The module embedding theorem

David Penneys, OSU

AMS JMM Special Session on Hopf algebras and tensor categories

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DEPARTMENT OF MATHEMATICS

QUANTUM SYMMETRIES

Summer Research Program

June 3-14, 2019 • Columbus, OH

This program aims to bring together researchers in quantum symmetries (e.g. tensor categories, operator algebras and subfactors, quantum groups and Hopf algebras, higher categories, topological and conformal field theories, etc.) for a two-week-long summer research program. Each morning will consist of 2 mini-courses and one research talk. The afternoons will be left open for research and collaboration amongst the participants.

MINI-COURSES

André Henriques, University of Oxford
Emily Peters, Loyola University Chicago
Jamie Vicary, University of Birmingham and University of Oxford
Makoto Yamashita, University of Oslo

RESEARCH TALKS

Michael Brannan, Texas A&M
Shawn Xingshan Cui, Virginia Tech
Colleen Delaney, UCSB
César Galindo, Universidad de los Andes
Scott Morrison, Australian National University
Claudia Pinzari, Universita' di Roma “La Sapienza”
Julia Plavnik, Indiana University
Chelsea Walton, University of Illinois Urbana-Champaign
Stuart White, University of Glasgow

All participants are asked to register online before March 1, 2019 at **go.osu.edu/qs-registration**.

For more information, visit **go.osu.edu/QuantumSymmetries2019**
or contact: Corey Jones: jones.6457@osu.edu • David Penneys: penneys.2@osu.edu

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DEPARTMENT OF MATHEMATICS

GRADUATE ADVISOR WORKSHOP

June 1-2, 2019 • The Ohio State University, Columbus, OH

How do I successfully advise graduate students through the PhD process? In this workshop, which is aimed at early career faculty making the transition to graduate advising, we will work collaboratively to understand the many components of the advisor-student relationship. Panels will be anchored by senior faculty who will share their experience on being an effective graduate advisor and best practices in mentoring.

Some experienced leading faculty participants include:
Benjamin Braun, University of Kentucky
Angela Gibney, Rutgers University
Phil Kutzko, University of Iowa
Fadil Santosa, University of Minnesota

Ohio State organizers:
María Angélica Cueto
Dave Penneys
Krystal Taylor
Daniel Thompson

Please register here: **go.osu.edu/gaw-registration** if you would like to participate.

Visit **go.osu.edu/gaw2019** for more information about this workshop.

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References

Today's talk focuses on the following recent articles:

- The Extended Haagerup fusion categories
  (with Grossman, Morrison, Peters, and Snyder)
  `arXiv:1810.06076`
- The module embedding theorem via towers of algebras
  (with Coles*, Huston, and Srinivas*)
  *graduating undergraduate researcher. Highly recommended!
  `arXiv:1810.07049`
- Unitary dual functors for unitary multitensor categories
  `arXiv:1808.00323`

Planar algebras and tensor categories

Definition/Folklore Theorem

The following are equivalent mathematical objects:

1. Finite depth subfactor planar algebras $\mathcal{P}_\bullet$,
2. Pairs $(\mathcal{C}, A)$ with $\mathcal{C}$ a unitary fusion category and $A \in \mathcal{C}$ a unitary Frobenius algebra object, and
3. Unitary 2×2 multifusion categories $\mathcal{D}$ such that $1_\mathcal{D} = 1_+ \oplus 1_-$ is a decomposition into simples and a choice of generating object $X = 1_+ \otimes X \otimes 1_-$.

$1 \to 2$ Take $\mathcal{C} = \text{Proj}_{\text{even}}(\mathcal{P}_\bullet)$ and $A =$  with multiplication 

$2 \to 3$ Take $\mathcal{D} = \begin{pmatrix} \mathcal{C} & \text{Mod}_A \\ {}_A\text{Mod} & {}_A\text{Mod}_A \end{pmatrix}$ and $X = A \in \text{Mod}_A$.

$3 \to 1$ Take $\mathcal{P}_{n,+} := \text{End}(X^{\text{alt}\,\otimes n})$ and $\mathcal{P}_{n,-} := \text{End}(\overline{X}^{\text{alt}\,\otimes n})$

Graph planar algebra embedding theorem

Given a bipartite graph Γ, there is a combinatorial object called the *graph planar algebra* [Jon00] which should be viewed as a target for planar algebra representations.

Theorem [JP11]

Every finite depth subfactor planar algebra embeds in the graph planar algebra of its principal graph.

Many exotic subfactors and fusion categories have been constructed by finding them inside graph planar algebras:

- Extended Haagerup [BMPS12]

Question (V.F.R. Jones ∼2001)

For which Γ does $\mathcal{P}_\bullet$ embed into $\mathcal{GPA}(\Gamma)_\bullet$?

The module embedding theorem

Theorem [GMP+18]

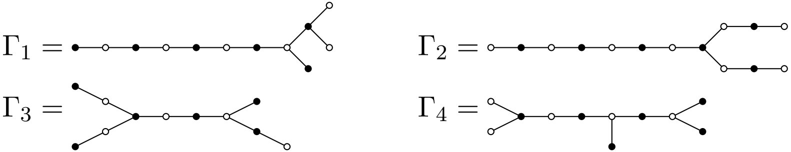
A finite depth subfactor planar algebra $\mathcal{P}_\bullet$ embeds in the graph planar algebra of a connected bipartite graph Γ if and only if Γ is the fusion graph for the generator $X \in \mathcal{D}$ acting on some module C^* category $\mathcal{M}$, where $\mathcal{D}$ is the corresponding 2×2 unitary multifusion category of $\mathcal{P}_\bullet$.

We use this theorem to construct 2 new unitary fusion categories Morita equivalent to the Extended Haagerup fusion categories.

The EH fusion categories

Theorem [GMP+18]

The Extended Haagerup subfactor planar algebra embeds exactly into the graph planar algebras of the following four graphs:



By the module embedding theorem, these embeddings correspond to modules for the 2×2 unitary multifusion category $\mathcal{D}$ corresponding to the EH subfactor planar algebra.

- Γ_1 and Γ_2 correspond to the two columns of $\mathcal{D}$ as modules.
- Γ_3 and Γ_4 give $\mathcal{EH}_3$ and $\mathcal{EH}_4$ (as in Noah's talk).

Cayley's theorem

In fact, the module embedding theorem is a categorification of Cayley's theorem from group theory:

Theorem (Cayley)

A group action $G \curvearrowright X$ is equivalent to a group homomorphism $G \to \mathrm{Aut}(X)$.

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Cayley's Theorem for multifusion categories

A module category $\mathcal{M}$ for $\mathcal{C}$ is equivalent to a tensor functor $\mathcal{C} \to \mathrm{End}(\mathcal{M})$.

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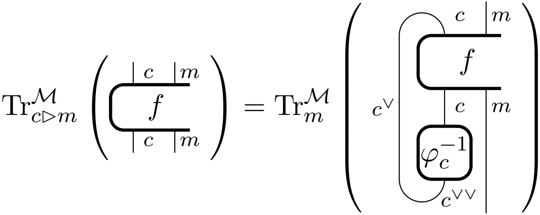
Cayley's Theorem for unitary multifusion categories

A module C^* category $\mathcal{M}$ for $\mathcal{C}$ is equivalent to a dagger tensor functor $\mathcal{C} \to \mathrm{End}^\dagger(\mathcal{M})$.

Pivotal modules for pivotal categories

When $\mathcal{C}$ is equipped with a pivotal structure φ, compatibility between a module $\mathcal{M}$ and φ is witnessed by a family of *compatible traces* on the endomorphism spaces of $\mathcal{M}$ [Sch13]



Cayley's theorem revisited

Facts [GMP+18]

- Traces on $\mathcal{M}$ (up to uniform scaling) correspond to pivotal structures on $\mathrm{End}(\mathcal{M})$.
- A trace on $\mathcal{M}$ is compatible if and only if the corresponding tensor functor $\mathcal{C} \to \mathrm{End}(\mathcal{M})$ is pivotal.

Cayley's Theorem for pivotal multifusion categories

A pivotal module category $(\mathcal{M}, \mathrm{Tr}^{\mathcal{M}})$ for $(\mathcal{C}, \varphi_{\mathcal{C}})$ is equivalent to a pivotal tensor functor $(\mathcal{C}, \varphi_{\mathcal{C}}) \to (\mathrm{End}(\mathcal{M}), \varphi_{\mathrm{Tr}^{\mathcal{M}}})$.

Dual functors vs. pivotal structures

Non-unitary case

Picking $\mathrm{ev}_c, \mathrm{coev}_c$ for every $c \in \mathcal{C}$ gives a dual functor $\mathcal{C} \to \mathcal{C}^{\mathrm{mop}}$. All dual functors are uniquely monoidally naturally isomorphic. If $\mathcal{C}$ has a pivotal structure, then all pivotal structures form a torsor over

$$\mathrm{Aut}_{\otimes}(\mathrm{id}_{\mathcal{D}}) \cong \mathrm{Hom}(\mathcal{U} \to \mathbb{C}^{\times})$$

where $\mathcal{U}$ is the *universal grading groupoid* of $\mathcal{C}$ [EGNO15, Pen18].

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Unitary case [Pen18]

Each unitary dual functor gives a canonical unitary pivotal structure

$$\varphi_c := (\mathrm{coev}_c^{\dagger} \otimes \mathrm{id}_{c^{\vee}}) \circ (\mathrm{id}_c \otimes \mathrm{coev}_{c^{\vee}}) : c \to c^{\vee\vee}.$$

Not all unitary dual functors are *unitarily* naturally isomorphic. Unitary dual functors form a torsor over $\mathrm{Hom}(\mathcal{U} \to \mathbb{R}_{>0})$.

- The trivial hom gives the unique unitary spherical structure. [LR97, Yam04, BDH14]

Unitary planar algebra correspondence

Fact [Pen18]

Unitary shaded planar algebras correspond to triples $(\mathcal{D}, X, \vee)$ where:

- $\mathcal{D}$ is unitary multifusion,
- $1_{\mathcal{D}} = 1_+ \oplus 1_-$ is a decomposition ($1_\pm$ need not be simple!) and $X = 1_+ \otimes X \otimes 1_-$ is a generator, and
- $\vee$ is a unitary dual functor on $\mathcal{D}$.

Example

Suppose $\Gamma = (V_+, V_-, E)$ is a finite connected bipartite graph. Form $\mathcal{M} = \mathsf{Hilb}_{\mathsf{fd}}(V_+) \oplus \mathsf{Hilb}_{\mathsf{fd}}(V_-)$, and observe Γ gives a dagger endofunctor of $\mathcal{M}$. The graph planar algebra of Γ corresponds to $\mathrm{End}^\dagger(\mathcal{M})$ with generator Γ and unitary dual functor induced from the Frobenius-Perron vertex weighting of Γ.

The module embedding theorem

Module embedding theorem [GMP+18]

Suppose $\mathcal{P}_\bullet$ is a finite depth subfactor planar algebra and let $\mathcal{D}$ be its 2×2 unitary multifusion category of projections with generator X. Equip $\mathcal{D}$ with its canonical spherical structure $\varphi_{\mathcal{D}}$. The following are equivalent:

- A pivotal module C^* category $(\mathcal{M}, \mathrm{Tr}^{\mathcal{M}})$ for $(\mathcal{D}, \varphi_{\mathcal{D}})$
- A pivotal tensor functor $(\mathcal{D}, \varphi_{\mathcal{D}}) \to (\mathrm{End}^\dagger(\mathcal{M}), \varphi_{\mathrm{Tr}^{\mathcal{M}}})$
- A planar algebra embedding $\mathcal{P}_\bullet \hookrightarrow \mathcal{GPA}(\Gamma_X)$.

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The original embedding theorem [JP11] was proved using towers of algebras. Summer 2018, I supervised some undergraduate researchers (Desmond Coles and Srivatsa Srinivas) to prove part of the above theorem using towers of algebras [CHPS18].

Thanks for listening!

Relevant preprints:

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