Distortion for multifactor bimodules and representations of multifusion categories

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joint with Bischoff, Charlesworth, Evington, and Giorgetti

('Unitary group' from 2018 AMS MRC Quantum symmetries: subfactors and fusion categories)

AMS JMM - Special Session on Advances in Operator Algebras

January 7, 2021

IPAM: Actions of tensor categories on C^*-algebras

Today, C^*-algebras stand at an analogous stage to vNAs in the '80s when Jones pioneered subfactor theory. This virtual workshop will bring together researchers at the interface of structure and classification of C^*-algebras and subfactor theory/tensor categories to set foundations for actions of tensor categories on C^*-algebras.

- Thurs 21 (8am - 11:30 am), Fri 22 Jan (8am - 10:30am): Expository overview talks by Courtney, Carrion, Szabo, Vaes, Yamashita on classification of simple nuclear C^*-algebras; group actions on the hyperfinite II_1 factor and on classifiable C^*-algebras; tensor categories associated to subfactors.
- Mon 25 Jan - Thurs 28 Jan (8am - 11am): Research talks, discussion of overview talks, ask expert sessions.

Participation is open to all. Current speakers and registration information can be found here:
`http://www.ipam.ucla.edu/programs/workshops/actions-of-tensor-categories-on-c-algebras/?tab=overview`

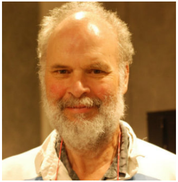
Subfactors

- A II_1 *factor* is an infinite dimensional von Neumann algebra with trivial center and a trace.
- A II_1 *subfactor* is a unital inclusion of type II_1 factors.

Jones' Index Rigidity Theorem [Jon83]

The index $[B : A] := \dim({}_A L^2 B)$ of a II_1 subfactor $A \subset B$ takes values in:

$$[B : A] \in \left\{4\cos^2(\pi/n)\middle| n \geq 3\right\} \cup [4, \infty].$$



We'll always assume $A \subset B$ has finite index.

The standard invariant

Definition

The *standard invariant* of $A \subset B$ is the collection of all $A - A$, $A - B$, $B - B$, and $B - A$ bimodules generated by $L^2 B$ under

- $\oplus$ direct sum
- $\boxtimes$ Connes' fusion relative tensor product (over A or B)
- $\subseteq$ sub-bimodules
- $\overline{\cdot}$ conjugates.

We can think of this as a *tensor category* of bimodules of $A \oplus B$.

$$\mathcal{C} = \mathcal{C}(A \subset B) := \begin{pmatrix} {}_A\mathcal{C}_A & {}_A\mathcal{C}_B \\ {}_B\mathcal{C}_A & {}_B\mathcal{C}_B \end{pmatrix} \subset \mathrm{Bim}(A \oplus B)$$

We call $A \subset B$ *finite depth* if this tensor category has finitely many isomorphism classes of simple bimodules. (A bimodule ${}_P H_Q$ is *simple* if $\mathrm{End}_{P-Q}(H) = \mathbb{C}$.)

Popa's Classification Theorem

Popa's Classification Theorem [Pop90]

A finite depth, finite index *hyperfinite* II_1 subfactor is completely determined by its standard invariant.

Corollary [Pop90, Izu17, Tom18, HP20]

Every unitary fusion category admits an essentially unique embedding $\mathcal{C} \hookrightarrow \mathrm{Bim}(R)$ where R is a non type I hyperfinite factor.

- This allows us to define a unitary fusion category as any collection of $R - R$ bimodules closed under $\oplus, \boxtimes, \subseteq, \bar{\cdot}$ with only finitely many isomorphism classes of simple bimodules.

Multifactors and multifusion categories

In the study of unitary fusion categories, we must naturally consider unitary *multifusion* categories, which may no longer have simple unit.

- We would like to define a unitary fusion category as any collection of $R^{\oplus n} - R^{\oplus n}$ bimodules closed under $\oplus, \boxtimes, \subseteq, \overline{\cdot}$ with only finitely many isomorphism classes of simple bimodules. But this requires an existence theorem for representations, which is one of our results!

Example

For $A \subset B$ finite depth, $\mathcal{C}(A \subset B)$ is a unitary multifusion.

$$\mathcal{C}(A \subset B) = \begin{pmatrix} {}_A\mathcal{C}_A & {}_A\mathcal{C}_B \\ {}_B\mathcal{C}_A & {}_B\mathcal{C}_B \end{pmatrix} \subset \mathrm{Bim}(A \oplus B)$$

Representing multifusion categories requires *multifactors*.

- A II_1 *multifactor* is a finite direct sum of II_1 factors.

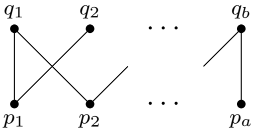
Multifactor inclusions

Notation for a II_1 multifactor inclusion $A \subset B$:

- $p_1, \ldots, p_a$ are the minimal central projections of A.
- $q_1, \ldots, q_b$ are the minimal central projections of B.
- $A_i := p_i A$ and $B_j := q_j B$.

A II_1 multifactor inclusion $A \subset B$

- has *finite index* if $p_i q_j A \subset p_i q_j B p_i$ has finite index for all i, j.
- is *connected* if $Z(A) \cap Z(B) = \mathbb{C}$. Inclusion graph has node for each p_i, q_j and edges when $p_i q_j \neq 0$:



$A \subset B$ is connected iff this graph is connected.

We'll always assume $A \subset B$ is finite index and connected.

Standard invariants of multifactor inclusions

The *standard invariant* of $A \subset B$ is the collection of all bimodules generated by $L^2 B$. This time, we can fuse over any A_i, B_j:

$$\mathcal{C}(A \subset B) = \begin{pmatrix} {}_{A_1}\mathcal{C}_{A_1} & \cdots & {}_{A_1}\mathcal{C}_{A_a} & {}_{A_1}\mathcal{C}_{B_1} & \cdots & {}_{A_1}\mathcal{C}_{B_b} \\ \vdots & & \vdots & \vdots & & \vdots \\ {}_{A_a}\mathcal{C}_{A_1} & \cdots & {}_{A_a}\mathcal{C}_{A_a} & {}_{A_a}\mathcal{C}_{B_1} & \cdots & {}_{A_a}\mathcal{C}_{B_b} \\ {}_{B_1}\mathcal{C}_{A_1} & \cdots & {}_{B_1}\mathcal{C}_{A_a} & {}_{B_1}\mathcal{C}_{B_1} & \cdots & {}_{B_1}\mathcal{C}_{B_b} \\ \vdots & & \vdots & \vdots & & \vdots \\ {}_{B_b}\mathcal{C}_{A_1} & \cdots & {}_{B_b}\mathcal{C}_{A_a} & {}_{B_b}\mathcal{C}_{B_1} & \cdots & {}_{B_b}\mathcal{C}_{B_b} \end{pmatrix}$$

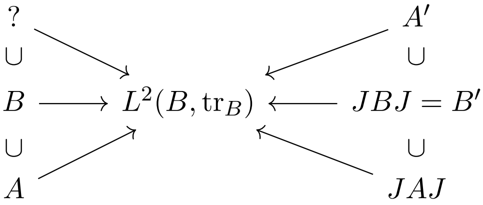
- $A \subset B$ has *finite depth* if $\mathcal{C}(A \subset B)$ has only finitely many isomorphism classes of simple bimodules.

Jones' basic construction

The basic construction is the main tool Jones used to prove his Index Rigidity Theorem.

Definition

The *basic construction* of $A \subset B$ is the II_1 multifactor $\langle B, A\rangle := JA'J$ acting on $L^2(B, \mathrm{tr}_B)$.



Here tr_B is the unique *Markov trace* [GdlHJ89].

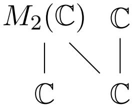
- A *downward basic construction* (if one exists) is a II_1 multifactor $M \subset A$ and an isomorphism $B \cong \langle A, M\rangle$.

Popa's theorem fails for multifactor inclusions

Example

Consider the finite dimensional (and hence finite depth) inclusion

$$\begin{array}{c} Q \\ \cup \\ P \end{array}$$



$$\rightsquigarrow \qquad \begin{array}{c} B = Q \otimes R \\ \cup \\ A = P \otimes R \end{array}$$

- $A \subset B$ does not admit a *downward basic construction* [Pop95]
- Taking the Jones tower $A_0 \subset A_1 \subset A_2 \subset A_3$, we get a Morita equivalent inclusion $A_2 \subset A_3$ with the same standard invariant which manifestly admits two downward basic constructions.

One quickly observes these inclusions have different distortions.

Modular distortion

If M, N are II_1 factors and ${}_M H_N$ is a bimodule, the *modular distortion* [notion due to André Henriques] of H is

$$\delta = \delta(H) := \left(\frac{\dim({}_M H)}{\dim(H_N)} \right)^{1/2}$$

If A, B are II_1 multifactors and ${}_A X_B$ is a bimodule, the modular distortion of X is the partially defined matrix in $M_{a\times b}(\mathbb{R}_{>0})$

$$\delta_{ij} := \delta(p_i X q_j) = \left(\frac{\dim({}_{A_i}(p_i X q_j))}{\dim((p_i X q_j)_{B_j})} \right)^{1/2} \qquad (p_i X q_j \neq 0)$$

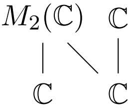
Theorem [BCEGP20]

When a connected bimodule ${}_A X_B$ has finite depth (generates a multifusion category) or more generally is *extremal*, δ extends *uniquely* to a fully defined matrix in $M_{a\times b}(\mathbb{R}_{>0})$ satisfying

$$\delta_{ij}\delta_{i'j'} = \delta_{ij'}\delta_{i'j} \qquad \rightsquigarrow \qquad \delta : \mathrm{smu}(M_{a+b}(\mathbb{C})) \to \mathbb{R}_{>0}$$

Back to our finite dimensional inclusion

$$\begin{matrix} Q \\ \cup \\ P \end{matrix}$$



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Back to our finite dimensional inclusion

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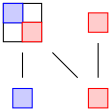

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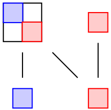
Back to our finite dimensional inclusion

$$\begin{array}{c} Q \\ \cup \\ P \end{array}$$

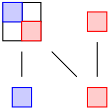


$$\rightsquigarrow \qquad \begin{array}{c} B = Q \otimes R \\ \cup \\ A = P \otimes R \end{array}$$

$$\delta({}_{A_0}L^2 A_{1\,A_1}) = \begin{pmatrix} \left(\dfrac{\dim_{\color{blue}\blacksquare}({\color{blue}\blacksquare}\square)}{\dim(\square\square)_{\boxplus}} \right)^{1/2} & ? \\ \left(\dfrac{\dim_{\color{red}\blacksquare}(\square{\color{red}\blacksquare})}{\dim(\square\square)_{\boxplus}} \right)^{1/2} & \left(\dfrac{\dim_{\color{red}\blacksquare}({\color{red}\blacksquare})}{\dim(\square)_{\square}} \right)^{1/2} \end{pmatrix}$$

Back to our finite dimensional inclusion

$$\begin{matrix} Q \\ \cup \\ P \end{matrix}$$



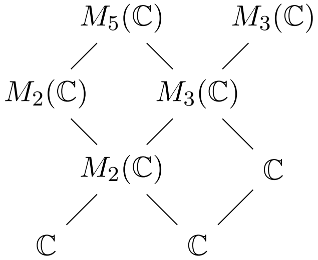
$$\rightsquigarrow \qquad \begin{matrix} B = Q \otimes R \\ \cup \\ A = P \otimes R \end{matrix}$$

$$\delta({}_{A_0}L^2 A_{1\,A_1}) = \begin{pmatrix} \left(\frac{2}{1/2}\right)^{1/2} & ? \\ \left(\frac{2}{1/2}\right)^{1/2} & \left(\frac{1}{1}\right)^{1/2} \end{pmatrix} = \begin{pmatrix} 2 & ? \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

Distortion changes under the basic construction

The basic construction up two levels is given by reflecting twice:

$$\begin{array}{c} P_3 \\ \cup \\ P_2 \\ \cup \\ P_1 \\ \cup \\ P_0 \end{array}$$



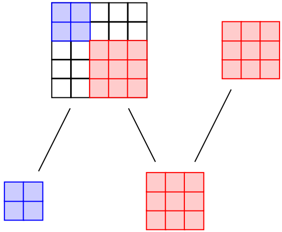
$\rightsquigarrow$

$$\begin{array}{c} A_3 = P_3 \otimes R \\ \cup \\ A_2 = P_2 \otimes R \\ \cup \\ A_1 = P_1 \otimes R \\ \cup \\ A_0 = P_0 \otimes R \end{array}$$

Distortion changes under the basic construction

The basic construction up two levels is given by reflecting twice:

$$P_3 \qquad\qquad\qquad A_3 = P_3 \otimes R$$

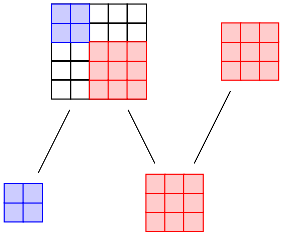


$$\cup \qquad \rightsquigarrow \qquad \cup$$

$$P_2 \qquad\qquad\qquad A_2 = P_2 \otimes R$$

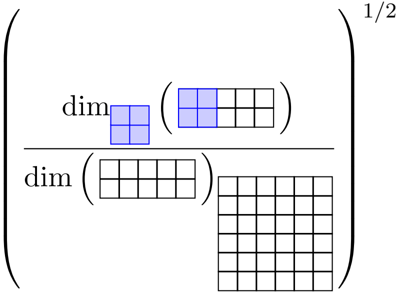
Distortion changes under the basic construction

The basic construction up two levels is given by reflecting twice:

P_3  $A_3 = P_3 \otimes R$

$\cup$ $\rightsquigarrow$ $\cup$

P_2 $A_2 = P_2 \otimes R$

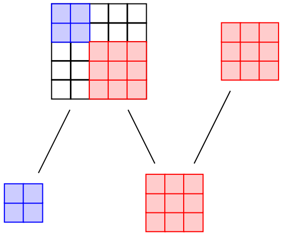
$\delta_{1,1} =$ 

$= \left(\frac{5/2}{2/5}\right)^{1/2} = \frac{5}{2}$

Distortion changes under the basic construction

The basic construction up two levels is given by reflecting twice:

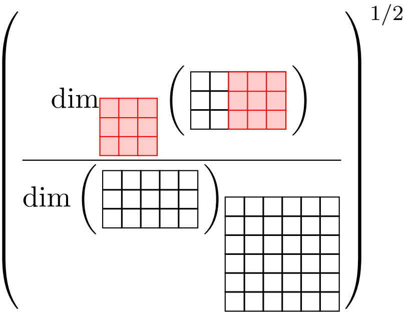
$$P_3 \quad\quad\quad A_3 = P_3 \otimes R$$



$$\cup \quad \rightsquigarrow \quad \cup$$

$$P_2 \quad\quad\quad A_2 = P_2 \otimes R$$

$\delta_{2,1} =$

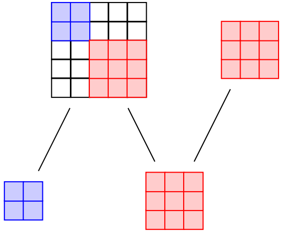


$$= \left(\frac{5/3}{3/5}\right)^{1/2} = \frac{5}{3}$$

Distortion changes under the basic construction

The basic construction up two levels is given by reflecting twice:

$$P_3 \qquad\qquad A_3 = P_3 \otimes R$$



$$\cup \qquad \rightsquigarrow \qquad \cup$$

$$P_2 \qquad\qquad A_2 = P_2 \otimes R$$

$$\delta({}_{A_2}L^2 A_{3\,A_3}) = \begin{pmatrix} 5/2 & ? \\ 5/3 & 1 \end{pmatrix} = \begin{pmatrix} 5/2 & 3/2 \\ 5/3 & 1 \end{pmatrix}$$

Distortion changes under the basic construction

Even though $A_0 \subset A_1$ and $A_2 \subset A_3$ are Morita equivalent inclusions and share a standard invariant, they have visibly different distortions:

$$\delta({}_{A_0}L^2 A_{1A_1}) = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \qquad \delta({}_{A_2}L^2 A_{3A_3}) = \begin{pmatrix} 5/2 & 3/2 \\ 5/3 & 1 \end{pmatrix}.$$

One calculates that for the Jones tower $(A_n)_{n\geq 0}$,

$$\delta({}_{A_{2n}}L^2 A_{2n+1A_{2n+1}}) = \begin{pmatrix} F_{2n+2}/F_{2n} & F_{2n+1}/F_{2n} \\ F_{2n+2}/F_{2n+1} & 1 \end{pmatrix} \xrightarrow{n\to\infty} \begin{pmatrix} \phi^2 & \phi \\ \phi & 1 \end{pmatrix},$$

where F_n is the n-th Fibonacci number ($F_0 = F_1 = 1$) and ϕ is the golden ratio.

Some results on distortion

In our article [BCEGP20], we give the following results on distortion:

- Quantitative measurement via distortion of when an inclusion admits a downward basic construction
- Formula for dynamical system of the distortion under taking basic construction
- Uniqueness of fixed point for this dynamical system and its relation to Popa's *homogeneity*

We then use distortion to prove 4 classification theorems for finite depth hyperfinite II_1 multifactor inclusions and representations of unitary multifusion categories.

One of our classification theorems

Theorem [BCEGP20]

The map which takes $A \subset B$ to the pair $(\mathcal{P}_\bullet^{A\subset B}, \mathrm{tr}_B^{\mathrm{Markov}}|_{Z(A)})$ gives a bijection

$$\frac{\left\{\begin{array}{l}\text{Finite depth, finite index connected hyperfinite } \mathrm{II}_1 \text{ multifactor inclusions } A \subset B\end{array}\right\}}{\varphi : B_1 \xrightarrow{\sim} B_2 \text{ taking } A_1 \text{ onto } A_2} \cong \frac{\left\{\begin{array}{l}\text{Pairs } (\mathcal{P}_\bullet, \tau) \text{ with } \mathcal{P}_\bullet \text{ a finite depth indecomposable unitary 2-shaded planar algebra and } \tau \text{ a faithful state on } \mathcal{P}_{0,+}\end{array}\right\}}{\varphi_\bullet : \mathcal{P}_\bullet^1 \xrightarrow{\sim} \mathcal{P}_\bullet^2 \text{ such that } \tau^2 \circ \varphi_{0,+} = \tau^1}.$$

- $\mathrm{tr}_B^{\mathrm{Markov}}$ is the unique *Markov trace* on $A \subset B$ [GdlHJ89].
- The Markov trace restricted to $Z(A)$ completely determines the distortion of ${}_A L^2 B_B$.

Thank you for listening!

Slides available at:
https://people.math.osu.edu/penneys.2/PenneysJMM2021.pdf

Article available at arXiv:2010.01067.

Marcel Bischoff, Ian Charlesworth, Samuel Evington, Luca Giorgetti, and David Penneys, *Distortion for multifactor bimodules and representations of multifusion categories*, 2020, arXiv:2010.01067.

Frederick M. Goodman, Pierre de la Harpe, and Vaughan F.R. Jones, *Coxeter graphs and towers of algebras*, Mathematical Sciences Research Institute Publications, 14. Springer-Verlag, New York, 1989, x+288 pp. ISBN: 0-387-96979-9, MR999799.

André Henriques and David Penneys, *Representations of fusion categories and their commutants*, 2020, arXiv:2004.08271.

Masaki Izumi, *A Cuntz algebra approach to the classification of near-group categories*, Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday, Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 46, Austral. Nat. Univ., Canberra, 2017, MR3635673 arXiv:1512.04288, pp. 222–343. MR 3635673

Vaughan F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25, MR696688, DOI:10.1007/BF01389127.

Sorin Popa, *Classification of subfactors: the reduction to commuting squares*, Invent. Math. **101** (1990), no. 1, 19–43, MR1055708, DOI:10.1007/BF01231494.

______, *Classification of subfactors and their endomorphisms*, CBMS Regional Conference Series in Mathematics, vol. 86, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1995, MR1339767. MR 1339767 (96d:46085)

Reiji Tomatsu, *Centrally free actions of amenable C*-tensor categories on von Neumann algebras*, 2018, arXiv:1812.04222.