Local topological order and topological holography

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Motivation

Given a (2+1)D lattice model with topological order, give a *model independent way* to extract/characterize its topological order.

Outline of strategy

- Start with a *net of projections* in a $(2+1)D$ *abstract quantum spin system*.
- Give *local topological order axioms* [JNPW23] generalizing TQO axioms [BHM10].
- Get a *holographically dual* (1+1)D abstract quantum spin system called the *boundary algebra*.
- The *'local' representation theory* of the boundary algebra remembers the bulk topological order.

Nets of algebras

Study *abstract quantum spin systems* via nets of algebras.

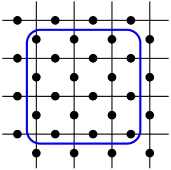
Definition: net of algebras

Local operator algebra $A(\Lambda)$ for each rectangle Λ on a lattice.

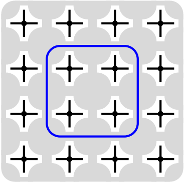
- $A(\emptyset) = \mathbb{C}1_A$
- $\Lambda \subset \Delta \Longrightarrow A(\Lambda) \subset A(\Delta)$
- $\Lambda \cap \Delta = \emptyset \Longrightarrow [A(\Lambda), A(\Delta)] = 0$
- $\bigcup_\Lambda A(\Lambda)$ generates A.

Usually assume *translation invariance*.

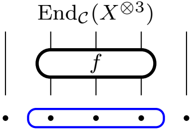
Examples



edge lattice in $\mathbb{R}^2$



$\mathbb{Z}^2$ lattice in $\mathbb{R}^2$



fusion spin chain on $\mathbb{Z}$ lattice in $\mathbb{R}$

Nets of projections

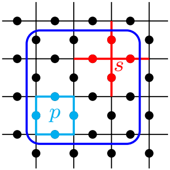
Instead of a local Hamiltonian, each rectangle Λ gets a projection.

$$p_\Lambda \in A(\Lambda), \qquad \Lambda \subset \Delta \Longrightarrow p_\Delta \le p_\Lambda$$

- $p_\Delta \le p_\Lambda$ means $p_\Delta \mathcal{H} \subset p_\Lambda \mathcal{H}$ when acting on Hilbert space $\mathcal{H}$

Example

If $H = \sum H_i$ is a frustration free commuting projector local Hamiltonian, p_Λ can be projection to the local ground state space.

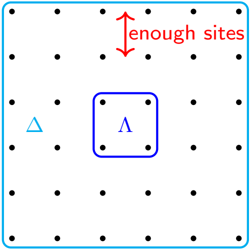


Toric Code

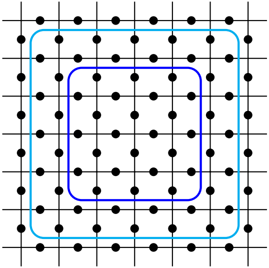
$$p_\Lambda = \prod_{s \subset \Lambda} \frac{I + A_s}{2} \cdot \prod_{p \subset \Lambda} \frac{I + B_p}{2}$$

Local topological order 1

(LTO1) Whenever $\Lambda \ll \Delta$ sufficiently, $p_\Delta A(\Lambda) p_\Delta = \mathbb{C} \cdot p_\Delta$.



$\Lambda \ll_2 \Delta$
Δ *completely surrounds* Λ
with surrounding constant 2



- Inspired by TQO axioms of [BHM10]
- (LTO1) implies both TQO axioms.
- Holds for Toric Code [AFH07], Kitaev's Quantum Double [Naa12, CDH$^+$20], Levin-Wen [QW20, JNPW23], Walker-Wang [JNP], ...

Proof sketch for Toric Code

4 steps for proof that $p_\Delta x p_\Delta \in \mathbb{C} \cdot p_\Delta$ for $x \in A(\Lambda) = \bigotimes_{\ell \in \Lambda} M_2(\mathbb{C})$:

1. Pauli matrices I, X, Y, Z span $M_2(\mathbb{C})$, so we may assume x is a *Pauli monomial*. (Notice A_s, B_p are also Pauli monomials.)
2. All Pauli monomials commute or anti-commute.
3. If x anticommutes with some A_s (or B_p) in Δ,

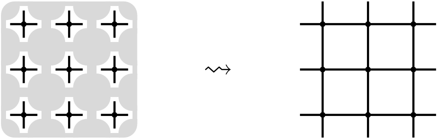
$$p_\Delta x p_\Delta = p_\Delta A_s x p_\Delta = -p_\Delta x A_s p_\Delta = -p_\Delta x p_\Delta$$

   so $p_\Delta x p_\Delta = 0$.
4. If x commutes with all A_s, B_p in Δ, then x is a product of A_s, B_p in Λ [AFH07]. (This part is an algorithm.)

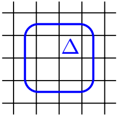
Proof sketch for Levin-Wen

3 steps for proof that $p_\Delta x p_\Delta \in \mathbb{C} \cdot p_\Delta$ for $x \in A(\Lambda)$

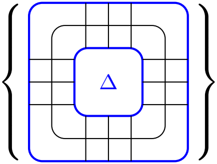
1. Compressing by p_Δ projects to the *skein module* [Kon14, GHK+24].



2. $p_\Delta x p_\Delta$ commutes with tube algebra action on $\partial\Delta$:



$\mathrm{Tube}_{\mathcal{C}}(\partial\Delta) =$

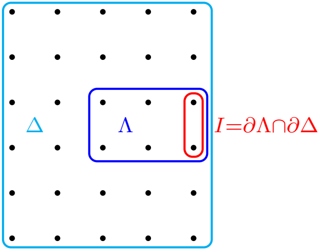


.

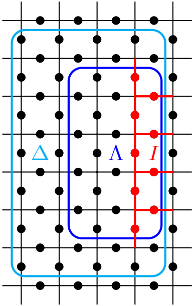
3. Skein module is an irreducible $\mathrm{Tube}_{\mathcal{C}}(\partial\Delta)$-module. (Irreducible iff trivial commutant.)

Local topological order 2

(LTO2) Whenever $\Lambda \Subset \Delta$ sufficiently, $p_\Delta A(\Lambda) p_\Delta = B(I) \cdot p_\Delta$.



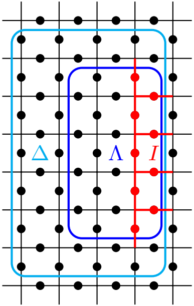
$\Lambda \Subset_2 \Delta$
Δ *weakly surrounds* Λ
with surrounding constant 2

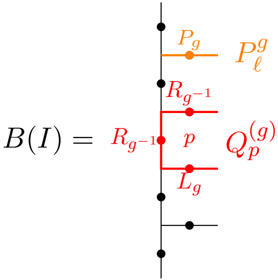


- $B(I)$ only depends on I and not on $\Lambda \Subset \Delta$.
- (Really $B(I)$ is supported on *sites near* I...)
- $B(I)$ has been calculated for Toric Code [JNPW23], Kitaev Quantum Double [CHK$^+$24], Levin-Wen [JNPW23], in terms of *fusion categorical nets*

Examples of boundary algebras I

Examples - Toric Code/Kitaev Quantum Double

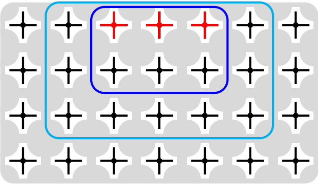


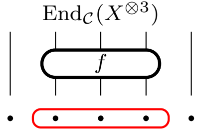


- $B(I)$ generated by *patch operators* [IW23, CW23]: projections onto $\mathbb{C} \cdot |g\rangle$ and summands of partial plaquette operators.
- $B(I)$ forms the fusion categorical net for $\mathcal{C} = \mathsf{Hilb}(G)$ with generator $X = \mathbb{C}[G]$
- Smooth boundary is UFC net for $\mathcal{C} = \mathsf{Rep}(G)$ with $X = \mathbb{C}^G$

Examples of boundary algebras II

Example - Levin-Wen





fusion categorical net

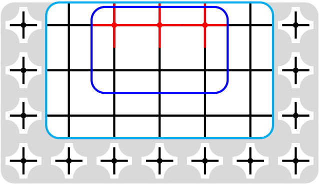
- Boundary algebra is fusion categorical net for $\mathcal{C}$ with $X = \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} c$
- *Proof sketch*: Commutant of right $\mathrm{End}_{\mathcal{C}}(X^{\otimes I})$ action is left $\mathrm{End}_{\mathcal{C}}(X^{\otimes J})$ action on skein module [Kon14, GHK+24]

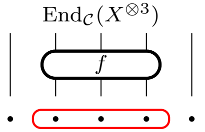
$$\mathrm{End}_{\mathcal{C}}(X^{\otimes J}) \curvearrowright \mathcal{C}(X^{\otimes I} \to X^{\otimes J}) \curvearrowleft \mathrm{End}_{\mathcal{C}}(X^{\otimes I})$$

- Gives examples of nets on S^1 which can fail strong additivity or Haag duality [SSS25], [PS, in progress, started last Friday]

Examples of boundary algebras II

Example - Levin-Wen





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Topological holography

Boundary algebras give a precise mathematical formulation of *topological holography* [IW23, CW23]:

- Boundary algebra is discrete (1+1)D AQFT holographically dual to the bulk theory.
- Bulk topological order should be characterized as *'local' representations/DHR bimodules* [Jon24] of the boundary algebra.

Definition - DHR bimodule

(Intense math slide 1/4)

A bimodule M for a net of algebras B which can be *localized* in any large enough region I: there are $m_1, \ldots, m_n \in M$ such that

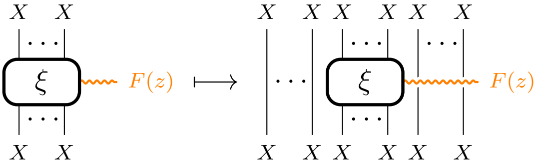
- every $m \in M$ can be written as $\sum m_i b_i$ for some $b_1, \ldots, b_n \in B$, and
- $m_i b = b m_i$ for all $b \in B(J)$ for $J \subset I^c$.

DHR bimodules for UFC nets

For $z \in Z(\mathcal{C})$, get a DHR bimodule M^z by taking inductive limits

$$M^z = \varinjlim M_n^z \qquad M_n^z = \mathrm{Hom}(X^{\otimes n} \to X^{\otimes n} \otimes F(z))$$

Get inclusions $M_n^z \hookrightarrow M_{j+n+k}^z$ by



Theorem [Jon24]

- DHR bimodules for a 1D net of algebras forms a braided W^*-tensor category. ($\mathrm{W}^* =$ von Neumann)
- DHR bimodules for the fusion categorical net for $\mathcal{C}$ is equivalent to $Z(\mathcal{C})$.

Cone von Neumann algebras

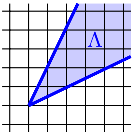
Given $\Lambda \ll \Delta$,

$$p_\Delta x p_\Delta = \psi(x) p_\Delta$$

for $\psi(x) \in \mathbb{C}$ independent of Δ beyond $\Lambda \ll \Delta$.

- $\psi : A \to \mathbb{C}$ is the unique state such that $\psi(p_\Lambda) = 1$ for all Λ.
- ψ is a pure state.
- For an LTO coming from a translation invariant frustration free local Hamiltonian, ψ is the unique translation invariant ground state [BR97].
- For a cone Λ,

$$A(\Lambda)'' \subset B(L^2(A, \psi))$$

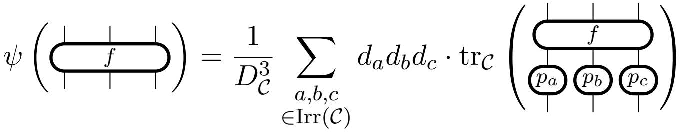


is a properly infinite *von Neumann factor*.

Type of von Neumann cone algebra

- For Toric Code and Quantum Double, we get a type II factor [Oga24]
- For Levin-Wen, we get a type II factor iff $d_c = 1$ for all $c \in \mathrm{Irr}(\mathcal{C})$ (all objects invertible). Otherwise it is type III.

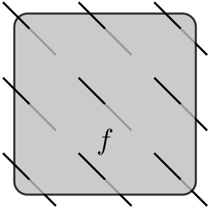
This follows by calculating ψ on the boundary algebra of Levin-Wen. For $\mathrm{End}_{\mathcal{C}}(X^{\otimes 3})$ where $X = \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} c$,



Get ψ tracial iff $d_c = 1$ for all $c \in \mathrm{Irr}(\mathcal{C})$.

Future directions

- Analyze boundary algebras of (3+1)D Walker-Wang using *braided fusion categorical nets* [JNP]



$$\in \mathrm{End}_{\mathcal{B}}(X^{\otimes \Lambda}), \qquad X = \bigoplus_{b \in \mathrm{Irr}(\mathcal{B})} b$$

- Connect to entanglement bootstrap techniques which use strict area law for entanglement entropy [SKK20, KLRS24, KR24]
- Connect DHR bimodules to AQFT superselection sectors [Naa12, Oga22]
- Adapt LTO to study *chiral* (2+1)D topological order
- Give model-independent operator algebraic way to get higher modular fusion categories of excitations

Thank you for listening!

Slides available at:
`https://people.math.osu.edu/penneys.2/talks/PenneysKITP2025.pdf`

- [JNPW23] *Local topological order and boundary algebras* To appear **Forum of Math. Sigma.** `arXiv:2307.12552`
- [GHK$^+$24] *Enriched string-net models and their excitations*. **Quantum.** `arXiv:2305.14068`
- [CHK$^+$24] *Boundary algebras of the Kitaev quantum double model*. **J. Math. Phys.** `arXiv:2309.13440`
- [JNP] *2D Braided fusion spin systems and topological holography*. Coming soon to an `arXiv` near you!

Shameless plug

Book with Giovanni Ferrer and Kyle Kawagoe in progress:

- *Unitary Quantum Symmetries Lite*

(low dimensional) higher linear algebra in unitary setting, applications to topological phases

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