The 2-category of tracial von Neumann algebras

INI mini-course

David Penneys

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Introduction

These notes were produced during and after a mini-course entitled "Introduction to subfactor theory" at the Isaac Newton Institute on their semester program on Operator algebras: subfactors and their applications. During this mini-course, I wanted to do something a bit different from my mini-course on subfactor theory at the 2014 Spring Institute on Noncommutative Geometry and Operator Algebras (NCGOA). I ended up giving an introduction to the 2-category of tracial von Neumann algebras, which admits direct sums, subobjects, and Connes fusion tensor products, together with its *bi-involutive* structure, including the adjoint and the conjugate. The rich structure of this 2-category leads to a powerful graphical calculus, where the adjoint and conjugate correspond to vertical and horizontal reflections which commute. Much of this material is adapted from [Bis97] and [Bur03].

In the mini-course, we then specialized to the case of II_1 factor bimodules to discuss dualizability. Here in the notes, we will try to stay in the most general case possible. Using the graphical calculus for the bi-involutive sub 2-category of dualizable bimodules, we can define the canonical planar algebra associated to a finite index II_1 subfactor.

I am now in the process of going over my mini-course here at The Ohio State University during open spots in the Quantum Algebra/Quantum Topology seminar. As we go along, I hope to revise and expand on these notes.

Most of the material in these notes is presented in the form of exercises. More challenging exercises are broken into multiple parts. It is my hope that by working through these exercises, one will become familiar with the material.

1 Tracial von Neumann algebras

1.1 von Neumann algebras

Definition 1.1. A *von Neumann algebra* is a $*$-closed subalgebra $A \subseteq B(H)$ such that $A = A''$, where for a subset $S \subset B(H)$,

$$S' = \{x \in B(H) | xs = sx \text{ for all } s \in S\}.$$

We will only work with separable Hilbert spaces in these lectures.

Exercise 1.2. Show that $S' = S'''$ for any subset $S \subset B(H)$.

Exercise 1.3. Show that if $S \subset T$, then $T' \subset S'$.

Thus von Neumann algebras come in pairs, A and A'. The *center* of a von Neumann algebra is $Z(A) = A' \cap A$, which is the center of both A and A'.

Definition 1.4. The *strong operator topology* or the *topology of pointwise convergence* on $B(H)$ is the topology induced by the seminorms $x \mapsto \|x\xi\|$ for $\xi \in H$. This means that for a net $(x_\lambda) \subset B(H)$ and $x \in B(H)$, $x_\lambda \to x$ in the strong operator topology if and only if $x_\lambda \xi \to x\xi$ for all $\xi \in H$.

Theorem 1.5 (von Neumann bicommutant)**.** *Suppose $A \subset B(H)$ is a unital $*$-closed subalgebra. Then the closure of A in the strong operator topology is equal to A''.*

Remark 1.6. The content of the above theorem is contained in the proof of the finite dimensional bicommutant theorem, which states that a unital $*$-closed subalgebra $A \subset M_n(\mathbb{C})$ satisfies $A = A''$. We refer the reader to [Jon10] for the proof of this finite dimensional statement, along with the adaptation to prove Theorem 1.5.

Definition 1.7. A *tracial* von Neumann algebra is a pair (A, tr_A) where A is a von Neumann algebra and $\operatorname{tr}_A : A \to \mathbb{C}$ is a faithful normal tracial state. This means:

- (state) tr_A is a linear functional with $\operatorname{tr}_A(1) = 1$ such that $\operatorname{tr}_A(a^*a) \geq 0$ for all $a \in A$.
- (faithful) $\operatorname{tr}_A(a^*a) = 0$ implies $a = 0$.
- (normal) if $(a_\lambda) \subset A$ is an increasing net of operators with $a_\lambda \nearrow a$, then $\operatorname{tr}_A(a_\lambda) \nearrow \operatorname{tr}_A(a)$.
- (tracial) $\operatorname{tr}_A(ab) = \operatorname{tr}_A(ba)$ for all $a, b \in A$.

Example 1.8. Suppose Γ is a countable group, and let $\ell^2\Gamma = \left\{\xi : \Gamma \to \mathbb{C} \middle| \sum_{g\in\Gamma} \|\xi(g)\|^2 < \infty\right\}$. The left regular action of Γ on $\ell^2\Gamma$ is given for $g \in \Gamma$ by $(\lambda_g\xi)(h) = \xi(g^{-1}h)$. One shows that this is a unitary representation $\lambda : \Gamma \to U(\ell^2\Gamma)$. The left regular von Neumann algebra is $L\Gamma = \{\lambda_g | g \in \Gamma\}''$. The trace on $L\Gamma$ is given by $\operatorname{tr}_{L\Gamma}(a) = \langle a\delta_e, \delta_e\rangle$, where δ_e is the indicator function at $e \in \Gamma$. For finite sums, $\operatorname{tr}_{L\Gamma}\left(\sum c_g \lambda(g)\right) = c_e$.

Definition 1.9. A *weight* on a von Neumann algebra A is an $\mathbb{R}^+$-linear map $\omega : A^+ \to [0, \infty]$, i.e., for all $a, b \in A^+$ and $\lambda \geq 0$, $\omega(\lambda a + b) = \lambda\omega(a) + \omega(b)$. Normailty and faithfulness for weights is defined similarly as above. A von Neumann algebra A is called *semifinite* if there exists a normal, faithful semifinite tracial weight $\operatorname{Tr}_A : A^+ \to [0, \infty]$, where the remaining properties are defined as follows:

- (tracial) $\operatorname{Tr}_A(a^*a) = \operatorname{Tr}_A(aa^*)$ for all $a \in A$.
- (semifinite) for all $a \in A^+$, there is a $0 \leq b \leq a$ such that $\operatorname{Tr}_A(b) < \infty$.

If A is semifinite with faithful normal semifinite trace Tr_A such that $\operatorname{Tr}_A(1) < \infty$, we may rescale Tr_A to get a faithful normal tracial state tr_A on A, and we call A *finite*.

Example 1.10. The von Neumann algebra $B(H)$ is always semifinite with trace

$$\mathrm{Tr}_{B(H)}(x) = \sum_{i=1}^{\infty} \langle x e_i, e_i \rangle$$

for $x \geq 0$, where $\{e_i\}$ is an orthonormal basis for H. Note that $\mathrm{Tr}_{B(H)}$ is independent of the choice of orthonormal basis.

Remark 1.11. From here onward, all traces are considered to be faithful normal tracial states, and all semifinite traces are considered to be faithful and normal unless stated otherwise.

The main distinction between tracial and finite von Neumann algebras is that tracial von Neumann algebras come with a trace, whereas such a trace exists for a finite von Neumann algebra.

1.2 The standard representation

Let (A, tr_A) be a tracial von Neumann algebra. We define its standard representation on $L^2A = L^2(A, \mathrm{tr}_A)$ via the GNS construction.

We define a sesquilinear form on A by $\langle a, b \rangle = \mathrm{tr}_A(b^*a)$. Since tr_A is faithful, there are no zero-length vectors. Define $L^2A = L^2(A, \mathrm{tr}_A)$ to be the completion of A in the 2-norm given by $\|a\|_2 = \mathrm{tr}_A(a^*a)^{1/2}$. We denote the image of $1 \in A$ in L^2A by Ω, which allows us to differentiate between the operator $a \in A$ and the vector $a\Omega \in L^2A$.

There is a left action of A on $A\Omega$ given by $\lambda_a(b) = ab\Omega$.

Exercise 1.12. Show that for all $a \in A$, $\lambda_a \in B(L^2A)$. Then show that $\lambda : A \to B(L^2A)$ is an injective normal unital $*$-algebra homomorphism.

Exercise 1.13. Show that Ω is cyclic and separating for $\lambda(A) \subset B(L^2A)$. That is, show that $\lambda(A)\Omega$ is dense in L^2A, and that $\lambda_a\Omega = \lambda_b\Omega$ if and only if $a = b$.

Definition 1.14. The standard representation of A is the left regular representation on $L^2(A)$.

There is also a right action of A on $A\Omega$ given by $\rho_a(b\Omega) = ba\Omega$.

Exercise 1.15. Do Exercise 1.12 for the right regular representation.
Hint: This uses the fact that tr_A *is a trace, and is not true for a general state!*

Since tr_A is a trace, the map $a\Omega \mapsto a^*\Omega$ is isometric, and thus extends to an anti-linear unitary J on L^2A called the *modular conjugation*.

From now on, we will identify A with $\lambda(A)$ on $B(L^2A)$. Consider the action of Ja^*J on L^2A:

$$Ja^*Jb\Omega = Ja^*b^*\Omega = Ja^*b^*\Omega = ba\Omega,$$

i.e., Ja^*J is right multiplication by a. Since right multiplication commutes with left multiplication, we have $JAJ \subseteq A' \cap B(L^2A)$.

Theorem 1.16. *$JAJ = A' \cap B(L^2A)$.*

The proof uses the following 4 exercises. One can perform these exercises by taking inner products against vectors in the dense subspace $A\Omega$.

Exercise 1.17. Show that $(JAJ)' = JA'J$.

Exercise 1.18. Show that for all $\eta, \xi \in L^2A$, $\langle \eta, \xi\rangle = \langle J\xi, J\eta\rangle$.

Exercise 1.19. Show that for all $x \in A'$, $Jx\Omega = x^*\Omega$.

Exercise 1.20. Show that for all $x \in A$ or A', $(JxJ)^* = Jx^*J$.

Proof of Theorem 1.16. Since we have already shown $JAJ \subseteq A' \cap B(L^2A)$, it remains to prove $A' \subseteq JAJ$. By Exercise 1.17, $(JAJ)' = JA'J$, so conjugating by J, it suffices to show $(JAJ)' \subseteq A$. Indeed, for $x, y \in A'$ and $a, b \in A$, we have

$$\begin{aligned}\langle xJyJa\Omega, b\Omega\rangle &= \langle Jya^*\Omega, x^*b\Omega\rangle = \langle Ja^*y\Omega, bx^*\Omega\rangle = \langle Jbx^*\Omega, a^*y\Omega\rangle\\ &= \langle JbJx\Omega, a^*y\Omega\rangle = \langle ax\Omega, Jb^*Jy\Omega\rangle = \langle ax\Omega, Jb^*y^*\Omega\rangle\\ &= \langle xa\Omega, Jy^*b^*\Omega\rangle = \langle xa\Omega, Jy^*Jb\Omega\rangle = \langle JyJxa\Omega, b\Omega\rangle.\end{aligned}$$

Hence x and JyJ commute, and thus $JyJ \in A'' = A$. The reader should verify where we used Exercises 1.18, 1.19, and 1.20 in the above calculation. □

1.3 Modules and bounded vectors

Let (A, tr_A) and (B, tr_B) be tracial von Neumann algebras.

Definition 1.21. A left A-module is a Hilbert space together with a normal $*$-homomorphism $\lambda A \to B(H)$. We write ${}_AH$ to denote that H has a left A action. A right B-module is a left B^{op}-module, i.e, it is a Hilbert space K together with a normal $*$-homomorphism $\rho : B^{\mathrm{op}} \to B(H)$. We write K_B to denote that K has a right B-action.

An $A - B$ bimodule is a Hilbert space H together with a left action $\lambda : A \to B(H)$ and a right action $\rho : B^{\mathrm{op}} \to B(H)$ which commute: $[\lambda(A), \rho(B^{\mathrm{op}})] = 0$. Note that this means $\lambda(A) \subseteq (B^{\mathrm{op}})'$. We write ${}_AH_B$ to denote that H has a left A-action and a right B-action.

Example 1.22. By Theorem 1.16, L^2A is an $A - A$ bimodule.

Example 1.23. Suppose we have an inclusion of tracial von Neumann algebras $(A \subseteq B, \mathrm{tr}_B)$. Then L^2B is a $C - D$ bimodule for $C, D \in \{A, B\}$.

Definition 1.24. Let H_B be a right B-module. A vector $\xi \in H_B$ is called *right B-bounded* if the map $b\Omega \mapsto \xi b$ extends to a bounded operator $L_\xi : L^2B \to H$. We denote the set of right B-bounded vectors by H_B°.

Suppose ${}_AK$ is a left A-module. A vector $\eta \in K$ is called *left A-bounded* if the map $a\Omega \mapsto a\eta$ extends to a bounded operator $R_\eta : L^2A \to K$. The set of left A-bounded vectors is denoted by ${}_AK^\circ$.

Suppose we have an $A - B$ bimodule ${}_AH_B$. The set of bi-bounded vectors ${}_AH^\circ \cap H_B^\circ$ is denoted simply by H°. By [Pop86, Lemma 1.2.2], H° is dense in H.

Exercise 1.25. Suppose $\xi \in H_B^\circ$.

(1) Show that L_ξ is right B-linear.

(2) Show that for all $x \in (B^{\text{op}})'$, $x\xi \in H_B^\circ$.

(3) Formulate and prove the analogous statements for left A-bounded vectors.

Definition 1.26. Suppose $\eta, \xi \in H_B^\circ$. By Exercise 1.25, $L_\eta^* L_\xi \in B(L^2B)$ commutes with the right B-action, and thus defines an element of B, denoted $\langle \eta|\xi\rangle_B$. The form $(\eta, \xi) \mapsto \langle \eta|\xi\rangle_B$ is a B-valued inner product which is B-linear on the right:

$$\langle \eta|\xi_1 b + \xi_2\rangle_B = \langle \eta|\xi_1\rangle_B b + \langle \eta|\xi_2\rangle_B.$$

Clearly $\langle \eta|\xi\rangle_B^* = \langle \xi|\eta\rangle_B$.

Exercise 1.27. Verify the right B-valued inner product is positive definite, i.e., for all $\xi \in H_B^\circ$, $\langle \xi|\xi\rangle_B \geq 0$, and it is equal to zero if and only if $\xi = 0$.
Hint: Show that for all $\xi \in H_B^\circ$, $\operatorname{tr}_B(\langle \xi|\xi\rangle_B) = \|\xi\|_H^2$.

Exercise 1.28. Show that the B-valued inner product is *completely positive*, i.e., for every n left B-bounded vectors $\xi_1, \dots, \xi_n \in H_B^\circ$, the operator $(\langle \xi_i|\xi_j\rangle_B)_{i,j=1}^n \in M_n(B)$ is positive.
Hint: Consider the right B-module $K_B = \bigoplus_{i=1}^n H_B$. Show that $\bigoplus_{i=1}^n H_B^\circ \subset K_B^\circ$, and calculate the right B-valued inner product for K_B.

Exercise 1.29. Show that for all $\eta, \xi \in H_B^\circ$ and $x \in (B^{\text{op}})'$, $\langle x\eta|\xi\rangle_B = \langle \eta|x^*\xi\rangle_B$.

Exercise 1.30. Show that ${}_A\langle \eta, \xi\rangle = JR_\eta^* R_\xi J$ defines a left A-valued inner product on a left A-module ${}_AK$ for left A-bounded vectors $\eta, \xi \in {}_AK^\circ$, which is A-linear on the left.
Hint: $R_\eta^* R_\xi$ commutes with the left A-action, so gives an element of $JAJ = A^{\text{op}}$, not A!

Proposition 1.31. *The space of right B-bounded vectors in L^2B is exactly $B\Omega$.*

Proof. That $B\Omega \subseteq L^2B_B^\circ$ is straightforward and left to the reader. Suppose $\xi \in L^2B_B^\circ$. The operator $L_\xi : L^2B \to L^2B$ is right B-linear (JBJ-linear) and thus defines an element $b \in (JBJ)' = B$. Now $\xi = L_\xi\Omega = \lambda_b\Omega = b\Omega$. $\square$

2 The 2-category of tracial von Neumann algebras

We now define a 2-category. We remind the reader that by '2-category', we mean a weak 2-category, also known as a bicategory.

Definition 2.1. The 2-category of tracial von Neumann algebras TvNA has

- Objects are tracial von Neumann algebras (A, tr_A). (Changing the trace changes the object!)
- 1-Morphisms are bimodules ${}_AH_B$.
- 2-Morphisms are bounded $A - B$ bilinear maps $f : {}_AH_B \to {}_AK_B$.

We will now go into the details of this 2-category.

2.1 Bimodules and fusion

Let (A, tr_A) and (B, tr_B) be tracial von Neumann algebras. We already saw that an $A-B$ bimodule is a Hilbert space H together with normal representations $\lambda : A \to B(H)$ and $\rho : B^{\mathrm{op}} \to B(H)$ such that $[\lambda(A), \rho(B)] = 0$. We will suppress the λ, ρ, and just write $a\xi b$ for the left and right action of $a \in A$ and $b \in B$ on $\xi \in H$.

Definition 2.2. Given ${}_AH_B$ and ${}_AK_B$, we can form the direct sum Hilbert space $H \oplus K$, which is again naturally an $A - B$ bimodule.

Definition 2.3. Suppose we have a right B-module H_B and a left B-module ${}_BK$. The *fusion* or the *relative tensor product* of H and K, denoted $H \boxtimes_B K$, is formed as follows. First, we take the tensor product of modules $H_B^\circ \otimes_B {}_BK^\circ$, which is algebraically spanned by symbols $\eta \otimes \xi$ for $\eta \in H_B^0$ and $\xi \in {}_BK^\circ$, subject to the relation that $\eta b \otimes \xi = \eta \otimes b\xi$. We endow this vector space with the sesquilinear form

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle_{H\boxtimes_B K} = \langle \langle \eta_2 | \eta_1 \rangle_B \xi_1, \xi_2 \rangle_K.$$

By Exercise 1.28, this sesquilinear form is positive semi-definite. In the usual way, we obtain a Hilbert space $H \boxtimes_B K$ as the completion of the vector space $H_B^\circ \otimes_B {}_BK^\circ / N_{\langle \cdot, \cdot \rangle}$, where

$$N_{\langle \cdot, \cdot \rangle} = \{ \xi \in H_B^\circ \otimes_B {}_BK^\circ | \langle \xi, \xi \rangle_{H\boxtimes_B K} = 0 \}$$

is the subspace of length zero vectors under the pseudonorm induced by the sesquilinear form.

For subspaces $D_H \subseteq H_B^\circ$ and $D_K \subseteq {}_BK^\circ$, we denote the image of $D_H \otimes_B D_K$ in $H \boxtimes_B K$ by $D_H \boxtimes_B G_K$. We denote the image of the vector $\eta \otimes \xi \in H_B^\circ \otimes_B {}_BK^\circ$ by $\eta \boxtimes \xi$.

Lemma 2.4 ([Bur03, Claim 3.2.15])**.** *Suppose $D_H \subseteq H_B^\circ$ is a dense subspace of H_B and $D_K \subseteq {}_BK^\circ$ is a dense subspace of ${}_BK$. Then $D_H \boxtimes_B D_K$ is dense in $H \boxtimes_B K$.*

Proof. Suppose $\eta \in H_B^\circ$ and $\xi \in {}_BK^\circ$. Take $(\xi_n) \subset D_K$ with $\xi_n \to \xi$. Then

$$\|\eta \boxtimes \xi - \eta \boxtimes \xi_n\|_{H\boxtimes_B K}^2 = \|\eta \boxtimes (\xi - \xi_n)\|_{H\boxtimes_B K}^2 = \langle \langle \eta|\eta\rangle_B (\xi - \xi_n), \xi - \xi_n \rangle_K \leq \|\langle \eta|\eta\rangle_B\|_\infty \cdot \|\xi - \xi_n\|_K^2 \to 0.$$

Similarly, we may approximate each $\eta \boxtimes \xi_n$ by $\eta_k \boxtimes \xi_n$ for some sequence $(\eta_k) \subset D_H$ with $\eta_k \to \eta$. $\square$

Now if H_B also has a left A-action, so does $H \boxtimes_B K$. Similarly, if ${}_BK$ also has a right C-action. This means that the fusion of ${}_AH_B$ and ${}_BK_C$ is naturally an $A - C$ bimodule.

Exercise 2.5. That the left A-action on $H \boxtimes_B K$ is normal can be verified as follows. (A similar series of exercises verifies normality for the right C-action.)

(1) Show that if $D \subset {}_AH$ is a dense subspace and $\omega_\zeta(a_\lambda) \nearrow \omega_\zeta(a)$ for every $\zeta \in D$, then $a_\lambda \nearrow a$.

(2) Show that if $a, b \geq 0$ in a tracial von Neumann algebra (B, tr_B), then $\operatorname{tr}_B(ab) \geq 0$. Moreover, if $a_\lambda \nearrow a$, then $\operatorname{tr}_B(a_\lambda b) \nearrow \operatorname{tr}_B(ab)$.

(3) Show that if $a_\lambda \nearrow a$ is an increasing net in A, then for all $\eta_1, \ldots, \eta_n \in H_B^\circ$, the matrices $(\langle a_\lambda \eta_i | \eta_j \rangle_B)_{i,j=1}^n \in M_n(B)$ are positive, and they increase to $(\langle a\eta_i | \eta_j \rangle_B)_{i,j=1}^n \in M_n(B)$.

(4) Show that for all $\sum_{i=1}^{n} \eta_i \boxtimes \xi_i \in H_B^\circ \boxtimes_B {}_B K^\circ$,

$$\sum_{i,j=1}^{n} \langle a_\lambda(\eta_i \boxtimes \xi_i), \eta_j \boxtimes \xi_j \rangle = \operatorname{tr}_{M_n(B)} \left((\langle a_\lambda \eta_i | \eta_j \rangle_B)_{i,j=1}^{n} (\langle \xi_i | \xi_j \rangle_B)_{i,j=1}^{n} \right)$$

(5) Deduce that for every vector state associated to a finite sum $\zeta = \sum_{i=1}^{n} \eta_i \boxtimes \xi_i \in H_B^\circ \boxtimes_B {}_B K^\circ$ and every increasing net $a_\lambda \nearrow a$ in A, $\omega_\zeta(a_\lambda) \nearrow \omega_\zeta(a)$. Then use (1) to conclude the left A-action on $H \boxtimes_B K$ is normal.

Exercise 2.6. Verify that for all $\eta_1, \eta_2 \in H_B^\circ$ and $\xi_1, \xi_2 \in {}_B K^\circ$,

$$\langle \langle \eta_2 | \eta_1 \rangle_B \xi_1, \xi_2 \rangle_K = \langle \eta_1 {}_B\langle \xi_1, \xi_2 \rangle, \eta_2 \rangle_H.$$

Hint: Show they are both equal to $\operatorname{tr}_B(\langle \eta_2 | \eta_1 \rangle_B \cdot {}_B\langle \xi_1, \xi_2 \rangle)$.

2.2 Intertwiners and the exchange relation

Suppose (A, tr_A) and (B, tr_B) are tracial von Neumann algebras, and ${}_A H_B$ and ${}_A K_B$ are two $A-B$ bimodules.

Definition 2.7. The space of *intertwiners* from H to K is the Banach space $\operatorname{Hom}_{A-B}(H, K)$ of continuous linear transformations which are $A-B$ bilinear. Note that $\operatorname{End}_{A-B}(H) = A' \cap (B^{\mathrm{op}})' \cap B(H)$ is a von Neumann algebra.

Exercise 2.8. Show that $A-B$ subbimodules ${}_A K_B$ of a given $A-B$ bimodule ${}_A H_B$ are in bijective correspondence with orthogonal projections $p_K \in \operatorname{End}_{A-B}(H)$. Under this correspondence, show that K is irreducible if and only if p_K is minimal in the von Neumann algebra $\operatorname{End}_{A-B}(H)$, i.e., $p_K \operatorname{End}_{A-B}(H) p_K = \mathbb{C} p_K$.

Exercise 2.9 (Roberts' 2×2 trick)**.** Show that the Banach space $\operatorname{Hom}_{A-B}(H, K)$ is isometrically isomorphic to $p_K \operatorname{End}_{A-B}(H \oplus K, H \oplus K) p_H$, where p_H is the projection with range $H \oplus 0$ and p_K is the projection with range $0 \oplus K$.

Exercise 2.10. Use Roberts' 2×2 trick from Exercise 2.9 to show that the Banach space $\operatorname{Hom}_{A-B}(H, K)$ is the dual space of a Banach space.

Definition 2.11. Suppose $f \in \operatorname{Hom}_{A-B}(H_1, H_2)$ and $g \in \operatorname{Hom}_{B-C}(K_1, K_2)$. We define the map $f \boxtimes g \in \operatorname{Hom}_{A-C}(H_1 \boxtimes_B K_1, H_2 \boxtimes_B K_2)$ by the unique extension of the map $\eta \boxtimes \xi \mapsto f(\eta) \boxtimes g(\xi)$ where $\eta \in H_1^\circ$ and $\xi \in K_1^\circ$.

Exercise 2.12. Show that the map $f \boxtimes g$ from Definition 2.11 is well-defined. That is, show that for all finite sums $\sum_i \eta_i \boxtimes \xi_i \in H_1^\circ \boxtimes_B K_1^\circ$,

$$\left\| \sum_i f(\eta_i) \boxtimes g(\xi_i) \right\|_{H_2 \boxtimes_B K_2} \leq \|f\| \cdot \|g\| \cdot \left\| \sum_i \eta_i \boxtimes \xi_i \right\|_{H_1 \boxtimes_B K_1}.$$

(1) Use Exercise 2.5 to show that $(\langle f(\eta_i) | f(\eta_j) \rangle_B)_{i,j} \in M_n(B)$ is positive, and $(\langle f(\eta_i) | f(\eta_j) \rangle_B)_{i,j} \leq \|f\|^2 (\langle \eta_i | \eta_j \rangle_B)_{i,j}$.

(2) Conclude that $\left\|\sum_i f(\eta_i) \boxtimes g(\xi_i)\right\|_{H_2 \boxtimes_B K_2} \leq \|f\| \cdot \left\|\sum_i \eta_i \boxtimes g(\xi_i)\right\|_{H_1 \boxtimes_B K_2}$.

(3) Repeat the argument for g.

Exercise 2.13 (Exchange relation)**.** Suppose $f_1 \in Hom_{A-B}(H_1, H_2)$, $f_2 \in \operatorname{Hom}_{A-B}(H_2, H_3)$, $g_1 \in \operatorname{Hom}_{B-C}(K_1, K_2)$, and $g_2 \in \operatorname{Hom}_{B-C}(K_2, K_3)$. Show that

$$(f_2 \boxtimes g_2) \circ (f_1 \boxtimes g_1) = (f_2 \circ f_1) \boxtimes (g_2 \circ g_1).$$

Hint: Show they are equal on simple vectors in $H_1^\circ \boxtimes_B K_1^\circ$.

2.3 Associators and unitors

We now define the associator and unitor natural isomorphisms for our 2-category. Of special importance is Popa's lemma saying that for every bimodule ${}_AH_B$, $H^\circ = {}_AH^\circ \cap H_B^\circ$ is dense in H, together with Lemma 2.4 which says that for all ${}_AH_B$ and ${}_BK_C$, $H^\circ \boxtimes_B K^\circ$ is dense in $H \boxtimes_B K$.

Definition 2.14 (Associators)**.** For ${}_AH_B$, ${}_BK_C$, and ${}_CL_D$, we define the *associator* unitary isomorphism $\alpha_{H,K,L} : H \boxtimes_B (K \boxtimes_C L) \to (H \boxtimes_B K) \boxtimes_C L$ as the extension of the map $H^\circ \boxtimes_A (K^\circ \boxtimes_C L^\circ) \to (H^\circ \boxtimes_B K^\circ) \boxtimes_C L^\circ$ given by $\xi \boxtimes (\eta \boxtimes \zeta) \mapsto (\xi \boxtimes \eta) \boxtimes \zeta$.

Exercise 2.15. Show that $\alpha_{H,K,L}$ is a well-defined, i.e., that $\alpha_{H,K,L}$ on $H^\circ \boxtimes_B (K^\circ \boxtimes_C L^\circ)$ is an isometry with dense range. Conclude that the associators α are unitary isomorphisms.
Hint: To show the map is well-defined, use Exercise 2.6 to show it's an isometry. To show it extends uniquely to $H \boxtimes_B (K \boxtimes_C L)$, use Lemma 2.4 twice – once for showing $D_{K \boxtimes_C L} = K^\circ \boxtimes_C L^\circ$ is dense in $K \boxtimes_C L$, and once for showing $H^\circ \boxtimes_B (K^\circ \boxtimes_C L^\circ)$ is dense in $H \boxtimes_B (K \boxtimes_C L)$. Finally, to show it has dense range, use Lemma 2.4 twice more.

Exercise 2.16. Prove that α is natural, i.e., if $f \in \operatorname{Hom}_{A-B}(H_1, H_2)$, $g \in \operatorname{Hom}_{B-C}(K_1, K_2)$, and $h \in \operatorname{Hom}_{C-D}(L_1, L_2)$, then $\alpha_{H_2,K_2,L_2} \circ (f \boxtimes (g \boxtimes h)) = ((f \boxtimes g) \boxtimes h) \circ \alpha_{H_1,K_1,L_1}$.

Exercise 2.17 (Pentagon axiom)**.** The associator α satisfies the following coherence axiom. For all tracial von Neumann algebras A, B, C, D, E (with distinguished traces) and all bimodules ${}_AG_B$, ${}_BH_C$, ${}_CK_D$, and ${}_DL_E$, the following diagram commutes:

$$\begin{tikzcd}
G \boxtimes_B (H \boxtimes_C (K \boxtimes_D L)) \arrow[r, "\alpha_{G,H,K\boxtimes L}"] \arrow[dd, "\mathrm{id}_G \boxtimes \alpha_{H,K,L}"] & (G \boxtimes_B H) \boxtimes_C (K \boxtimes_D L) \arrow[dr, "\alpha_{G\boxtimes H,K,L}"] & \\
& & ((G \boxtimes_B H) \boxtimes_C K) \boxtimes_D L \\
G \boxtimes_B ((H \boxtimes_C K) \boxtimes_D L) \arrow[r, "\alpha_{G,H\boxtimes L,K}"] & (G \boxtimes_B (H \boxtimes_C K)) \boxtimes_D L \arrow[ur, "\alpha_{G,H,K}\boxtimes \mathrm{id}_L"] &
\end{tikzcd}$$

Hint: Check both composites on simple vectors in $G^\circ \boxtimes_B (H^\circ \boxtimes_C (K^\circ \boxtimes_D L^\circ))$, which is dense in $G \boxtimes_B (H \boxtimes_C (K \boxtimes_D L))$ by three uses of Lemma 2.4.

Definition 2.18 (Unitors)**.** For ${}_AH_B$, we define a unitary isomorphism $\lambda_H^A \in \operatorname{Hom}_{A-B}(L^2A \boxtimes H, H)$ as the extension of the linear map $A\Omega \boxtimes H^\circ \to H$ by $a\Omega \boxtimes \xi \mapsto a\xi$. Similarly, we define $\rho_H^B \in \operatorname{Hom}_{A-B}(H \boxtimes L^2B, H)$ as the extension of $\xi \boxtimes b\Omega \mapsto \xi b$.

Exercise 2.19. Verify that the unitors λ, ρ are well-defined unitary isomorphisms.

Exercise 2.20. Verify that the unitors λ, ρ are natural isomorphisms.

Exercise 2.21 (Triangle axiom)**.** The associator α and unitors λ, ρ satisfy the following coherence axiom. For all tracial von Neumann algebras A, B, C (with distinguished traces) and all bimodules ${}_AH_B$, ${}_BK_C$, the following diagram commutes:

$$\begin{tikzcd}
H \boxtimes_B (L^2B \boxtimes_B K) \arrow[rr, "\alpha_{H,L^2B,K}"] \arrow[dr, "\mathrm{id}_H \boxtimes \lambda^B_K"'] & & (H \boxtimes_B L^2B) \boxtimes_B K \arrow[dl, "\rho^B_H \boxtimes \mathrm{id}_K"] \\
& H \boxtimes_B K &
\end{tikzcd}$$

Definition 2.22. A *2-category* C consists of the following data:

- A collection of objects $A, B, C, \ldots$.
- For each pair of objects A, B, a category $\mathsf{Mor}(A, B)$ whose objects are called 1-morphisms and whose morphisms are called 2-morphisms. Given two 1-morphisms $H, K \in \mathsf{Mor}(A, B)$, we denote the set of 2-morphisms from H to K by $\mathrm{Hom}(H, K)$.
- For each object A, a distinguished 1-morphism $1_A \in \mathsf{Mor}(A, A)$ called the identity 1-morphism.
- For each three objects A, B, C, a bi-functor $- \circ - : \mathsf{Mor}(A, B) \times \mathsf{Mor}(B, C) \to \mathsf{Mor}(A, C)$ called *horizontal composition*. Here, we are writing composition from left to right, in the reverse order of what is usually used in mathematics!
- For each quadruple of objects A, B, C, D, an *associator* natural isomorphism

$$\begin{tikzcd}
\mathsf{Mor}(A,B) \times \mathsf{Mor}(B,C) \times \mathsf{Mor}(C,D) \arrow[r, "(-\circ-)\times \mathrm{id}"] \arrow[d, "\mathrm{id} \times (-\circ-)"] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & \mathsf{Mor}(A,C) \times \mathsf{Mor}(C,D) \arrow[d, "-\circ-"] \\
\mathsf{Mor}(A,B) \times \mathsf{Mor}(B,D) \arrow[r, "-\circ-"] & \mathsf{Mor}(A,D).
\end{tikzcd}$$

  This means for all $H \in \mathsf{Mor}(A, B)$, $K \in \mathsf{Mor}(B, C)$, and $L \in \mathsf{Mor}(C, D)$, we have an isomorphism $\alpha_{H,K,L} : H \circ (K \circ L) \to (H \circ K) \circ L$ which is natural in all three variables.

- For each pair of objects A, B, a pair of natural isomorphisms $\lambda^A : \mathrm{const}_{1_A} \circ \mathrm{id}_{\mathsf{Mor}(A,B)} \Rightarrow \mathrm{id}_{\mathsf{Mor}(A,B)}$ and $\rho^B : \mathrm{id}_{\mathsf{Mor}(A,B)} \circ \mathrm{const}_{1_B} \Rightarrow \mathrm{id}_{\mathsf{Mor}(A,B)}$ called *unitors*. Here, $\mathrm{const}_{1_A} \circ \mathrm{id}_{\mathsf{Mor}(A,B)} : \mathsf{Mor}(A, B) \to \mathsf{Mor}(A, B)$ is the functor $H \mapsto 1_A \circ H$ and $\mathrm{const}_{1_B} \Rightarrow \mathrm{id}_{\mathsf{Mor}(A,B)} : \mathsf{Mor}(A, B) \to \mathsf{Mor}(A, B)$ is the functor $H \mapsto H \circ 1_B$. This means that for every $H \in \mathsf{Mor}(A, B)$, we have isomorphisms $\lambda^A_H : 1_A \circ H \to H$ and $\rho^B_H : H \circ 1_B \to H$ which are natural in H.

This data must satisfy the following axioms:

- (Pentagon) The associators satisfy the pentagon axiom from Exercise 2.17.

- (Triangle) The associator and unitors satisfy the triangle axiom from Exercise 2.21.

We remind the reader that our notion of 2-category is a weak 2-category, which is also known as a bicategory.

Remark 2.23. The reader may be familiar with the notion of a monoidal category. We note that the above definition for a 2-category, when restricted to a 2-category with only one object, is exactly the definition of a monoidal category.

Theorem 2.24. *There is a 2-category* TvNA *whose objects are tracial von Neumann algebras, whose 1-morphisms are bimodules, and whose 2-morphisms are intertwiners.*

Proof. We verify the data and axioms from Definition 2.22. For each pair of tracial von Neumann algebras (A, tr_A) and (B, tr_B), we have take $\mathsf{Mor}(A, B)$ to be the category whose objects are $A - B$ bimodules and whose morphisms are the intertwiners. This means the 1-morphisms in TvNA are bimodules, and the 2-morphisms are intertwiners. The distinguished 1-morphism $1_A \in \mathsf{Mor}(A, A)$ is $L^2A = L^2(A, \mathrm{tr}_A)$. The horizontal composition functor $\mathsf{Mor}(A, B) \times \mathsf{Mor}(B, C) \to \mathsf{Mor}(A, C)$ is the Connes fusion tensor product $- \boxtimes_B -$ from Definition 2.11, which is a bi-functor by Exercise 2.13, and the fact that $\mathrm{id}_H \boxtimes \mathrm{id}_K$ is easily verified to be $\mathrm{id}_{H \boxtimes K}$. The associator and unitor natural isomorphisms were defined in Definition 2.14 and 2.18 respectively. That they satisfy the pentagon and triangle axioms was verified in Exercises 2.17 and 2.21. $\square$

2.4 Involution 1: the adjoint

We now define the first of two involutions on TvNA, which is called the *adjoint*.

Definition 2.25. Recall from Definition 2.7 that $\mathrm{Hom}_{A-B}(H, K)$ is the space of continuous linear transformations $H \to K$ which are $A - B$ bilinear. Since H, K are Hilbert spaces, every $f \in \mathrm{Hom}_{A-B}(H, K)$ has an adjoint $f^* \in \mathrm{Hom}_{A-B}(K, H)$.

Exercise 2.26. Verify that the adjoint satisfies the following axioms.

(1) For all $f \in \mathrm{Hom}_{A-B}(H, K)$, $f^{**} = f$.

(2) For all $f_1 \in Hom_{A-B}(H_1, H_2)$ and $f_2 \in \mathrm{Hom}_{A-B}(H_2, H_3)$, $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$.

(3) For all $f \in \mathrm{Hom}_{A-B}(H_1, H_2)$ and $g \in \mathrm{Hom}_{B-C}(K_1, K_2)$, $(f \boxtimes g)^* = f^* \boxtimes g^*$.

Definition 2.27. An *adjoint* on a 2-category C is a map $* : \mathrm{Hom}(H, K) \to \mathrm{Hom}(K, H)$ for all 1-morphisms H, K in C that satisfies the axioms from Exercise 2.26, and such that the associators and unitors are unitary isomorphisms. (A 2-morphism $f \in \mathrm{Hom}(H, K)$ is called *unitary* if f is invertible with inverse f^*.)

A 2-category C with adjoint $*$ is called a C*-*2-category* if for all 1-morphisms H, K in C,

- for all $f \in \mathrm{Hom}(H, K)$, there is a $g \in \mathrm{Hom}(H, H)$ such that $f^* \circ f = g^* \circ g$, and
- the function $\| \cdot \| : \mathrm{Hom}(H, K) \to [0, \infty]$ given by
$$\|f\|^2 = \sup \{ |\lambda| \geq 0 \mid f^* \circ f - \lambda \, \mathrm{id}_H \text{ is not invertible} \}$$
defines a norm on $\mathrm{Hom}(H, K)$ which is sub-multiplicative for composition, the normed spaces $\mathrm{Hom}(H, K)$ are complete with respect to this norm, and the norm satisfies the C*-axiom $\|f^* \circ f\| = \|f\|^2$.

A C*-2-category is called a W*-*2-category* if for all 1-morphisms H, K in C, the 2-morphism Banach space $\mathrm{Hom}(H, K)$ is the dual space of a Banach space.

Remark 2.28. Note that being a C*-2-category or a W*-2-category is a property of a 2-category with adjoint, and not extra structure.

Exercise 2.29. Suppose C is a 2-category with adjoint which admits finite direct sums of 1-morphisms. Show that C is a C*/W*-2-category respectively if and only if for every 1-morphism H in C, $\mathrm{End}(H)$ is a C*/W*-algebra.
Use Roberts' 2×2 trick from Exercise 2.9.

Corollary 2.30. *The 2-category* TvNA *with its adjoint forms a* W*-*2-category.*

Proof. We saw in Theorem 2.24 that TvNA is a 2-category. By Exercises 2.15, 2.19, and 2.26, $*$ is an adjoint on TvNA. Now TvNA is a W*-2-category by Exercise 2.10 or 2.29. Note that TvNA admits *arbitrary* direct sums of bimodules, and for all bimodules ${}_AH_B$, $\mathrm{End}_{A-B}(H) = A' \cap (B^{\mathrm{op}})' \cap B(H)$ is a von Neumann algebra. □

2.5 Involution 2: the conjugate

We now define the second of two involutions, which is called the *conjugate*.

Definition 2.31. Recall that for every Hilbert space H, its conjugate Hilbert space $\overline{H}$ is $\left\{\overline{\xi}\middle|\xi \in H\right\}$ with addition and scalar multiplication given by $\lambda\overline{\eta} + \overline{\xi} = \overline{\overline{\lambda}\eta + \xi}$ for $\lambda \in \mathbb{C}$ and $\eta, \xi \in H$. The inner product on $\overline{H}$ is given by

$$\langle \overline{\eta}, \overline{\xi} \rangle_{\overline{H}} := \langle \xi, \eta \rangle_H.$$

If H is an $A - B$ bimodule, then $\overline{H}$ is naturally a $B - A$ bimodule with $b \cdot \overline{\xi} \cdot a := \overline{a^*\xi b^*}$.

Now if $f \in \mathrm{Hom}_{A-B}(H, K)$, we can define $\overline{f} \in \mathrm{Hom}_{B-A}(\overline{H}, \overline{K})$ by $\overline{f}(\overline{\xi}) = \overline{f(\xi)}$. One verifies:

- For all ${}_AH_B$, $\overline{\mathrm{id}_H} = \mathrm{id}_{\overline{H}}$.
- For all $f \in \mathrm{Hom}_{A-B}(H, K)$ and $g \in \mathrm{Hom}_{A-B}(K, L)$, $\overline{g \circ f} = \overline{g} \circ \overline{f}$.

We have several canonical natural isomorphisms. For each tracial von Neumann algebra (A, tr_A), L^2A has a canonical *real structure* $r_A \in \mathrm{Hom}_{A-A}(L^2A, \overline{L^2A})$ given by the extension of the map $a\Omega \mapsto \overline{a^*\Omega}$. For each $A - B$ bimodules ${}_AH_B$, we have a canonical isomorphism $\varphi_H \in \mathrm{Hom}_{A-B}(H, \overline{\overline{H}})$ given by $\xi \mapsto \overline{\overline{\xi}}$. Finally, for each $A - B$ bimodule ${}_AH_B$ and $B - C$ bimodule ${}_BK_C$, we have an isomorphism $\nu_{K,H} \in \mathrm{Hom}_{C-A}(\overline{K} \boxtimes_B \overline{H}, \overline{H \boxtimes_B K})$ given by the extension of $\overline{\xi} \boxtimes_B \overline{\eta} \in \overline{K}^\circ \boxtimes_B \overline{H}^\circ$ maps to $\overline{\eta \boxtimes \xi}$.

Exercise 2.32. Show that the inner product on $\overline{H}$ is linear in the first variable and conjugate-linear in the second variable.

Exercise 2.33. Verify that the maps $r_A \in \mathrm{Hom}_{A-A}(L^2A, \overline{L^2A})$, $\varphi_H \in \mathrm{Hom}_{A-B}(H, \overline{\overline{H}})$, and $\nu_{K,H} \in \mathrm{Hom}_{C-A}(\overline{K} \boxtimes_B \overline{H}, \overline{H \boxtimes_B K})$ are well-defined unitary isomorphisms.

Exercise 2.34. Verify that the maps r, φ, ν are natural isomorphisms.

Exercise 2.35. Verify that the maps r, φ, ν satisfy the following axioms:

(1) (associative) For all bimodules ${}_AH_B$, ${}_BK_C$ and ${}_CL_D$, $\nu_{L,H\boxtimes_B K} \circ (\mathrm{id}_{\overline{L}} \boxtimes_C \nu_{K,H}) = \nu_{K\boxtimes_C L,H} \circ (\nu_{L,K} \boxtimes_B \mathrm{id}_{\overline{H}})$.

(2) (unital 1) For all tracial von Neumann algebras A, $\varphi_{L^2A} = \overline{r_A} \circ r_A$.

(3) (unital 2) For all bimodules ${}_AH_B$, $\nu_{H,L^2B} \circ (\mathrm{id}_{\overline{H}} \boxtimes r_B) = \mathrm{id}_{\overline{H}} = \nu_{L^2A,H} \circ (r_A \boxtimes \mathrm{id}_{\overline{H}})$.

(4) (monoidal) For all bimodules ${}_AH_B, {}_BK_C$, $\varphi_{H\boxtimes_B K} = \overline{\nu_{K,H}} \circ \nu_{\overline{H},\overline{K}} \circ (\varphi_H \otimes \varphi_K)$.

Remark 2.36. It is possible to define the conjugation $\overline{\cdot}$ for TvNA such that $\varphi_H = \mathrm{id}_H$ for all bimodules ${}_AH_B$. To do so, instead of defining $\overline{H}$ to be the space of symbols $\left\{\overline{\xi}\middle|\xi \in H\right\}$, we define $\overline{H}$ to be the same underlying set H, but with the conjugate Hilbert space structure. This means we define a new scalar multiplication on H given by $\lambda \cdot \eta := \overline{\lambda}\eta$, and we define a new inner product on H given by $\langle \eta, \xi\rangle_{\overline{H}} := \langle \xi, \eta\rangle_H$. At first glance, this new inner product seems to be linear in the second variable, but we are using the conjugate scalar multiplication, and thus it is linear in the first variable. Since complex conjugation is its own inverse, we see that taking the conjugate twice exactly recovers our original Hilbert space H.

In the sequel, we will assume that $\varphi_H = \mathrm{id}_H$ for all ${}_AH_B$.

Exercise 2.37. Verify that the conjugate Hilbert space structure defined in Remark 2.36 actually endows H with the structure of a Hilbert space.

Definition 2.38. A conjugation $\overline{\cdot}$ on a 2-category C is the data of an anti-linear functor $\overline{\cdot} : \mathsf{Mor}(A, B) \to \mathsf{Mor}(A, B)$ for every pair of objects A, B, together with the data of

- for every object A, an isomorphism $r_A : 1_A \to \overline{1_A}$,
- for all objects A, B, a natural isomorphism $\varphi : \mathrm{id}_{\mathsf{Mor}(A,B)} \Rightarrow \overline{\overline{\cdot}}$, and
- for all objects A, B, C, a natural isomorphism

$$\begin{tikzcd}
\mathsf{Mor}(C,B) \times \mathsf{Mor}(B,A) \arrow[rr, "- \circ -"] \arrow[d, "(\overline{\cdot})\times(\overline{\cdot})"'] & & \mathsf{Mor}(C,A) \arrow[d, "\overline{\cdot}"] \\
\mathsf{Mor}(C,B) \times \mathsf{Mor}(B,A) \arrow[rr, "- \circ -"] \arrow[urr, phantom, "\overset{\nu}{\Longrightarrow}"] & & \mathsf{Mor}(C,A).
\end{tikzcd}$$

  This means for all $H \in \mathsf{Mor}(A, B)$ and $K \in \mathsf{Mor}(B, C)$, we have an isomorphism $\nu_{K,H} : \overline{K} \circ \overline{H} \to \overline{H \circ K}$ which is natural in H and K.

which satisfy the relations of Exercise 2.35.

Definition 2.39. A 2-category C with adjoint and conjugation is called *bi-involutive* if

- the conjugation natural isomorphisms r, φ, ν are unitary, and

- the two involutions commute. This means for every objects A, B, all 1-morphisms $H, K \in \mathsf{Mor}(A, B)$, and every 2-morphism $f \in \mathrm{Hom}(H, K)$, we have $\overline{f^*} = \overline{f}^*$.

Corollary 2.40. *The 2-category* TvNA *with its adjoint and conjugate is bi-involutive.*

Proof. That the maps φ, ν, r are unitary follows from Exercise 2.33. Suppose $f \in \mathrm{Hom}_{A-B}(H, K)$. For all $\eta \in H^\circ$ and $\xi \in K^\circ$,

$$\langle \overline{f}(\overline{\eta}), \overline{\xi} \rangle_{\overline{K}} = \langle \overline{f(\eta)}, \overline{\xi} \rangle_{\overline{K}} = \langle \xi, f(\eta) \rangle_K = \langle f^*(\xi), \eta \rangle_H = \langle \overline{\eta}, \overline{f^*(\xi)} \rangle_{\overline{H}} = \langle \overline{\eta}, \overline{f^*}(\overline{\xi}) \rangle_{\overline{H}}.$$

We conclude that $\overline{f}^* = \overline{f^*}$, and thus TvNA is bi-involutive. $\square$

2.6 Graphical calculus

TODO:

3 Factors, subfactors, and dualizability

TODO:

References

[Bis97] Dietmar Bisch, *Bimodules, higher relative commutants and the fusion algebra associated to a subfactor*, Operator algebras and their applications (Waterloo, ON, 1994/1995), 13-63, Fields Inst. Commun., 13, Amer. Math. Soc., Providence, RI, 1997, MR1424954, (preview at google books).

[Bur03] Michael Burns, *Subfactors, planar algebras, and rotations*, Ph.D. thesis at the University of California, Berkeley, 2003, arXiv:1111.1362.

[Jon10] Vaughan F. R. Jones, *Von neumann algebras*, 2010, http://math.berkeley.edu/~vfr/MATH20909/VonNeumann2009.pdf.

[Pop86] Sorin Popa, *Correspondences*, INCREST Preprint, 1986, available at http://www.math.ucla.edu/~popa/popa-correspondences.pdf.