Classification of $\mathbb{Z}/2\mathbb{Z}$-quadratic unitary fusion categories

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Quadratic fusion categories

Definition

A fusion category is called *quadratic* if it has exactly 2 orbits of simple objects under the action of the group of invertible objects.

- Simples $G \cup \{g\rho\}_{g \in G}$, G a finite group
- Quadratic fusion relation:

$$\rho \otimes \rho \cong \text{ some } g\text{'s } \oplus \text{ some } h\rho\text{'s}$$

This says that $\mathrm{FPdim}(\rho)$ lies in a quadratic field extension of $\mathbb{Q}$.

Examples of quadratic fusion categories

Example: Fib

Simples are $\{1\} \cup \{\tau\}$ with $\tau \otimes \tau \cong 1 \oplus \tau$.

Example: Ising

Simples are $\{1, \psi\} \cup \{\sigma\}$ with $\psi \otimes \psi \cong 1$ and $\sigma \otimes \sigma \cong 1 \oplus \psi$.

Example: $\mathcal{TY}$

Simples are $A \cup \{\rho\}$ with A an abelian group and $\rho \otimes \rho \cong \bigoplus_{a \in A} a$.

Example: near group of type $A + k|A|$

Simples are $A \cup \{\rho\}$ with A an abelian group and

$$\rho \otimes \rho \cong k|A|\rho \oplus \bigoplus_{a \in A} a.$$

Classifying fusion categories by rank

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- To date, *unitary* fusion categories are classified up to rank 3 [Ost13]. (This article classifies pseudounitary fusion categories up to rank 3.)
- Rank 4 appears out of reach at this time.

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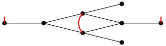
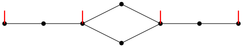
- To date, fusion categories are classified up to rank 2 [Ost03].
- To date, *unitary* fusion categories are classified up to rank 3 [Ost13]. (This article classifies pseudounitary fusion categories up to rank 3.)
- Rank 4 appears out of reach at this time.
- In her high school research project [Lar14], Hannah Larson gave a finite list of possible rank 4 fusion rings for pseudounitary fusion categories with a dual pair of simples, i.e., there is a simple c such that $c \ncong c^\vee$.

Classification of $\mathbb{Z}/2\mathbb{Z}$-quadratic UFCs [EMIP21]

Rank 3:

- 2 UFCs $\mathcal{TY}(\mathbb{Z}/2\mathbb{Z}, \chi, \pm)$ [Jon83, TY98]
- 3 UFCs with Rep(S_3) fusion rules [Izu17]
- 2 $\mathrm{Ad}(E_6)$ UFCs [BN91, Izu01, HH09]

Rank 4:

- 8 pointed UFCs Hilb$(\mathbb{Z}/4\mathbb{Z}, \omega)$ and Hilb$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \omega)$
- 2 UFCs Fib $\boxtimes$ Hilb$(\mathbb{Z}/2\mathbb{Z}, \omega)$ for $\omega \in H^3(\mathbb{Z}/2\mathbb{Z}, U(1))$
- $\mathrm{Ad}(SU(2)_6) = \mathrm{Ad}(A_7)$
- 2 even parts of the $\mathcal{S}' =$  PAs [LMP15, Izu18]
- even part of $2D2 =$  PA [MP15, Izu18].

This completes Larson's fusion ring classification to the classification of rank 4 UFCs with a dual pair of simple objects.

Step 1 of the proof

An associativity argument gives 3 cases for the fusion ring for a $\mathbb{Z}/2\mathbb{Z}$-quadratic fusion category

1. simple objects: $\mathbf{1}, \alpha, \rho$; fusion rules determined by: $\rho^2 \cong 1 \oplus m\rho \oplus \alpha$.
2. simple objects: $\mathbf{1}, \alpha, \rho, \alpha\rho$, ρ not self-dual; fusion rules determined by: $\rho^2 \cong m\rho \oplus n\alpha\rho \oplus \alpha$.
3. simple objects: $\mathbf{1}, \alpha, \rho, \alpha\rho$, ρ self-dual; fusion rules determined by: $\rho^2 \cong 1 \oplus m\rho \oplus n\alpha\rho$.

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3. simple objects: $\mathbf{1}, \alpha, \rho, \alpha\rho$, ρ self-dual; fusion rules determined by: $\rho^2 \cong 1 \oplus m\rho \oplus n\alpha\rho$.

   Status: at 2014 AMS MRC, a group[1] adapted [Lar14] in the pseudounitary setting to show $m = n \leq 2$.

---

[1] Ryan Johnson, Siu-Hung Ng, David Penneys, Jolie Roat, Matthew Titsworth, and Henry Tucker. This calculation is included as an appendix.

Step 2 of the proof

TODO: Classify case 2 and 3 in unitary setting for $m = n \in \{1, 2\}$.

- Step 2: We give a generalization of Ostrik's theorem on formal codegrees [Ost09, Ost13].

Definition

Given a fusion category $\mathcal{C}$ and an irreducible representation V of $K_0(\mathbb{C})$, the element

$$\alpha_V := \sum_{c \in \mathrm{Irr}(\mathcal{C})} \mathrm{Tr}_V(c) c^\vee \in K_0(\mathcal{C})$$

lies in the center of $K_0(\mathcal{C})$. It acts by a scalar f_V on V called the *formal codegree* of V, and it acts by zero on any other irreducible representation $V' \not\cong V$.

Ostrik's theorem on formal codegrees

For a spherical fusion category $\mathcal{C}$, given an irrep V of $K_0(\mathcal{C})$, $K_0(Z(\mathcal{C}))$ acts on V via the forgetful map $K_0(Z(\mathcal{C})) \to K_0(\mathcal{C})$. The image of $K_0(Z(\mathcal{C}))$ lies in $\mathrm{End}_{K_0(\mathcal{C})}(V) = \mathbb{C}$, giving a character of $K_0(Z(\mathcal{C}))$, which corresponds to a unique simple $\Gamma_V \in \mathrm{Irr}(Z(\mathcal{C}))$.

Theorem [Ost13]

The assignment $V \mapsto \Gamma_V$ is an embedding $\mathrm{Irr}(\mathsf{Rep}(K_0(\mathcal{C}))) \hookrightarrow \mathrm{Irr}(Z(\mathcal{C}))$. The image is the simples $\Gamma \in \mathrm{Irr}(Z(\mathcal{C}))$ which lie under $\mathcal{I}(1_{\mathcal{C}})$, where $\mathcal{I} : \mathcal{C} \to Z(\mathcal{C})$ is the induction functor adjoint to the forgetful functor $\mathcal{F} : Z(\mathcal{C}) \to \mathcal{C}$. Moreover,

$$\dim(\Gamma_V) = \frac{\dim \mathcal{C}}{f_V} \qquad \text{and} \qquad \dim \mathrm{Hom}(\mathcal{I}(1) \to \Gamma_V) = \dim(V).$$

This theorem is used in essential ways in the classification of pivotal fusion categories of rank 3 [Ost13] and Larson's results [Lar14].

Generalization of formal codegrees

Theorem [EMIP21, Thm. C]

Let $\mathcal{C}$ be a spherical fusion category, and let A be the tube algebra of $\mathcal{C}$. Fix $X \in \mathrm{Irr}(\mathcal{C})$. There is a bijective correspondence between equivalence classes of irreducible representations (V, π) of $A_{X \leftarrow X}$ and isomorphism classes of simple subobjects $\Gamma_V \subset \mathcal{I}(X) \in Z(\mathcal{C})$. The formal codegree f_V of (V, π) with respect to Tr_X is a scalar, and the categorical dimension of Γ_V is given by

$$\dim(\Gamma_V) = \frac{\dim(\mathcal{C})}{f_V \dim(X)}.$$

Moreover, if $Y \in \mathrm{Irr}(\mathcal{C})$ and ${}_X\pi_Y$ is the action of $A_{X \leftarrow X}$ on $A_{X \leftarrow Y}$, then

$$\dim(\mathcal{C}(Y \to \mathcal{F}(\Gamma_V))) = \dim(\mathrm{Hom}(\pi_V \to {}_X\pi_Y)).$$

Step 3 of the proof

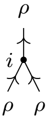
Use skein theory to perform the classification.

Example

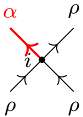
Consider the fusion rules for $m = 1, 2$:

$$\alpha \otimes \alpha \cong \mathbf{1} \qquad\qquad \rho \otimes \rho \cong \mathbf{1} \oplus m\rho \oplus m\alpha\rho.$$

Generating vertices for $0 \leq i < m$:



$\in \mathcal{C}_m(\rho \otimes \rho \to \rho)$

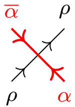


$\in \mathcal{C}_m(\rho \otimes \rho \to \alpha\rho)$

Generating isomorphisms, with Frobenius-Schur indicators $\lambda_\alpha, \lambda_\rho$:

$\in \mathcal{C}_m(\alpha \to \overline{\alpha})$

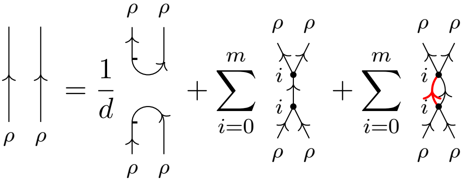
$\in \mathcal{C}_m(\rho \to \overline{\rho})$

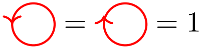


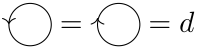
$\in \mathcal{C}_m(\rho \otimes \alpha \to \overline{\alpha} \otimes \rho)$.

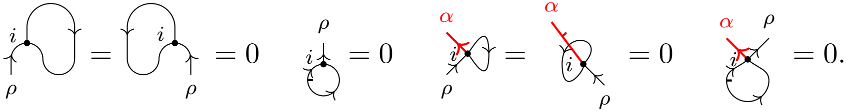
Basic semisimplicity relations

$$\alpha \otimes \alpha \cong \mathbf{1} \qquad \rho \otimes \rho \cong \mathbf{1} \oplus m\rho \oplus m\alpha\rho.$$





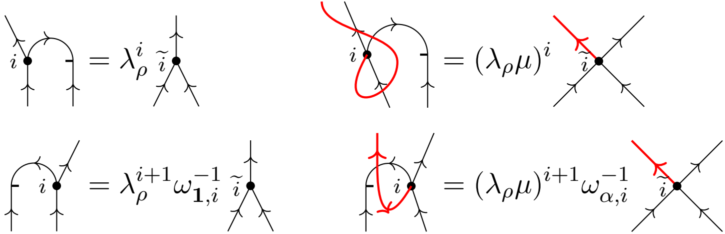




Frobenius relations

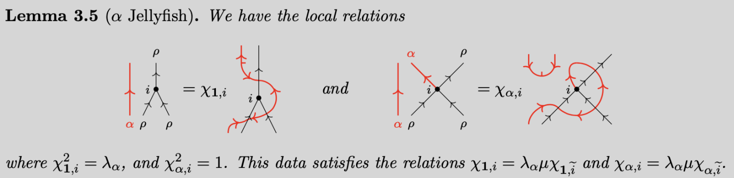
Theorem

There exist orthonormal bases of $\mathcal{C}(\rho\rho \to \rho)$ and $\mathcal{C}(\rho\rho \to \alpha\rho)$ such that



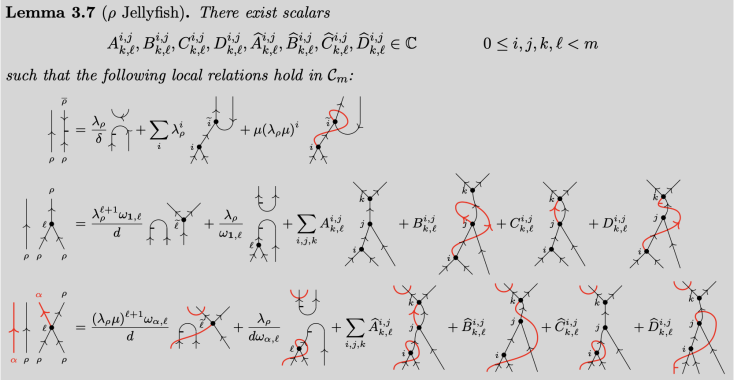
for some scalars (definitions omitted).

Jellyfish relations



Lemma 3.5 (α Jellyfish)**.** *We have the local relations*

where $\chi_{\mathbf{1},i}^2 = \lambda_\alpha$, *and* $\chi_{\alpha,i}^2 = 1$. *This data satisfies the relations* $\chi_{\mathbf{1},i} = \lambda_\alpha \mu \chi_{\mathbf{1},\widetilde{i}}$ *and* $\chi_{\alpha,i} = \lambda_\alpha \mu \chi_{\alpha,\widetilde{i}}$.



Lemma 3.7 (ρ Jellyfish)**.** *There exist scalars*

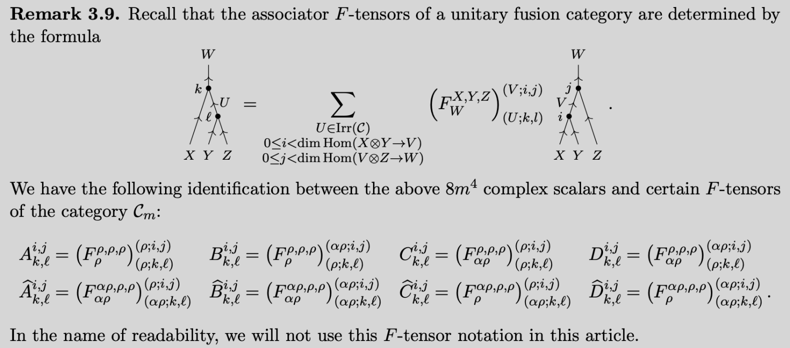
$$A^{i,j}_{k,\ell}, B^{i,j}_{k,\ell}, C^{i,j}_{k,\ell}, D^{i,j}_{k,\ell}, \widehat{A}^{i,j}_{k,\ell}, \widehat{B}^{i,j}_{k,\ell}, \widehat{C}^{i,j}_{k,\ell}, \widehat{D}^{i,j}_{k,\ell} \in \mathbb{C} \qquad 0 \le i,j,k,\ell < m$$

such that the following local relations hold in $\mathcal{C}_m$*:*

Plus lots of tetrahedral symmetries!

6j symbols

The scalars $A^{i,j}_{k,\ell}, B^{i,j}_{k,\ell}, C^{i,j}_{k,\ell}, D^{i,j}_{k,\ell}, \widehat{A}^{i,j}_{k,\ell}, \widehat{B}^{i,j}_{k,\ell}, \widehat{C}^{i,j}_{k,\ell}, \widehat{D}^{i,j}_{k,\ell}$ for $0 \leq i, j, k, \ell < m$ are 6j-symbols for the UFC.



Remark 3.9. Recall that the associator F-tensors of a unitary fusion category are determined by the formula

$$[\text{tree}] = \sum_{\substack{U \in \mathrm{Irr}(\mathcal{C}) \\ 0 \leq i < \dim \mathrm{Hom}(X \otimes Y \to V) \\ 0 \leq j < \dim \mathrm{Hom}(V \otimes Z \to W)}} \left(F^{X,Y,Z}_{W}\right)^{(V;i,j)}_{(U;k,l)} \; [\text{tree}] .$$

We have the following identification between the above $8m^4$ complex scalars and certain F-tensors of the category $\mathcal{C}_m$:

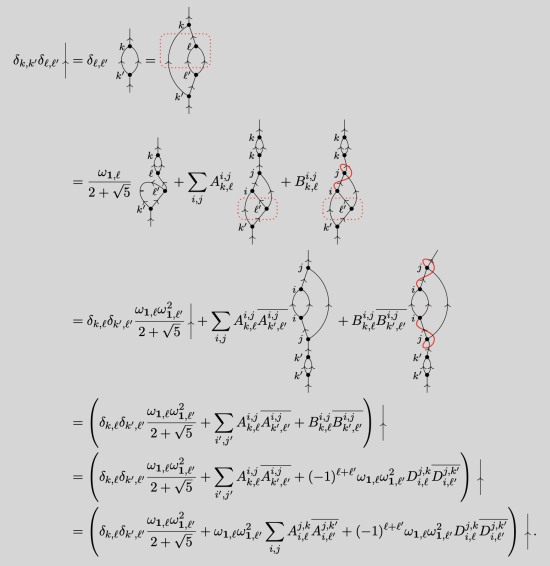
$$A^{i,j}_{k,\ell} = \left(F^{\rho,\rho,\rho}_{\rho}\right)^{(\rho;i,j)}_{(\rho;k,\ell)} \quad B^{i,j}_{k,\ell} = \left(F^{\rho,\rho,\rho}_{\rho}\right)^{(\alpha\rho;i,j)}_{(\rho;k,\ell)} \quad C^{i,j}_{k,\ell} = \left(F^{\rho,\rho,\rho}_{\alpha\rho}\right)^{(\rho;i,j)}_{(\rho;k,\ell)} \quad D^{i,j}_{k,\ell} = \left(F^{\rho,\rho,\rho}_{\alpha\rho}\right)^{(\alpha\rho;i,j)}_{(\rho;k,\ell)}$$

$$\widehat{A}^{i,j}_{k,\ell} = \left(F^{\alpha\rho,\rho,\rho}_{\alpha\rho}\right)^{(\rho;i,j)}_{(\alpha\rho;k,\ell)} \quad \widehat{B}^{i,j}_{k,\ell} = \left(F^{\alpha\rho,\rho,\rho}_{\alpha\rho}\right)^{(\alpha\rho;i,j)}_{(\alpha\rho;k,\ell)} \quad \widehat{C}^{i,j}_{k,\ell} = \left(F^{\alpha\rho,\rho,\rho}_{\rho}\right)^{(\rho;i,j)}_{(\alpha\rho;k,\ell)} \quad \widehat{D}^{i,j}_{k,\ell} = \left(F^{\alpha\rho,\rho,\rho}_{\rho}\right)^{(\alpha\rho;i,j)}_{(\alpha\rho;k,\ell)} .$$

In the name of readability, we will not use this F-tensor notation in this article.

Example evaluation in 2 ways

Now evaluate **lots** of diagrams in two ways to solve for the scalars!



$$\delta_{k,k'}\delta_{\ell,\ell'} = \delta_{\ell,\ell'} =$$

$$= \frac{\omega_{\mathbf{1},\ell}}{2+\sqrt{5}} + \sum_{i,j} A^{i,j}_{k,\ell} + B^{i,j}_{k,\ell}$$

$$= \delta_{k,\ell}\delta_{k',\ell'}\frac{\omega_{\mathbf{1},\ell}\omega^2_{\mathbf{1},\ell'}}{2+\sqrt{5}} + \sum_{i,j} A^{i,j}_{k,\ell}\overline{A^{i,j}_{k',\ell'}} + B^{i,j}_{k,\ell}\overline{B^{i,j}_{k',\ell'}}$$

$$= \left(\delta_{k,\ell}\delta_{k',\ell'}\frac{\omega_{\mathbf{1},\ell}\omega^2_{\mathbf{1},\ell'}}{2+\sqrt{5}} + \sum_{i',j'} A^{i,j}_{k,\ell}\overline{A^{i,j}_{k',\ell'}} + B^{i,j}_{k,\ell}\overline{B^{i,j}_{k',\ell'}}\right)$$

$$= \left(\delta_{k,\ell}\delta_{k',\ell'}\frac{\omega_{\mathbf{1},\ell}\omega^2_{\mathbf{1},\ell'}}{2+\sqrt{5}} + \sum_{i',j'} A^{i,j}_{k,\ell}\overline{A^{i,j}_{k',\ell'}} + (-1)^{\ell+\ell'}\omega_{\mathbf{1},\ell}\omega^2_{\mathbf{1},\ell'}D^{j,k}_{i,\ell}\overline{D^{j,k'}_{i,\ell'}}\right)$$

$$= \left(\delta_{k,\ell}\delta_{k',\ell'}\frac{\omega_{\mathbf{1},\ell}\omega^2_{\mathbf{1},\ell'}}{2+\sqrt{5}} + \omega_{\mathbf{1},\ell}\omega^2_{\mathbf{1},\ell'}\sum_{i,j} A^{j,k}_{i,\ell}\overline{A^{j,k'}_{i,\ell'}} + (-1)^{\ell+\ell'}\omega_{\mathbf{1},\ell}\omega^2_{\mathbf{1},\ell'}D^{j,k}_{i,\ell}\overline{D^{j,k'}_{i,\ell'}}\right).$$

Thank you for listening!

Slides available at:
`https://people.math.osu.edu/penneys.2/talks/PenneysJMM2022.pdf`

Cain Edie-Michell, Masaki Izumi, and David Penneys.
Classification of $\mathbb{Z}/2\mathbb{Z}$-quadratic unitary fusion categories. 2021.
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Synoptic chart video available at:
`https://people.math.osu.edu/penneys.2/Synoptic.mp4`

Animator: undergraduate researcher Quinn Kolt, Spring 2022
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