C^*-algebras, right correspondences, and Q-systems, part I: What is a Q-system?

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Overview

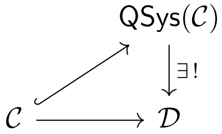
- Unitary tensor categories (UTCs) encode quantum symmetry and act on operator algebras via unitary tensor functors

$$\mathbf{H} : \mathcal{C} \to \mathsf{Bim}(A)$$

- A finite index II_1 subfactor $A \subset B$ can be viewed a triple $(\mathcal{C}, Q, \mathbf{H})$ where $\mathcal{C}$ is a UTC, $Q \in \mathcal{C}$ is a **Q-system**, and $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}(N)$ is an action.

$$A \subset A \rtimes_{\mathbf{H}} Q = B$$

- Q-systems in UTCs are **higher idempotents**, and we can take a **higher idempotent completion**.



Unitary tensor categories (UTCs) by example

Let A be a unital (simple) $\mathrm{C}*$-algebra. The UTC $\mathsf{Bim_{fgp}}(A)$ has:

- objects Hilbert C^* right A-modules X equipped with
  - right A-valued inner product $\langle\cdot|\cdot\rangle_A : \overline{X} \otimes_{\mathbb{C}} X \to A$
  - a left A-action $A \to \mathrm{End}(X_A)$ by adjointable operators

$$\langle a \rhd \eta|\xi\rangle_A = \langle\eta|a^* \rhd \xi\rangle_A \qquad \forall\, \eta, \xi \in X, \forall\, a \in A.$$

  We require X to be finitely generated projective as both a left and right A-module.

- morphisms $f : X \to Y$ are adjointable $A - A$ bimodular operators, i.e., $\exists\, f^\dagger : Y \to X$ such that

$$\langle f(\eta)|\xi\rangle_A = \langle\eta|f^\dagger(\xi)\rangle_A \qquad \forall\, \eta, \xi \in X.$$

Properties of UTCs

- (linear) $\mathrm{Hom}(X \to Y)$ finite dimensional vector space
- (Cauchy complete) admits finite direct sums $\bigoplus X_i$, and has all subobjects $X \subseteq Y$
- (C^*) $\mathrm{End}(X)$ is a C^* algebra under $\dagger$
- (tensor) Can take $X \boxtimes_A Y$ of bimodules and $f \boxtimes g$ of intertwiners. All coherence isomorphisms are *unitary* ($u^{-1} = u^\dagger$).
- (rigid) every object admits left and right duals (for $\mathsf{Bim}_{\mathsf{fgp}}(A)$, see [KW00, KPW04])
- (simple unit) $\mathrm{End}_{\mathcal{C}}({}_A A_A) = \mathbb{C}$ when A is simple

Where do UTCs come from?

1. Subfactor standard invariants $A \subset B \rightsquigarrow \mathcal{C}(A \subset B)$
2. Compact groups $G \rightsquigarrow \mathsf{Rep}(G)$
3. Discrete/compact quantum groups (Tannaka-Krein duality)

$$\mathbb{G} \qquad \rightsquigarrow \qquad (\mathsf{Rep}(\mathbb{G}), \mathbf{F} : \mathsf{Rep}(\mathbb{G}) \to \mathsf{Hilb})$$

4. Generators and relations [VV19]
5. Constructions of new UTCs from existing UTCs

Many people care about UTCs because of physics

- conformal field theory ($\mathsf{Rep}(\mathcal{A})$ of a conformal net)
- unitary fusion categories give Turaev-Viro TQFTs
- unitary modular categories give Reshetikhin-Turaev TQFTs
- topological phases of matter (UMTCs)

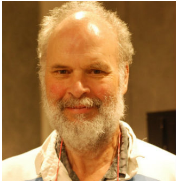
Subfactors

- A II_1 **factor** is an infinite dimensional von Neumann algebra with trivial center and a trace. (Eg: $L\Gamma := \mathbb{C}[\Gamma]'' \subset B(\ell^2\Gamma)$)
- A II_1 **subfactor** is a unital inclusion of type II_1 factors.

Jones' Index Rigidity Theorem [Jon83]

The index $[B : A] := \dim({}_A L^2 B)$ of a II_1 subfactor $A \subset B$ takes values in:

$$[B : A] \in \left\{4\cos^2(\pi/n)\middle| n \geq 3\right\} \cup [4, \infty].$$



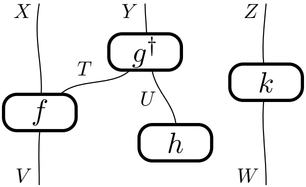
Example

Given a finite index II_1 subfactor $A \subset B$, the UTC ${}_A\mathcal{C}_A$ is the category of $A - A$ bimodules generated by $L^2 B$ under

- $\oplus$ direct sum
- $\boxtimes$ Connes' fusion relative tensor product over A
- $\subseteq$ sub-bimodules
- $\overline{\cdot}$ conjugates

2D graphical calculus for UTCs

0. Objects denoted by labelled strands, oriented bottom to top.
1. 1-morphisms denoted by coupons



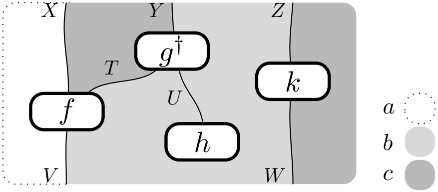
$f : V \to X \otimes T$
$g : Y \to T \otimes U$
$h : 1_{\mathcal{C}} \to U$
$k : W \to Z$

- vertical stacking is composition
- horizontal juxtaposition is $\otimes$
- vertical reflection is $\dagger$
- suppress unit $1_{\mathcal{C}}$ and all coheretors α, λ, ρ

2D graphical calculus for $\mathrm{C}^*/\mathrm{W}^*$ 2-categories

A tensor category is a 2-category with one object. For 2-categories, we have a dimension shift.

0. shadings for regions to denote objects
1. 1-morphisms denoted by strands
2. 2-morphisms denoted by coupons



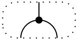
$$
\begin{aligned}
T &: c \to b \\
U &: b \to b \\
V &: a \to b \\
W &: b \to c \\
X &: a \to c \\
Y &: c \to b \\
Z &: b \to c
\end{aligned}
\qquad
\begin{aligned}
f &: V \Rightarrow X \otimes T \\
g &: Y \Rightarrow T \otimes U \\
h &: 1_{\mathcal{C}} \Rightarrow U \\
k &: W \Rightarrow Z
\end{aligned}
$$

Example

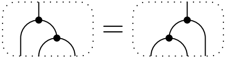
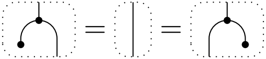
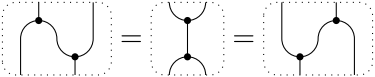
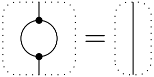
There is a 2-category C^*Alg whose objects are unital C^*-algebras, 1-morphisms are right Hilbert C^*-correspondences, and 2-morphisms adjointable intertwiners.

The standard invariant via Q-systems

The **standard invariant** of a finite index II_1 subfactor $A \subset B$ is the UTC ${}_A\mathcal{C}_A$ of $A - A$ bimodules generated by L^2B with the **Q-system** ${}_AL^2B_A$.

 : $L^2B \boxtimes_A L^2B \xrightarrow{\text{multiplication}} L^2B$

 : $L^2A \xrightarrow{\text{unit}} L^2B$

- (associative) 
- (unital) 
- (Frobenius) 
- (separable) 

Subfactors as triples

In work with Corey Jones [JP19, JP20] building on [Lon94, Müg03], finite index II_1 subfactors $A \subset B$ are triples $(\mathcal{C}, Q, \mathbf{H})$:

1. Unitary tensor category $\mathcal{C}$,
2. Q-system $Q \in \mathcal{C}$,
3. Fully faithful unitary tensor functor $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}(A)$.

The *standard invariant* of $A \subset B$ is the pair $(\mathcal{C}, Q)$ (forget $\mathbf{H}$).

This effectively splits subfactor classification into 2 parts:

1. (Analytic problem) Classify actions of unitary tensor categories $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}(A)$.
2. (Algebraic problem) Classify Q-systems in $\mathcal{C}$.

Classification of subfactors

Example

$R \subset R \rtimes G$ for finite G 'remembers' G, so classifying hyperfinite subfactors is hopeless. Focus on some notion of 'smallness.'

Strategy for small index hyperfinite II_1 subfactor classification:

1. Classify standard invariants $(\mathcal{C}, Q)$ with $\dim(Q)$ small
2. Determine how many subfactors give each standard invariant.

Popa's Subfactor Reconstruction Theorem [Pop90, Pop95]

Every standard invariant comes from a subfactor. If the standard invariant is *amenable*, the subfactor can be taken to be hyperfinite.

Amenability

Amenability arises in *two places* when subfactors can be classified:

1. Restrict to subfactors of the amenable II_1 factor R
2. Only consider amenable standard invariants $(\mathcal{C}, Q)$.

Non-amenable embedding questions

- How many embeddings $\mathrm{Ad}(A_3 * A_4) \hookrightarrow$ Bim(R)?
- How many embeddings $TLJ(d) \hookrightarrow$ Bim(R) for $d > 2$?

Question

When can we classify actions of UTCs on classifiable C^*-algebras?

- Motivation for our article is to allow straightforward adaptation of subfactor techniques for actions of UTCs on C^*-algebras.

Idempotent completion example: K-theory

Recall the definition of $K_0(A)$ for a unital C^*-algebra.

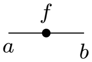
1. Look at the C^*-category $\mathsf{Mod}_{\mathsf{fgp}}(A)$ of finitely generated projective Hilbert C^* A-modules.
2. $\mathsf{Mod}_{\mathsf{fgp}}(A)$ admits all finite direct sums.
3. $K_0(A) := K_0(\mathsf{Mod}_{\mathsf{fgp}}(A))$, the Grothendieck group of $\mathsf{Mod}_{\mathsf{fgp}}(A)$.

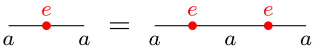
- $\mathsf{Mod}_{\mathsf{fgp}}(A)$ is *Cauchy complete*: it has finite direct sums and all projections *split*: given a projection $p \in \mathrm{End}(M_A)$, $pM_A \in \mathsf{Mod}_{\mathsf{fgp}}(A)$.

$$M_A \underset{i}{\overset{p}{\rightleftarrows}} pM_A \qquad\qquad p \circ i = \mathrm{id}_{pM} \qquad i \circ p = p$$

Here, p is a *retract* and i is a *section*.

Q-systems are higher categorical idempotents

- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

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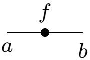
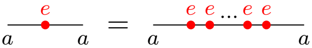
- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

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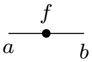
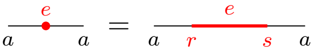
- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

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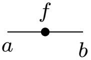
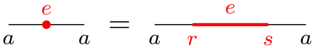
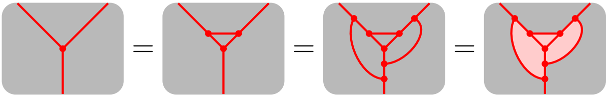
- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

- A Q-system is a unitary (co)unital *higher categorical idempotent*.



Now strands and tri/univalent vertices can replicate freely.

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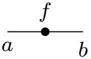
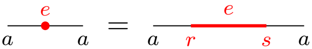
- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

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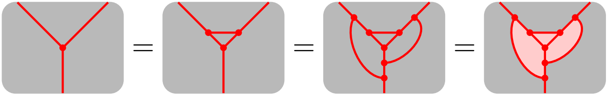


Now strands and tri/univalent vertices can replicate freely.

Definition based on [Yam04, EGNO15, BKLR15, CR16, NY16, DR18, GY20]

The *Q-system completion* $\mathrm{QSys}(\mathcal{C})$ of a $\mathrm{C}^*/\mathrm{W}^*$ 2-category $\mathcal{C}$ has

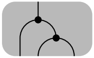
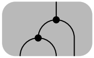
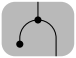
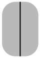
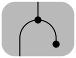
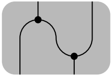
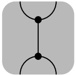
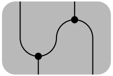
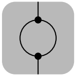
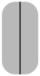
- objects are Q-systems,
- 1-morphisms are unitarily separable bimodules, and
- 2-morphisms are intertwiners.

Q-systems

Recall that a Q-system in a $\mathrm{C}^*/\mathrm{W}^*$ 2-category $\mathcal{C}$ is a 1-morphism $Q \in \mathcal{C}(b \to b)$ together with

 $: Q \otimes_b Q \xrightarrow{\text{multiplication}} Q$  $: 1_b \xrightarrow{\text{unit}} Q$

such that the following relations hold:

- (associative)  $=$ 
- (unital)  $=$  $=$ 
- (Frobenius)  $=$  $=$ 
- (unitarily separable)  $=$ 

Frobenius actually follows from associative, unital, and unitarily separable by [LR97]; see [BKLR15, Lem. 3.7].

Q-systems in C*Alg

Example

Suppose $A \subset B$ is a unital inclusion of C^*-algebras equipped with a completely positive $A - A$ bimodular map $E : B \to A$. Equip B with structure of $A - A$ bimodule by

$$\langle b_1 | b_1 \rangle_A := E(b_1^* b_2).$$

When B_A is **finitely generated projective (fgp)**, we can modify the multiplication on B to get a Q-system.

By our article [CPJP], a Q-system ${}_A Q_A$ in C*Alg is equivalent to:

- a direct sum decomposition $A = A_1 \oplus A_2$ of C^*-algebras, and
- a unital inclusion $(A_1 \subset B, E)$ where B_A is fgp.

Theorem [CPJP] cf. [GY20]

$\mathsf{C^*Alg}, \mathsf{W^*Alg}, \mathsf{vNA}$ are Q-system complete (Q-systems split).

Corollary [CPJP] cf. [GY20]

Can induce action $\mathcal{C} \to \mathsf{Bim}(A) \subset \mathsf{R} \in \{\mathsf{C^*Alg}, \mathsf{W^*Alg}, \mathsf{vNA}\}$

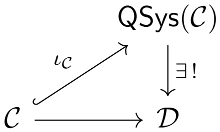
$$\mathsf{QSys}(\mathcal{C}) \to \mathsf{QSys}(\mathsf{Bim}(A)) \to \mathsf{QSys}(\mathsf{R}) \xrightarrow{\cong} \mathsf{R}$$

Followup results with Quan Chen:

Theorem [CP] cf. [DR18]

QSys is a 3-functor on $\mathrm{C}^*/\mathrm{W}^*$ 2-categories.

Universal property for Q-system completion [CP] cf. [DR18]



for every †-2-functor from $\mathcal{C}$ to a Q-system complete $\mathcal{D}$.

Thank you for listening!

Slides available at:
`https://people.math.osu.edu/penneys.2/PenneysGPOTS2021.pdf`

Articles in preparation, expected Summer 2021:

- Q-system completion for C^* 2-categories (with Quan Chen, Roberto Hernandez Palomares, and Corey Jones)
- Q-system completion is a 3-functor (with Quan Chen)

Marcel Bischoff, Yasuyuki Kawahigashi, Roberto Longo, and Karl-Henning Rehren, *Tensor categories and endomorphisms of von Neumann algebras—with applications to quantum field theory*, SpringerBriefs in Mathematical Physics, vol. 3, Springer, Cham, 2015, MR3308880 DOI:10.1007/978-3-319-14301-9. MR 3308880

Quan Chen and David Penneys, *Q-system completion is a 3-functor*, In preparation.

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