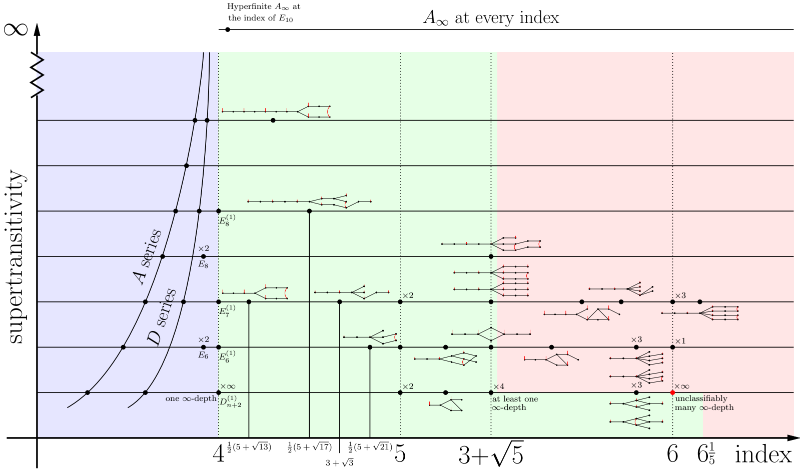
Small index subfactor planar algebra classification program

$$
\begin{tikzcd}[column sep=huge]
\begin{array}{c}\text{subfactor}\\ A \subset B\end{array} \arrow[r, bend left] & \begin{array}{c}\text{planar algebra}\\ (P_+, P_-)\end{array} \arrow[l, bend left] \arrow[r, bend left] & \begin{array}{c}\text{principal graphs}\\ (\Gamma_+, \Gamma_-)\end{array} \arrow[l, bend left, dotted, "??"]
\end{tikzcd}
$$

Focuses of the classification program:

- Enumerate graph pairs and apply obstructions.
- Construct examples when graphs survive.
- Place exotic examples into families.

Known small index subfactor planar algebras



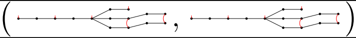
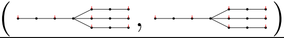
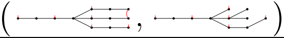
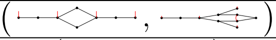
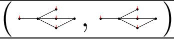
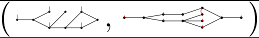
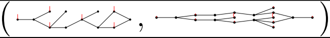
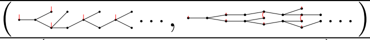
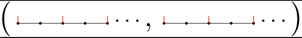
Theorem (Afzaly-Morrison-P 2015)

We know all non Temperley-Lieb subfactor planar algebras up to index $5\frac{1}{4} > 3+\sqrt{5}$, the next composite index above 4.

Subfactor planar algebras at index $3+\sqrt{5}$

Corollary (Afzaly-Morrison-P 2015)

At $3+\sqrt{5}$, we have only the following subfactor planar algebras:

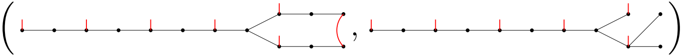
name	Principal graphs	#	Constructed
4442		1	[MP15], Izumi
$3^{\mathbb{Z}/2\times\mathbb{Z}/2}$		1	Izumi, [MP15]
$3^{\mathbb{Z}/4}$		2	Izumi, [PP13]
$2D2$		2	Izumi, [MP14]
$A_3 \otimes A_4$		1	$\otimes$
fish 2		2	BH
fish 3		2	[IMP13]
$A_3 * A_4$		2	[BJ97]
A_∞		1	[Pop93]

- 1-supertransitive case known by [Liu13, IMP13, LMP15]

The extended Haagerup subfactor planar algebra

(Bigelow-Morrison-Peters-Snyder [BMPS12])

The extended Haagerup subfactor planar algebra has principal graphs

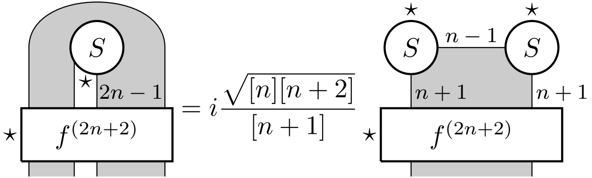


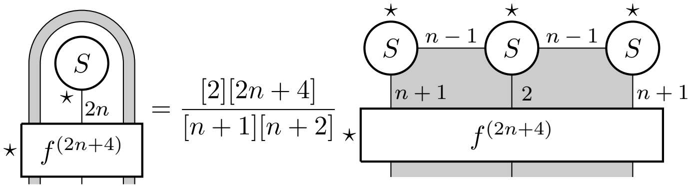
- Last remaining possible graph in Haagerup's classification to index $3 + \sqrt{3}$ [Haa94] by work of Asaeda-Yasuda [AY09].
- Largest known supertransitivity outside the A and D series. High supertransitivity is exceedingly rare!
- Planar algebra constructed using Bigelow's jellyfish algorithm.

Jellyfish relations

Theorem (Bigelow-Morrison-Peters-Snyder [BMPS12])

The Haagerup and extended Haagerup subfactor planar algebras have a generator $S \in P_{n,+}$ where $n = 4, 8$ respectively satisfying:

- **(Jellyfish)** We have the following 2 jellyfish relations for S:
  - 

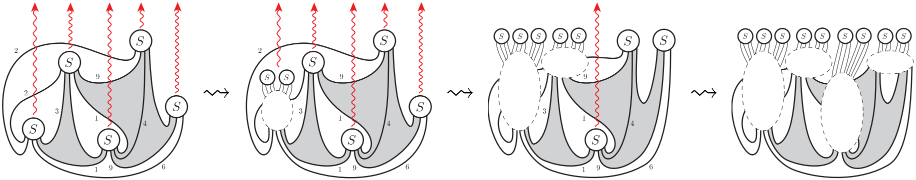
  - 

- **(Capping)** Capping any two strings of S gives zero.
- **(Absorption)** $S^2 = f^{(n)} \in TL_{n,+}$.

The jellyfish algorithm

We can evaluate all closed diagrams as follows:

1. First, pull all generators to the outside using the jellyfish relations



2. Second, reduce the number of generators using the capping and absorption relations.

Consistency and positivity

Theorem [Jones-P [JP11], Morrison-Walker]

Every subfactor planar algebra embeds in the graph planar algebra of its principal graph.

This serves two purposes:

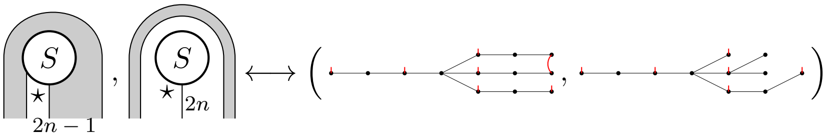
1. To show that an abstractly presented planar algebra is non-zero, we give a representation.
2. Graph planar algebras are always finite dimensional, spherical, and positive. Thus any evaluable planar subalgebra is a subfactor planar algebra!

Spokes and jellyfish

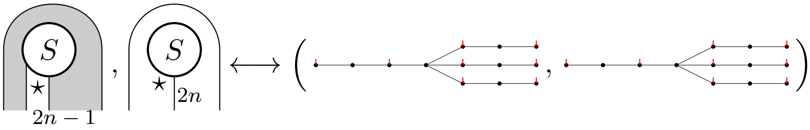
Assume all generators of $P_\bullet$ are at the same depth n.

Theorem (Bigelow-P [BP14], based on [Pop95])

- $P_\bullet$ has 2-strand jellyfish relations $\Leftrightarrow$ one graph is a spoke.



- $P_\bullet$ has 1-strand jellyfish relations $\Leftrightarrow$ both graphs are spokes.

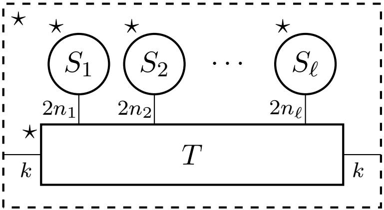


- Algorithms to construct examples in [MP15, PP13]

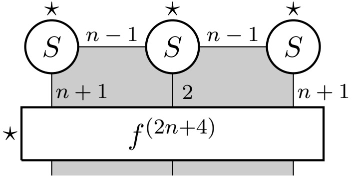
Trains

Definition

A train on a set of generators $\mathcal{S}$ is a diagram of the form



where $S_1, \dots, S_\ell \in \mathcal{S}$ and T is a single Temperley-Lieb diagram.

- 

$\in \operatorname{span}(\mathrm{trains}_{2n+4,+}(S))$.

Universal jellyfish algorithm

By [BP14], a version of the jellyfish algorithm is universal for finite depth subfactors.

Universal Jellyfish Algorithm (Morrison-P [MP14])

Let $\mathcal{P}_\bullet$ be a shaded planar algebra in which closed loops are multiples of the empty diagram. Let $\{\mathcal{S}_{k,+} | k = 0, 1, \ldots, n\}$ be a collection of finite subsets $\mathcal{S}_{k,+} \subset \mathcal{P}_{k,+}$ such that $\mathcal{S}_{n,+}$ generates $\mathcal{P}_\bullet$ as a planar algebra. Denote $\mathcal{S}_{\leq k} = \bigcup_{j \leq k} \mathcal{S}_{j,+}$.

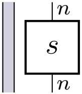
If the sets $\mathcal{S}_{k,+}$ satisfy the *finitely many* conditions on the next slide, then, for $k \leq n$,

$$\mathcal{P}_{k,+} = \mathrm{trains}_{k,+}\left(\mathcal{S}_{\leq k}\right).$$

In particular, if $\mathcal{S}_{0,+} = \emptyset$, $\mathcal{P}_\bullet$ is evaluable.

Conditions for universal jellyfish algorithm

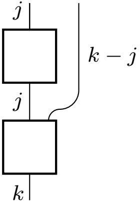
(1) **(Jellyfish)** $\mathcal{S}_{n,+}$ has 2-strand jellyfish relations, i.e., for all $s \in \mathcal{S}_{n,+}$,



$$\in \operatorname{span}(\operatorname{trains}_{n+2,+}(\mathcal{S}_n)).$$

(2) **(Capping)** For $j < n$, adding capping any element of $\mathcal{S}_{j+1,+}$, except on the left, gives an element of $\operatorname{span}(\mathcal{S}_{j,+}) \oplus \mathcal{TL}_{j,+}$.

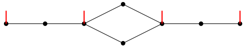
(3) **(Absorption)** For all $j \leq k \leq n$, applying the tangle



to an element in $\mathcal{S}_{k,+} \times \mathcal{S}_{j,+}$, or the reflection to an element in $\mathcal{S}_{j,+} \times \mathcal{S}_{k,+}$, gives an element in $\operatorname{span}(\mathcal{S}_{k,+}) \oplus \mathcal{TL}_{k,+}$.

The 2D2 subfactor

While running the enumerator, we found the possible principal graph

$2D2 =$ 

which could possibly pair with 3 other graphs.

- Izumi constructed a 2D2 subfactor, but uniqueness was unknown.

Theorem (Morrison-P)

There is exactly one subfactor with principal graph 2D2.

- We show there is a 1-parameter family of possible generators in the graph planar algebra, and this family always generates isomorphic planar $*$-subalgebras.

Economical generating set for 2D2

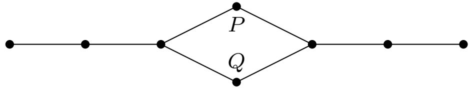
Theorem (Morrison-P [MP14])

Every finite depth subfactor planar algebra has a universal jellyfish presentation.

- We could just pick bases for the $P_{k,+}$ for $k \leq n = \mathrm{depth}(P_\bullet)$. But this is terribly inefficient!
- The key is to pick an economical generating set to make computations less expensive.

Example: 2D2

For the 2D2 subfactor planar algebra, we pick the generating set:



$\mathcal{S}_{n,+}$ $\emptyset$ $\emptyset$ $\emptyset$ T $(E_{i,j})$

where $T = P - Q$ and $(E_{i,j})$ is a system of 2×2 matrix units.

Thank you for listening!

Slides available at

http://www.math.ucla.edu/~dpenneys/PenneysGPOTS2015.pdf

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