Composing topological domain walls and anyon mobility

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New Frontiers: Interactions between Quantum Physics and Mathematics

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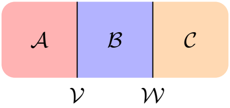
THE OHIO STATE UNIVERSITY

A higher-categorical puzzle

Our goal is to study *concatenation* and *fusion rules* of domain walls between (2+1)D topological orders (TOs).



We start with the following 'higher categorical puzzle':

- Low-energy topological excitations described by UMTCs $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and Witt-equivalences $\mathcal{V}, \mathcal{W}$. These are objects and 1-morphisms in the **4-category** of braided fusion categories.
- (2+1)D topologically ordered systems should form a symmetric monoidal **3-category**. (Warning: anomalies multiply!)
- While composing 2 equivalences yields an equivalence, concatenation of indecomposable domain walls does not always give an indecomposable domain wall.

Answer in anomaly-free setting: string nets

An answer is well understood for anomaly-free topological orders using string-nets built from unitary fusion categories (UFCs) [LW05, LLB21] and bimodule categories [KK12].

- UFCs form a symmetric monoidal 3-category called $\mathbf{UFC}$ [Hau17, JFS17, DSPS20].

cat level	projector model ingredient	topol. excitations
2D bulk	$\mathcal{X} \in \mathbf{UFC}$	UMTC $Z(\mathcal{X})$
1D domain wall	bimodule category ${}_{\mathcal{X}}\mathcal{M}_{\mathcal{Y}}$	Witt-eq $\mathrm{End}({}_{\mathcal{X}}\mathcal{M}_{\mathcal{Y}})$
0D point defect	bimodule functor	-
local operators	bimodule transformations	-

- Some fusion rules for domain walls were calculated in [BBJ19a, BBJ19b].

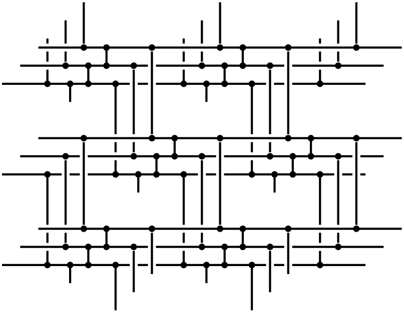
Outline

We propose an answer to this puzzle using *enriched UFCs* [MP19] which can be used to describe *anomalous* (2+1)D topologically ordered systems [JF20].

1. Using enriched UFCs, anyons and local operators appear as higher morphisms in a 3-category $\mathbf{UFC}^{\mathcal{A}}$ of topological orders with anomaly described by $\mathcal{A}$.
2. We can use this 3-category to decompose the concatenation of domain walls into superselection sectors and to characterize anyon mobility through domain walls using **tunneling operators**.
3. Using tunneling operators, we can identify the indecomposable summands of a composite domain wall.

Anomalies of (2+1)D topological orders

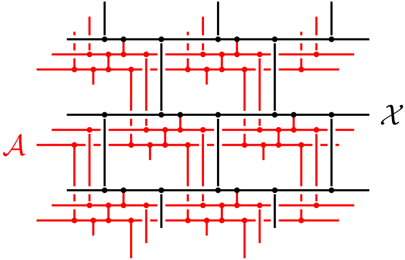
- (2+1)D topologically ordered systems carry an *anomaly* described by an invertible (3+1)D TQFT [JF20], equivalently a *Witt-class* of UMTC [BJSS21].
- This anomaly can be viewed as an obstruction to realizing the system by a (2+1)D commuting projector local Hamiltonian.
- The *Walker-Wang model* [WW12] gives a realization as a (2+1)D boundary of a (3+1)D commuting projector local Hamiltonian.



Enriched UFCs

Pick a representative UMTC $\mathcal{A}$ of the Witt-class for the anomaly.

- An $\mathcal{A}$-enriched fusion category $\mathcal{X}$ is exactly the data needed to couple the $\mathcal{X}$-string-net model to an $\mathcal{A}$-Walker-Wang bulk.



$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[dr, dashed] & Z(\mathcal{X}) \arrow[d, "\text{Forget}"] \\ & \mathcal{X} \end{tikzcd}$$

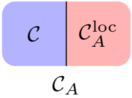
$$Z(\mathcal{X}) \cong \mathcal{A} \boxtimes Z^{\mathcal{A}}(\mathcal{X})$$

- The low-energy topological excitations are described by the **enriched center/Müger centralizer** $Z^{\mathcal{A}}(\mathcal{X}) := \mathcal{A}' \subset Z(\mathcal{X})$ [KZ18b, Müg03].

Chiral example

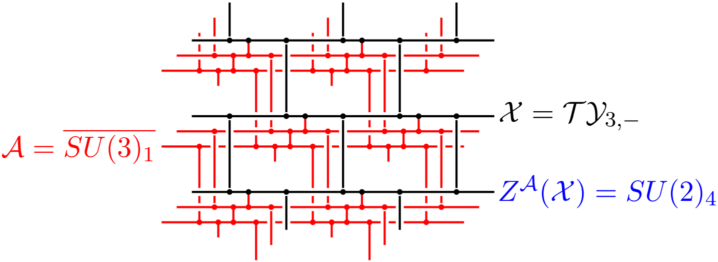
By [BEK01, DMNO13], for a condensable algebra A in a UMTC $\mathcal{C}$,

$$Z(\mathcal{C}_A) \cong \mathcal{C} \boxtimes \overline{\mathcal{C}_A^{\mathrm{loc}}}.$$



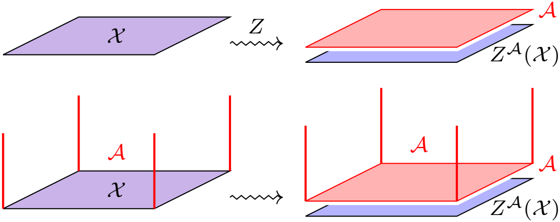
For $\mathcal{C} = SU(2)_4$ and $A = 1 \oplus g$ where g is the boson:

- $\mathcal{C}_A^{\mathrm{loc}} = SU(3)_1$
- $\mathcal{C}_A = \mathcal{TY}_{3,-}$ is a UFC with 4 simples $0, 1, 2$ and σ where $\mathbb{Z}/3 = \{0, 1, 2\}$ and $\sigma^2 = 0 \oplus 1 \oplus 2$.
- $Z(\mathcal{TY}_{3,-}) \cong SU(2)_4 \boxtimes \overline{SU(3)_1}$



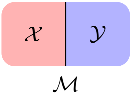
Bilayer visualization

Since $Z(\mathcal{X}) \cong Z^{\mathcal{A}}(\mathcal{X}) \boxtimes \mathcal{A}$, attaching an $\mathcal{A}$-Walker-Wang bulk to $\mathcal{X}$ trivializes the $\mathcal{A}$-layer of topological order, leaving only $Z^{\mathcal{A}}(\mathcal{X})$



Domain walls: enriched bimodules

Domain walls between $\mathcal{A}$-enriched UFCs $\mathcal{X}, \mathcal{Y}$ can be described by $\mathcal{A}$-enriched bimodule categories



Here, $\mathcal{M}$ is an $\mathcal{X} - \mathcal{Y}$ bimodule category with data identifying the left and right central $\mathcal{A}$-actions.

- $\mathcal{A}$-enriched UFCs, bimodules, functors, and natural transformations form a *3-category* $\mathbf{UFC}^{\mathcal{A}}$
- $\mathrm{End}^{\mathcal{A}}({}_{\mathcal{X}}\mathcal{M}_{\mathcal{Y}})$ is a Witt-equivalence $Z^{\mathcal{A}}(\mathcal{X}) \sim Z^{\mathcal{A}}(\mathcal{Y})$.
  In particular, $\mathrm{End}^{\mathcal{A}}({}_{\mathcal{X}}\mathcal{X}_{\mathcal{X}}) = Z^{\mathcal{A}}(\mathcal{X})$

cat level	projector model ingredient	topol. excitations
2D bulk	$\mathcal{X} \in \mathbf{UFC}^{\mathcal{A}}$	UMTC $Z^{\mathcal{A}}(\mathcal{X})$
1D domain wall	$\mathcal{A}$-enriched bimodule ${}_{\mathcal{X}}\mathcal{M}_{\mathcal{Y}}$	Witt-eq $\mathrm{End}^{\mathcal{A}}({}_{\mathcal{X}}\mathcal{M}_{\mathcal{Y}})$
0D point defect	$\mathcal{A}$-centered bimod functor	-
local operators	bimodule transformations	-

Equivalences

There are two notable equivalences between the $\mathcal{A}$-enriched UFC viewpoint and the UMTC viewpoint.

- Have an equivalence of truncated 1-categories [KZ18a, KZ21]

$$\left\{\begin{matrix}\mathcal{A}\text{-enriched UFCs} \\ \text{and bimodules}\end{matrix}\right\}_{\leq 1} \xrightarrow[\simeq]{Z^{\mathcal{A}},\mathrm{End}^{\mathcal{A}}} \left\{\begin{matrix}\text{UMTCs and Witt} \\ \text{equivalences}\end{matrix}\right\}_{\leq 1}$$

For a UMTC $\mathcal{C}$, the **fusion 2-category** $\mathsf{Mod}(\mathcal{C})$ describes generalized categorical symmetries cf. [KLW+20, CW22] and [NCRSS21, Mul22].

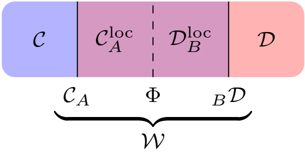
Proposition [HBJP]

There is an equivalence of fusion 2-categories

$$\mathsf{Bim}^{\mathcal{A}}(\mathcal{X}) \cong \mathsf{Mod}(Z^{\mathcal{A}}(\mathcal{X})).$$

Concatenations of elementary domain walls

Every topological boundary $\mathcal{W}$ between topological orders $\mathcal{C}, \mathcal{D}$ can be obtained by concatenating condensation boundaries and invertible boundaries cf. [DNO13, §3].

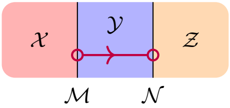


By the folding trick, $\mathcal{W}$ corresponds to a **Lagrangian algebra** $L(A, \overline{B}, \Phi) \in \mathcal{C} \boxtimes \overline{\mathcal{D}}$ where $A \in \mathcal{C}$ and $B \in \mathcal{D}$ are condensable algebras and $\Phi : \mathcal{C}_A^{\mathrm{loc}} \to \mathcal{D}_B^{\mathrm{loc}}$ is a braided equivalence.

$$ {}_{\mathcal{C}}\mathcal{W}_{\mathcal{D}} \longleftrightarrow L(A, \overline{B}, \Phi) \in \mathcal{C} \boxtimes \overline{\mathcal{D}} $$

- [Kon14] gives another way to decompose a domain wall as 2 condensation boundaries in the reverse order.

Decomposing composite domain walls



$\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathbf{UFC}^{\mathcal{A}}$

$$\mathrm{End}^{\mathcal{A}}({}_{\mathcal{X}}\mathcal{M}_{\mathcal{Y}}) \longleftrightarrow L(A, \overline{B_1}, \Phi) \in Z^{\mathcal{A}}(\mathcal{X}) \boxtimes \overline{Z^{\mathcal{A}}(\mathcal{Y})}$$
$$\mathrm{End}^{\mathcal{A}}({}_{\mathcal{Y}}\mathcal{N}_{\mathcal{Z}}) \longleftrightarrow L(B_2, \overline{C}, \Phi) \in Z^{\mathcal{A}}(\mathcal{Y}) \boxtimes \overline{Z^{\mathcal{A}}(\mathcal{Z})}.$$

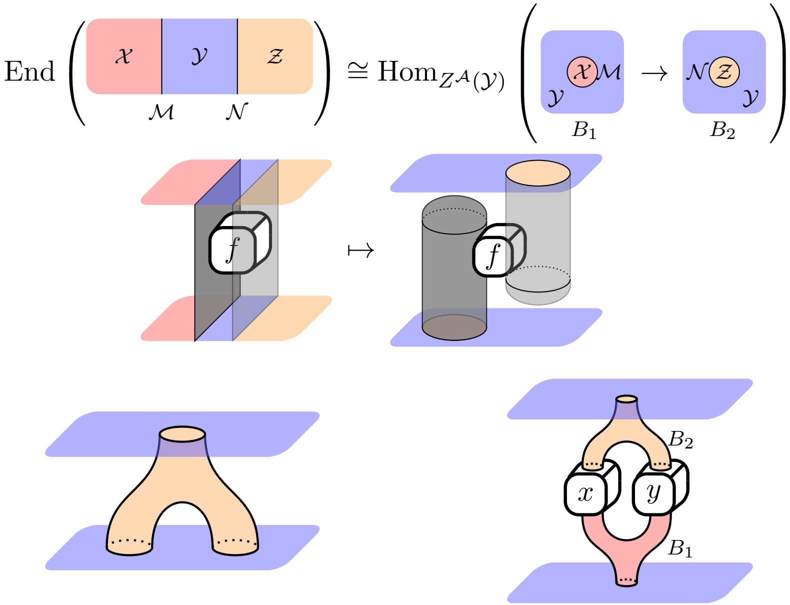
A *short string operator* creates an antiparticle-particle pair $\overline{c}, c$ in the $Z^{\mathcal{A}}(\mathcal{Y})$ bulk region and condenses one of each at the left and right walls, which creates no topological excitations.

Theorem [HBJP]

The algebra of short string operators is $\mathrm{Hom}_{Z^{\mathcal{A}}(\mathcal{Y})}(B_1 \to B_2)$, which is a finite dimensional abelian C^*-algebra, i.e., $\mathbb{C}^n$. The minimal projections correspond to the superselection sectors (indecomposable summands) of the composite domain wall.

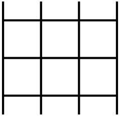
3-dualizability and the short string algebra

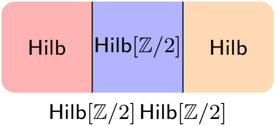
The proof uses *3-dualizability* and *unitarity* in $\mathbf{UFC}^{\mathcal{A}}$.



Basic example: Toric Code [Kit03, BK05, KK12, BBJ19a]

Consider a vertical strip of $\mathbb{Z}/2$ Toric Code, sandwiched between two smooth gapped boundaries to vacuum where m is condensed.





$\mathcal{A} = \mathsf{Hilb}$

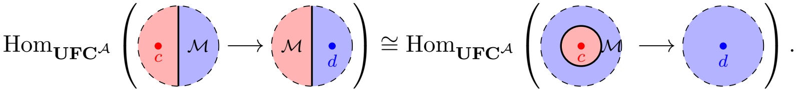
$\mathcal{X} = \mathcal{Z} = \mathsf{Hilb}$, $\mathcal{Y} = \mathsf{Hilb}[\mathbb{Z}/2]$, $\mathcal{M} = \mathcal{N} = \mathsf{Hilb}[\mathbb{Z}/2]$.

- The composite bimodule category $\mathcal{M} \boxtimes_{\mathcal{Y}} \mathcal{N}$ is $\mathsf{Hilb} \oplus \mathsf{Hilb}$ as a $\mathsf{Hilb} - \mathsf{Hilb}$ bimodule, i.e., the direct sum of two copies of the trivial domain wall from the trivial topological order to itself.
- Two copies arises because, with appropriate boundary conditions (close to a sphere), the GSD is the dimension of the short string algebra, which counts the number of e-lines running parallel between the two boundaries.

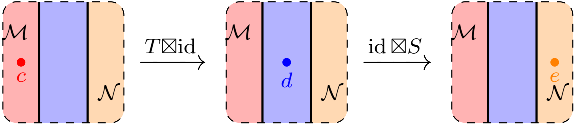
Anyon mobility: tunneling operators

- Particle-antiparticle anyon pairs are created using string operators.
- Anyons can be transported in the (2+1)D bulk by *hopping operators* [HGW18].
- Anyons can be transported across domain walls by *tunneling operators*.

The space of tunneling operators across the domain wall ${}_{\mathcal{X}}\mathcal{M}_{\mathcal{Y}}$ from the anyon $c \in \mathrm{Irr}(Z^{\mathcal{A}}(\mathcal{X}))$ to the anyon $d \in \mathrm{Irr}(Z^{\mathcal{A}}(\mathcal{Y}))$ is



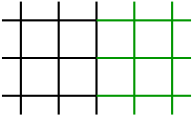
By 3-dualizability and semisimplicity in $\mathbf{UFC}^{\mathcal{A}}$, concatenating tunneling operators works well.



Basic example: boundary between $\mathbb{Z}/4$ and $\mathbb{Z}/2$ Toric Code

Consider a lattice where each edge has $\mathbb{C}^4$-spins:

$\mathbb{Z}/4$ TC



$\mathbb{Z}/2$ TC

The Hamiltonian is given by

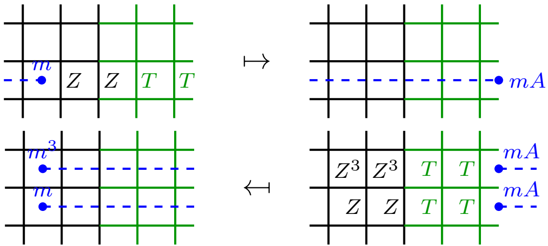
$$H = -\sum_{v/v} A_v - \sum_{p} B_p - \sum_{q} B'_q - K \sum_{\ell} C_\ell \qquad K \gg 1$$

$$A_v = \sum_{k=1}^{4} \begin{array}{ccc} & (Z^k)^\dagger & \\ Z^k & \overset{v}{+} & (Z^k)^\dagger \\ & Z^k & \end{array} \qquad B'_q = \frac{1}{2}\left(1 + \begin{array}{ccc} & X^2 & \\ X^2 & \boxed{q} & X^2 \\ & X^2 & \end{array}\right)$$

$$B_p = \sum_{j=1}^{4} \begin{array}{ccc} & X^j & \\ X^j & \boxed{p} & (X^j)^\dagger \\ & (X^j)^\dagger & \end{array} \qquad C_\ell = \frac{1}{2}\left(1 + \underline{Z^2}\right)$$

Boundary Toric Code tunneling operators

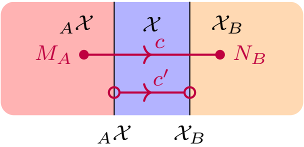
In the green region, m^2 has been condensed. This corresponds to TO $\mathcal{D}(\mathbb{Z}/4)_A^{\mathrm{loc}} \cong \mathcal{D}(\mathbb{Z}/2)$ where $A = 1 \oplus m^2$.
We can transport m in the black region by Z. There is a unique choice of tunneling operator in the green region given by $T = Z + Z^3$ (up to scalar).



When we transport the anyon mA back to the black region, there are 2 choices: mA can tunnel to become m or m^3. Each tunneling operator is unique up to phase.

Short string operators act on tunneling operators

Consider two condensation boundaries from condensing $A, B \in Z^{\mathcal{A}}(\mathcal{X})$. (At the level of $\mathcal{A}$-enriched UFCs, ${}_A\mathcal{X}$ is again $\mathcal{A}$-enriched UFCs, and $Z^{\mathcal{A}}({}_A\mathcal{X}) \cong Z^{\mathcal{A}}(\mathcal{X})_A^{\mathrm{loc}})$)



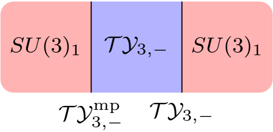
We can act by the minimal projections in $\mathrm{Hom}_{Z^{\mathcal{A}}(\mathcal{Y})}(B_1 \to B_2)$ onto superselection sectors.

- The space of all tunneling operators decomposes as the direct sum of subspaces of tunneling operators for each summand.
- Looking at these subspaces allows us to identify the summands in examples. It also gives a strategy to compute fusion rules for concatenating domain walls.
- All point defects between superselection sectors 'come from' anyons in the middle bulk region.

Chiral example

Recall $\mathcal{TY}_{3,-}$ has 3 invertible simples $0, 1, 2$ and another simple σ such that $\sigma^2 = 0 \oplus 1 \oplus 2$.

Consider the following composite domain wall from $SU(3)_1$ to itself:



$$\mathcal{A} = \overline{SU(3)_1}$$

Here, $Z^{\mathcal{A}}(SU(3)_1) = SU(3)_1$ and $Z^{\mathcal{A}}(\mathcal{TY}_{3,-}) = SU(2)_4$.

- The composite domain wall decomposes as $\mathcal{I} \oplus \mathcal{F}$ where $\mathcal{I}$ is the identity domain wall and $\mathcal{F}$ is the domain wall which flips 1 and 2.

Thank you for listening!

Slides available at:
`https://people.math.osu.edu/penneys.2/talks/PenneysHarvard2022.pdf`

Peter Huston, Fiona Burnell, Corey Jones, and David Penneys.
Composing topological domain walls and anyon mobility.
Coming soon!

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