Introduction to subfactors

NCGOA mini-course

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Introduction

This mini-course will focus on the theory of II_1-subfactors. I will try to point out differences in the type II_1 and type III theories at several points along the way.

The aim will be to prepare everyone for Scott's course on "Problems in the classification of small index subfactors." Since this workshop also focus on conformal field theory (CFT) and vertex operator algebras, of special importance will be the theory of finite depth subfactors (which is highly related to completely rational CFTs).

There will be three main parts to this mini-course:

(1) II_1-subfactors and their standard invariants

(2) Planar algebras and Temperley-Lieb

(3) Graph planar algebra embedding and evaluability

The basic overview is as follows. By a theorem of Ocneanu and Popa, finite depth subfactors of the hyperfinite II_1-factor are in one-to-one correspondence with their standard invariants, which have the structure of subfactor planar algebras. (There is a more general correspondence for the infinite depth case, but it is much more technical and requires more care.)

From this correspondence, subfactor planar algebras give a generators and relations approach to the construction of subfactors. Graph planar algebras are a particularly useful combinatorial tool to construct subfactor planar algebras, since they have all the requisite characteristics to be subfactor planar algebras, except for evaluability.

The canonical Temperley-Lieb planar subalgebra of a subfactor planar algebra together with the annular Temperley-Lieb action gives insight into planar algebraic structure and reveals characteristics about possible skein theories. These tools help us look for generators and relations in the combinatorial graph planar algebra to find evaluable planar subalgebras, which are necessarily subfactor planar algebras.

1 II_1-subfactors and their standard invariants

1.1 von Neumann algebras

Definition 1.1. A von Neumann algebra is a $*$-closed subalgebra $A \subseteq B(H)$ such that $A = A''$, where for a subset $S \subset B(H)$,

$$S' = \{x \in B(H) | xs = sx \text{ for all } s \in S\}.$$

We will only work with separable Hilbert spaces in these lectures.

Exercise 1.2. Show that $S' = S'''$ for any subset $S \subset B(H)$.

Exercise 1.3. Show that if $S \subset T$, then $T' \subset S'$.

Thus von Neumann algebras come in pairs, A and A'. The center of a von Neumann algebra is $Z(A) = A' \cap A$, which is the center of both A and A'.

Definition 1.4. A von Neumann algebra A is called a factor if $Z(A) = \mathbb{C}1$.

There are 3 types of factors:

(1) A is type I_n if $A \cong B(H)$ where $\dim(H) = n$ with $n \in \mathbb{N} \cup \{\infty\}$.

(2) A is type II if A has a normal, semifinite tracial weight $\mathrm{tr} : A^+ \to [0, \infty]$. A is called type II_1 if this trace can be normalized so $\mathrm{tr}(1_A) = 1$. Otherwise A is called type II_∞.

(3) If A is not type I or type II, then A is type III.

The factors that appear in chiral CFTs are type III. By theorems of Izumi and Popa, the theory of finite depth subfactors (and its applications to completely rational CFT) are the same for type II_1 and type III, so we will focus mainly on type II_1.

Fact 1.5. A II_1-factor has a unique normalized trace.

Example 1.6. Suppose Γ is a countable icc group (all conjugacy classes infinite except for the class of $e \in G$). Consider the left regular action of G on $\ell^2(G)$ by $(\lambda(g)f)(h) = f(g^{-1}h)$. Then $L(G) = \lambda(G)''$ is a II_1-factor with $\mathrm{tr}_A(x) = \langle a\delta_e, \delta_e \rangle$, where δ_e is the indicator function at $e \in G$. For finite sums, $\mathrm{tr}_A\left(\sum c_g \lambda(g)\right) = c_e$.

Definition 1.7. A II_1-subfactor is called hyperfinite if it generated by an increasing union of finite dimensional algebras.

In fact, there is only one hyperfinite II_1-factor R up to isomorphism by Murray and von Neumann. In fact, $R \cong L(S_\infty)$, where S_∞ is the group of finite permutations of $\mathbb{N}$, and also $R \cong \bigotimes_{i=1}^\infty M_2(\mathbb{C})$.

1.2 The standard representation

Given a II_1-factor A and its (unique) trace tr_A, we define its standard representation on $L^2(A, \mathrm{tr}_A)$ via the GNS construction.

We define a sesquilinear form on A by $\langle a, b\rangle = \mathrm{tr}_A(b^*a)$. Since tr_A is faithful, there are no zero-length vectors. Define $L^2(A) = L^2(A, \mathrm{tr}_A)$ to be the completion of A in the 2-norm given by $\|a\|_2 = \mathrm{tr}_A(a^*a)^{1/2}$. To differentiate between an element of A and a vector in $L^2(A)$, we will write $a \in A$ and $\widehat{a} \in L^2(A)$.

Now A acts on $\widehat{A}$ by left multiplication: $a \cdot \widehat{b} = \widehat{ab}$. Since tr_A is a state on A, this action is bounded, and thus the action extends to an action of A on $L^2(A)$. Moreover, the operator norm on $L^2(A)$ agrees with the operator norm on A coming from any other representation.

Definition 1.8. The standard representation of A is the left regular representation on $L^2(A)$.

Since tr_A is a trace, the map $\widehat{a} \mapsto \widehat{a^*}$ is isometric, and thus extends to an anti-linear unitary J on $L^2(A)$ called the <u>modular conjugation</u>.

We now consider the action of Ja^*J on $L^2(A)$:

$$Ja^*J\widehat{b} = Ja^*\widehat{b^*} = J\widehat{a^*b^*} = \widehat{ba},$$

i.e., Ja^*J is right multiplication by a. Since right multiplication commutes with left multiplication, we have $JAJ \subset A' \cap B(L^2(A))$.

Fact 1.9. $JAJ = A' \cap B(L^2(A))$.

In the type III case, the above story is more complicated. First, there is no trace, so we must pick a faithful state φ. One can then do the GNS construction to get a faithful action of A on $L^2(A, \varphi)$, but the map $a \mapsto \widehat{a^*}$ is no longer isomteric, and is actually unbounded. However, it is closable, and one can define a modular conjugation J using the polar decomposition. The construction is also independent of the choice of state up to isomorphism using Connes' cocycle derivative.

1.3 II_1-subfactors, index, and the basic construction

We now consider a II_1-subfactor, i.e., an inclusion of II_1-factors $A \subseteq B$. Since A, B have unique traces $\mathrm{tr}_A, \mathrm{tr}_B$ respectively, we must have $\mathrm{tr}_B|_A = \mathrm{tr}_A$.

Definition 1.10 ([Jon83])**.** The index $[B : A]$ of the subfactor $A \subset B$ is the von Neumann dimension of $L^2(B)$ as a left A-module.

Fact 1.11 ([PP86])**.** The index $[B : A]$ is finite if and only if B is a finitely generated projective left A-module. In this case,

$${}_AB \cong \bigoplus_{i=1}^{n} {}_AB \oplus {}_AAp$$

for some $n \in \mathbb{N}$ and a projection $p \in A$, and $[B : A] = n + \mathrm{tr}_A(p)$.

Fact 1.12. When $[B : A] < \infty$, A' is also a II_1-factor.

Let us now look more closely at the standard representation of B on $L^2(B)$ in the presence of a finite index subfactor $A \subseteq B$. We immediately see the action of five II_1-factors:

$$\begin{tikzcd}
?? \arrow[dr] & & A' \arrow[dl] \\
B \arrow[r] & L^2(B) & JBJ = B' \arrow[l] \\
A \arrow[ur] & & JAJ, \arrow[ul]
\end{tikzcd}$$

and we see there should be one more II_1-factor in this story.

Definition 1.13. The basic construction of $A \subseteq B$ is the II_1-factor $JA'J$ acting on $L^2(B)$.

Example 1.14. Suppose $\alpha : G \to \mathrm{Aut}(A)$ is an outer action where G is a finite group. The basic construction of $A^G \subseteq A$ is isomorphic to $A \rtimes G$. The basic construction of $A \subseteq A \rtimes G$ is given by $A \otimes B(\ell^2(G))$.

Just as von Neumann algebras come in pairs A, A', we see that subfactors also come in pairs: $A \subseteq B$ and $B' \subseteq A'$, which is conjugate to $B = JB'J \subseteq JA'J$.

Alternatively, since $\mathrm{tr}_B |A = \mathrm{tr}_A$, the natural inclusion $A \subset B$ gives rise to a canonical inclusion $L^2(A) \subset L^2(B)$. We define the Jones projection $e_A : L^2(B) \to L^2(B)$ to be the orthogonal projection with range $L^2(A)$.

Fact 1.15 ([Jon83]). $JA'J = \langle B, e_A \rangle$, the von Neumann algebra generated by B and e_A.

Fact 1.16. For each $b \in B$, $e_A(\widehat{b}) \in \widehat{A}$.

By the preceding fact, we define the conditional expectation $E_A : B \to A$ by $E_A(b) = a$ if $e_A(\widehat{b}) = \widehat{a}$, which is completely-positive, faithful, and $A - A$ bilinear.

Fact 1.17 ([Jon83]). $[\langle B, e_A \rangle : B] = [B : A]$, and $\mathrm{tr}_{\langle B, e_A \rangle}(e_A x) = [B : A]^{-1} \mathrm{tr}_B(x)$ for all $x \in B$.

Hence we may iterate the basic construction of $A_0 = A \subseteq B = A_1$ to get the Jones tower of II_1-factors $(A_n)_{n \geq 0}$ where $[A_{n+1} : A_n] = [B : A]$, and $A_{n+1} = \langle A_n, e_n \rangle$, where $e_n : L^2(A_n) \to L^2(A_n)$ is the orthogonal projection with range $L^2(A_{n-1})$.

Now the first sign that something genuinely interesting is going on is the following proposition.

Proposition 1.18 ([Jon83]). *The projections $(e_i)_{i \geq 0}$ satisfy the Jones-Temperley-Lieb relations [TL71] for $d = [B : A]^{-1/2}$:*

(1) $e_i^2 = e_i^* = e_i$,

(2) $e_i e_j = e_j e_i$ *when* $|i - j| > 1$, *and*

(3) $e_i e_{i \pm 1} e_i = d^{-2} e_i$.

Hence a subfactor gives a semisimple quotient of the Temperley-Lieb algebra $TL_n(d)$ for $d = [B : A]^{1/2}$ for every $n \geq 0$, which we will discuss in the second lecture, leading to Jones' famous index rigidity theorem.

Remark 1.19. Given an inclusion of factors of arbitrary type $A \subset B$ together with a conditional expectation $E : B \to A$, there is a canonical operator valued weight $E^{-1} : A' \to B'$, and the Kosaki index $E^{-1}(1)$ is a central element of the extended positive cone of B, i.e., in $[1, \infty]$. For II_1-factors, this index agrees with the Jones index. In this case, it is possible to define a Jones projection in an analogous way.

It is also possible to do a downward basic construction of II_1-factors [Jon83, PP86]. One picks a projection $e_0 \in A_1$ with $E_{A_0}(p) = [B : A]^{-1}$, and we set $A_{-1} = A_0 \cap \{e_0\}'$. We have $[A_0 : A_{-1}] = [B : A]$, and one may iterate this process to get a Jones tunnel which is no longer canonical. However, A_{-n} is unique up to conjugation by a unitary in A_{-n+1}.

For a type III subfactor $A \subset B$ (actually, we only need an inclusion of properly infinite factors), it is possible to find a representation which is jointly standard for both A and B. In such a representation, Longo's canonical enomorphism [Lon89], which is analogous to the Jones tunnel, is given by $\gamma(x) = J_A J_B x J_B J_A$ where the J's are the respective modular conjugations. In fact, γ maps B into A and is well-defined up to an inner automorphism of A.

1.4 The standard invariant

We now define the standard invariant of a finite index II_1-subfactor $A \subseteq B$, which we assume to be irreducible for simplicity, i.e., $A' \cap B = \mathbb{C}1$. In this case, the standard invariant is a unitary, pivotal 2-category generated by a single 1-morphism.

Definition 1.20. The standard invariant of $A \subseteq B$ is the unitary, pivotal 2-category $\mathcal{C}$ defined as follows. The 0-morphisms of $\mathcal{C}$ are A and B. The 1-morphisms are the bimodules generated by ${}_AL^2(B)_B$ and ${}_BL^2(B)_A$, and 1-composition is given by Connes' relative tensor product of bimodules. The 2-morphisms are bimodule intertwiners. The two forms of 2-composition are given by tensor product of intertwiners and the usual composition of intertwiners.

Remark 1.21. By a theorem of Jones [Jon08], we can also define the standard invariant algebraically using the 1-morphisms ${}_AB_B$ and ${}_BB_A$. In the notes, we'll use Hilbert bimodules for completeness, so certain extra structure, like unitarity, is transparent. In the lectures, I'll use the algebraic formalism for convenience and clarity of presentation.

We will expand the details of the above definition before getting to the pivotal and unitary structure. $L^2(B)$ can be seen as a left and right bimodule over A and B. This gives 4 different flavors of bimodules: $A - A$, $A - B$, $B - A$, and $B - B$.

The 1-morphisms are all bimodules generated by ${}_AL^2(B)_B$ and ${}_BL^2(B)_A$, which means that they are all bimodules which appear as summands of some alternating tensor product of ${}_AL^2(B)_B$ and ${}_BL^2(B)_A$.

Fact 1.22 ([PP86, JS97]). $L^2(A_n) \cong \bigotimes_A^n L^2(B)$, the relative tensor product, and $A_n \cong \bigotimes_A^n B$, the algebraic tensor product.

Hence these summands correspond to projections in the $C - D$ endomorphisms of the $L^2(A_n)$ for $C, D \in \{A, B\}$, which are also known as the higher relative commutants.

Fact 1.23 (Semi-simplicity [Jon83]). For every $n \geq 0$, $A_0' \cap A_n$ is finite dimensional, and consequently, each $L^2(A_n)$ splits as a finite direct sum of $C - D$ bimodules for C, D.

In fact, we have the following correspondence between projections in the higher relative commutants and Hilbert $C-D$ bimodules.

$$\begin{aligned}
p \in A_0' \cap A_{2n} &\longleftrightarrow pL^2(A_n)\ A-A \text{ bimodule}\\
p \in A_0' \cap A_{2n-1} &\longleftrightarrow pL^2(A_n)\ A-B \text{ bimodule}\\
p \in A_1' \cap A_{2n-1} &\longleftrightarrow pL^2(A_n)\ B-B \text{ bimodule}\\
p \in A_1' \cap A_{2n} &\longleftrightarrow pL^2(A_n)\ B-A \text{ bimodule}
\end{aligned}$$

For a Hilbert $C-D$ bimodule H, the <u>dual/contragredient bimodule</u> $\overline{H}$ is given by the conjugate Hilbert space $\overline{H} = \{\overline{\xi} | \xi \in H\}$ together with the $D-C$ action given by $d \cdot \overline{\xi} \cdot c = \overline{c^*\xi d^*}$. Note that $\overline{\overline{H}} = H$.

Fact 1.24 (Duality)**.** Since A_n is a $*$-algebra, for every $C-D$ summand $H \subseteq L^2(A_n)$, $\overline{H} \subseteq L^2(A_n)$.

The bimodules $L^2(C)$ and $L^2(D)$ act as the identity bimodules under relative tensor product:

$$L^2(C) \otimes_C H \cong H \cong H \otimes_D L^2(D).$$

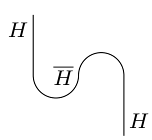
In fact, the dual bimodule ${}_D\overline{H}_C$ comes with two canonical maps $\mathrm{coev}_H : L^2(C) \to H \otimes_D \overline{H}$ and $\mathrm{ev}_H : \overline{H} \otimes_D H \to L^2(D)$. Representing these morphisms as cups and caps (reading upwards):

$\mathrm{ev}_H =$ 

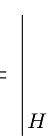
and $\mathrm{coev}_H =$ 

they satisfy:

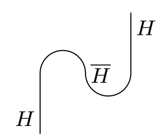
- the zig-zag identity:



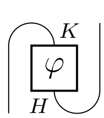
$$= (\mathrm{id}_H \otimes_A \mathrm{ev}_H) \circ (\mathrm{coev}_H \otimes_A \mathrm{id}_H) = \mathrm{id}_H =$$



$$= (\mathrm{ev}_{\overline{H}} \otimes_A \mathrm{id}_H) \circ (\mathrm{id}_H \otimes_A \mathrm{coev}_{\overline{H}}) =$$

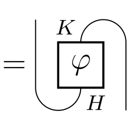


- pivotality:



$$= (\mathrm{ev}_K \otimes \mathrm{id}_{\overline{H}})(\mathrm{id}_{\overline{K}} \otimes \varphi \otimes \mathrm{id}_{\overline{H}})(\mathrm{id}_{\overline{K}} \otimes \mathrm{coev}_H)$$

$$= (\mathrm{id}_{\overline{H}} \otimes \mathrm{ev}_{\overline{K}})(\mathrm{id}_{\overline{H}} \otimes \varphi \otimes \mathrm{id}_{\overline{K}})(\mathrm{coev}_{\overline{H}} \otimes \mathrm{id}_{\overline{K}}) =$$



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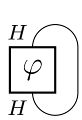
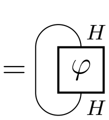
- sphericality:



$$= \mathrm{ev}_{\overline{H}} \circ (\varphi \otimes \mathrm{id}_{\overline{H}}) \circ \mathrm{coev}_H = \mathrm{ev}_H \circ (\mathrm{id}_{\overline{H}} \otimes \varphi) \circ \mathrm{coev}_{\overline{H}} =$$



In summary, we have:

Fact 1.25. Taking duals extends to a $\otimes$-contravariant 2-functor $\overline{\cdot} : \mathcal{C} \to \mathcal{C}$ such that $\overline{\overline{\cdot}} \cong \mathrm{id}_{\mathcal{C}}$.

Remark 1.26. Sphericality is equivalent to the subfactor being extremal, which is ensured by irreducibility. Extremality is not necessary to get a unitary, pivotal 2-category, but this will only happen in non-irreducible infinite depth cases, since finite depth subfactors are always extremal.

Taking adjoint of intertwiners gives an anti-linear involution $*$ on $\mathcal{C}$ which preserves all 0 and 1-morphisms, i.e., for all $C-D$ bimodules X, Y, $* : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(Y, X)$ given by $f \mapsto f^*$ is an involution of period 2. Moreover, $*$ is unitary:

- for each H, the duality is compatible with the adjoint, i.e., $\mathrm{coev}_{\overline{H}} = \mathrm{ev}_H^*$ and $\mathrm{ev}_{\overline{H}} = \mathrm{coev}_H^*$, and
- for each X, Y, the map $\langle f, g\rangle = \mathrm{tr}(g^* \circ f)$ on $\mathrm{Hom}(X, Y)$ is a positive definite inner product.

This can be seen from the fact that $\mathrm{Hom}(X, Y) \cong pL^2(A_n)q$ for some n and some projections p, q in the intertwiner space.

1.5 The principal graphs

From the unitary, pivotal 2-category $\mathcal{C}$, we define a combinatorial invariant, which consists of two bipartite graphs with a bit of extra structure.

Definition 1.27. The principal graph Γ_+ is the bipartite graph defined as follows. The even vertices are the isomorphism classes of simple $A-A$ bimodules in $\mathcal{C}$. A bimodule X is simple if $\mathrm{End}(X) \cong \mathbb{C}1$. The odd vertices are the isomorphism classes of simple $A-B$ bimodules in $\mathcal{C}$. There are $\dim(\mathrm{Hom}_{A-B}(X \otimes_A L^2(B), Y))$ edges from ${}_AX_A$ to ${}_AY_B$.

The dual principal graph Γ_- is defined similarly using $B-B$ and $B-A$ bimodules. The dual principal graph is also the principal graph for the inclusion $B' \subseteq A'$ or its conjugate $B \subseteq \langle B, e_A\rangle$.

We also record the data of the duality on $A-A$ and $B-B$ vertices by marking tags on the even vertices, and we order the odd vertices to encode the duality between the $A-B$ and $B-A$ vertices by lexicographic order.

Remark 1.28. The principal graphs (Γ_+, Γ_-) are invariants of the standard invariant $\mathcal{C}$ and the 1-morphism $L^2(B)$. We will see in the second lecture that they can be computed directly from the associated planar algebra.

Each of $\Gamma_\pm$ has a distinguished vertex corresponding to the trivial bimodules $L^2(A)$ and $L^2(B)$. The depth of a vertex is its distance to the trivial, which corresponds to the minimal n such that $X \subseteq L^2(A_n)$. Since A_n is a $*$-algebra, by Fact 1.24, the dual of an even vertex of $\Gamma_\pm$ is a vertex of $\Gamma_\pm$ at the same depth. The dual of an odd vertex of $\Gamma_\pm$ is a vertex of $\Gamma_\mp$ at the same depth.

A subfactor is said to have finite depth if the principal graph is finite.

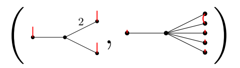
Fact 1.29. The principal graph is finite if and only if the dual principal graph is finite. In this case, the difference in their depth is at most one.

Example 1.30. Consider the outer action $\alpha : G \to \mathrm{Aut}(B)$. The standard invariant and principal graph of $A = B^G \subseteq B$ have the following structure.

The $A - A$ vertices correspond to irreducible representations of G. The full 2-subcategory generated by A is equivalent to $\mathsf{Rep}(G)$, where duality is contragredient.

The $B - B$ vertices correspond to the elements of G. The full 2-subcategory generated by B is equivalent to $\mathsf{Vec}(G)$, G-graded vector spaces, where duality is the group inverse law.

For $G = S_3$, the principal graphs are given by



One major importance of the standard invariant is the following theorem.

Theorem 1.31. *If $A \subseteq B$ is a finite depth hyperfinite II_1-subfactor, then the standard invariant $\mathcal{C}(A \subseteq B)$ is a complete invariant of $A \subseteq B$.*

By theorems of Ocneanu and Popa [Ocn88, Pop90], given a unitary, pivotal 2-category $\mathcal{C}$ with finitely many simple objects, generated by a 1-morphism X, one can reconstruct a hyperfinite II_1-subfactor $A \subset B$ with the same standard invariant.

The principal graphs $\Gamma_\pm$ also have some additional properties and structure that make them particularly useful.

Fact 1.32. $\dim(A_0' \cap A_n)$ is equal to the number of loops of length $2n$ on $\Gamma_\pm$ starting at $\star$.

We saw that irreducible summands of $L^2(A_n)$ correspond to minimal projections in the higher relative commutants. The traces of these projections give a <u>dimension function</u> dim on the vertices of $\Gamma_\pm$ which satisfies

$$d \dim(Y) = \sum_{Z \in V(\Gamma)} N_{Y,X}^{Z} \dim(Z)$$

for all simple $A - A$ bimodules Y and simple $A - B$ bimodules Z, where $N_{Y,X}^{Z}$ is the number of copies of Z inside $Y \otimes_A X$, and $d = \dim(X) = [B : A]^{1/2}$. The above condition says that the dimension function gives an eigenvector for the adjacency matrix of $\Gamma_\pm$.

When $\Gamma_\pm$ is finite, then this dimension is the unique <u>Frobenius-Perron eigenvector</u> with positive entries, normalized so that the dimension of $L^2(A)$ is 1 [BH12]. In this case, we write FPdim instead of dim.

Remark 1.33. In the infinite depth case, it is important to distinguish between the Frobenius-Perron dimension function FPdim corresponding to the norm of the adjacency matrix of $\Gamma_\pm$ and the quantum dimension function dim which may not agree. In fact, they agree if and only if the subfactor is amenable [Pop94]. Finite depth subfactors are amenable by uniqueness of the Frobenius-Perron dimension function [Jon86, EK98].

Fact 1.34. The index $[B : A] \geq \|\Gamma_\pm\|^2$, where the norm of $\Gamma_\pm$ is the operator norm of the adjacency matrix acting on ℓ^2 of the vertices [Pop94]. If $\Gamma_\pm$ is finite, then $[B : A] = \|\Gamma_\pm\|^2$, and the norm of $\Gamma_\pm$ is the largest eigenvalue of the adjacency matrix [Jon86].

The principal graphs are much less information than the standard invariant, which is exactly why they are so useful. One of the main motivating questions in subfactor theory is the following.

Question 1.35. *Given a pair of bipartite graphs* (Γ_+, Γ_-)*, are they the principal graphs of a finite index subfactor? If so, for how many subfactors?*

Remark 1.36. In the type III case, we work with sectors instead of bimodules [Lon89, Lon89]. A sector from B to A is an equivalence class of endomorphisms from $B \to A$, where two endomorphisms are equivalent if they are conjugate by a unitary. There are corresponding intertwiner spaces which are finite dimensional.

In the language of sectors, composition of 1-morphisms is just composition of endomorphisms, which is much simpler than Connes' relative tensor product of Hilbert bimodules. However, there is a conservation of difficulty in that it is more difficult to define the direct sum of sectors than the direct sum of Hilbert bimodules.

2 Planar algebras and Temperley-Lieb

The standard invariant of a finite index subfactor forms a particularly nice planar algebra, called a subfactor planar algebra. Conversely, by results of Popa, every subfactor planar algebra arises as the standard invariant of some subfactor. When the subfactors are finite depth, there is a one-to-one correspondence between hyperfinite II_1-subfactors and subfactor planar algebras.

Some material in this section has been adapted from [Pet10, BMPS12].

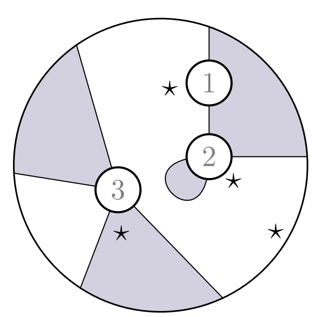
2.1 Planar algebras

Definition 2.1. A shaded planar tangle consists of:

- a large output disk,
- a finite number of disjoint input disks
- an even number of marked points on each of the input and output disks,
- non-intersecting strings that connect the marked points of the input disks to the output disk,
- a distinguished interval on each disk, and
- a checkerboard shading on the regions.

We consider shaded planar tangles to be equivalent if they are isotopic.

Example 2.2. Here is an example of a shaded planar tangle:



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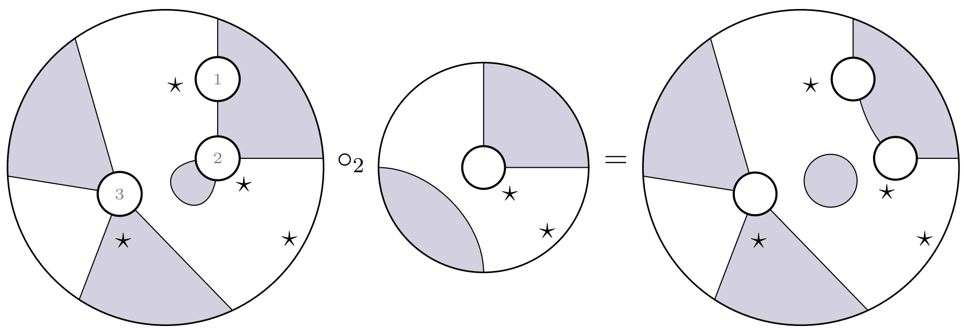
Definition 2.3. The shaded planar operad consists of all shaded planar tangles together with the operation of composition. Planar tangles can be composed by placing one planar tangle inside an interior disk of another, lining up the marked points, and connecting endpoints of strands. The numbers of endpoints, the distinguished intervals, and the shadings must match up appropriately.

Example 2.4. Here is an example of composition of tangles:



Definition 2.5. A shaded planar algebra is an algebra over the shaded planar operad. This consists of:

- a collection of $\mathbb{C}$-vector spaces $\mathcal{P}_\bullet = (\mathcal{P}_{n,\pm})$, and
- an action of shaded planar tangles as multilinear maps amongst the vector spaces. If a tangle T has input disks $D_1, \ldots, D_r$ and output disk D_0 each with $2k_i$ marked points and distinguished interval in an unshaded/shaded region, which corresponds to $+/-$ respectively, we get a linear map

$$Z(T) : \bigotimes_{i=1}^{r} \mathcal{P}_{k_i,\pm_i} \longrightarrow \mathcal{P}_{k_0,\pm_0}.$$

Moreover, the assignment $T \mapsto Z(T)$ must satisfy:

(1) isotopy invariance: isotopic tangles produce the same multilinear maps,

(2) identity: the identity tangle (which only has radial strings) acts as the identity transformation, and

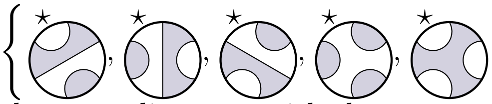
(3) naturality: the gluing of tangles corresponds to composition of multilinear maps. When we glue tangles, we match up the points along the boundary disks making sure the distinguished intervals marked by $\star$ align.

We call $\mathcal{P}_\bullet$ a planar $*$-algebra if each $\mathcal{P}_{n,\pm}$ has an anti-linear map $*$ which satisfies:

(4) reflection: If T^* is the reflection of T, then $(Z(T)(\xi_1, \ldots, \xi_r))^* = Z(T^*)(\xi_1^*, \ldots, \xi_r^*)$.

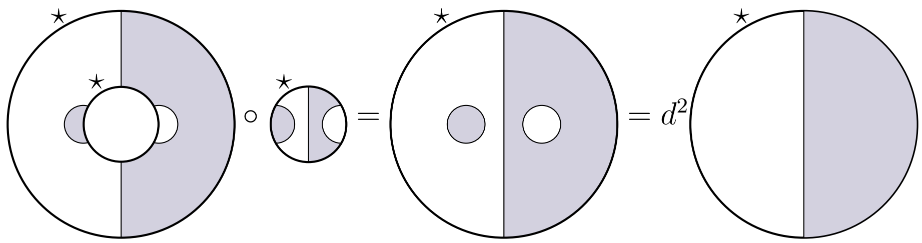
2.2 The Temperley-Lieb planar algebra

The most basic example of a planar algebra is the Temperley-Lieb planar algebra with parameter d (this algebra was introduced in [TL71] and formulated diagrammatically by Kauffman in [Kau87]). The vector spaces $\mathcal{TL}_{n,\pm}$ have a diagrammatic basis consisting of non-crossing pairings on $2n$ points on a circle. The number of such pictures is the n^{th} Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

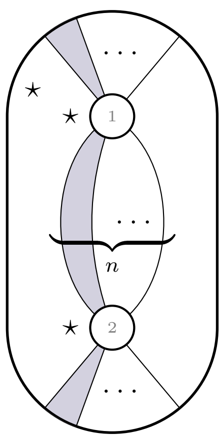
Example 2.6. $\mathcal{TL}_{3,+} = \mathbb{C} - \text{span}\left\{\right.$  $\left.\right\}$.

$\mathcal{TL}_{3,-}$ has a basis consisting of the same diagrams with the reverse shading, where the intervals marked $\star$ meet shaded regions.

The action of shaded planar tangles on this basis is straightforward: put the pictures inside the tangle, smooth all strings, and trade closed circles by multiplicative factors of d.

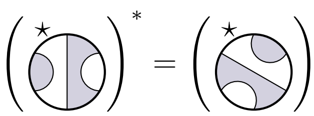
Example 2.7. 

Each vector space $\mathcal{TL}_{n,+}$ forms an associative algebra using the following multiplication tangle:

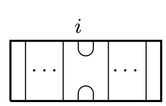

$: \mathcal{TL}_{n,+} \otimes \mathcal{TL}_{n,+} \to \mathcal{TL}_{n,+},$

and similarly for $\mathcal{TL}_{n,-}$ using the reverse shaded tangle. Associativity follows from the fact that tangle composition is associative on the nose. (Note that this is only true when we identify isotopic tangles. This is not always the case in more general notions of planar algebras and diagrammatic calculus for higher categories.)

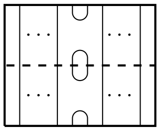
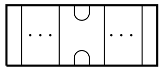
Moreover, the algebra $\mathcal{TL}_{n,\pm}$ becomes a $*$-algebra under the involution given by the anti-linear extension of reflection about a horizontal line.

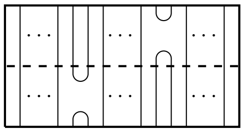
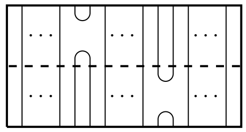
Example 2.8. .

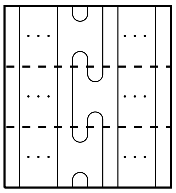
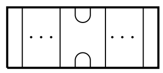
Exercise 2.9. For $i = 1, \ldots, n-1$, show that the elements of $\mathcal{TL}_{n,\pm}$ given by

$E_i =$ 

satisfy the following relations:

(1) $E_i^2 =$  $= d$  $= dE_i = dE_i^*$,

(2) $E_iE_j =$  $=$  $= E_jE_i$ if $|i-j| > 1$, and

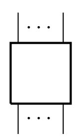
(3) $E_iE_{i\pm1}E_i =$  $=$  $= E_i$.

Remark 2.10. Drawing the diagrammatic elements as rectangles, multiplication corresponds to stacking of diagrams. When we draw diagrams as rectangles, the distinguished interval is always on the left. Sometimes we omit the output disk, which is assumed to be large. When it is omitted, again the distinguished interval is on the left.

Definition 2.11. We define $TL_n(d)$ to be the vector space $\mathcal{TL}_{n,+}$ with the structure of a $*$-algebra given by the multiplication and involution defined above, where the loop parameter is d.

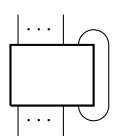
Exercise 2.12. Show that the map $E_i \mapsto de_i$ is a $*$-algebra isomorphism from $TL_n(d)$ to the abstract $\mathbb{C}$-algebra generated by $n-1$ projections $e_1, \dots, e_{n-1}$ satisfying the Jones-Temperley-Lieb relations from Proposition 1.18 [Jon83] with parameter d.

Definition 2.13. The inclusion tangle

$i_n =$ 

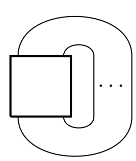
is a unital, injective $*$-algebra homomorphism $TL_n(d) \to TL_{n+1}(d)$.

The conditional expectation tangle

$\mathcal{E}_{n+1} =$ 

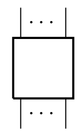
is a surjective map of $\mathbb{C}$-vector spaces $TL_{n+1}(d) \to TL_n(d)$.

The trace tangle

$\mathrm{tr}_n =$ 

defines a map $TL_n(d) \to TL_0(d)$. Note that $TL_0(d) \cong \mathbb{C}$ via the map which sends the empty diagram to 1. (In fact, this map is an isomorphism of complex $*$-algebras.) Using tr_n, we define a sesquilinear form on $TL_n(d)$ by $\langle x, y\rangle_n = \mathrm{tr}_n(xy^*)$.

Of course, the identity map id_n is given by the following diagram:



Exercise 2.14. Draw diagrams to show that the maps $i_n, \mathcal{E}_{n+1}$, tr_n, and id_n satisfy the following relations:

(1) $\mathcal{E}_{n+1} \circ i_n = d\,\mathrm{id}_n$,

(2) $\mathrm{tr}_{n+1} = \mathrm{tr}_n \circ \mathcal{E}_{n+1}$,

(3) $(i_n \circ i_{n-1} \circ \mathcal{E}_n(x))E_n = E_n i_n(x) E_n$ for all $x \in TL_n(d)$,

(4) $\mathrm{tr}_n(xy) = \mathrm{tr}_n(yx)$ for all $x, y \in TL_n(d)$,

(5) (Markov property) $\mathrm{tr}_{n+1}(i_n(x) \cdot E_n) = \mathrm{tr}_n(x)$ for all $x \in TL_n(d)$, and

(6) $\mathrm{tr}_n(\mathcal{E}_{n+1}(x) \cdot y) = \mathrm{tr}_{n+1}(x \cdot i_n(y))$ for all $x \in TL_{n+1}(d)$ and $y \in TL_n(d)$.

Exercise 2.15. Show that for d large, the diagrammatic basis vectors of $TL_n(d)$ have positive length.

Hint: Look at the matrix of inner products of the diagrams and use diagonal dominance.

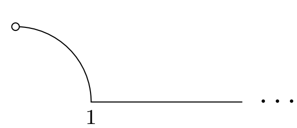
Fact 2.16. Given a subfactor $A \subseteq B$ with index $[B : A] = d^2$, we get a $*$-representation $TL_n(d) \to A_0' \cap A_n$ by $E_i \mapsto de_i$ which preserves the trace up to the scalar d^n. (This follows from the fact that the Markov trace is unique up to a scalar.) Since the relative commutants are finite dimensional C*-algebras with a faithful trace tr_n, the image is a positive-definite semi-simple quotient of $TL_n(d)$.

2.3 Jones-Wenzl idempotents

We are now interested in finding the $d \in \mathbb{C}$ for which the sesquilinear form $\langle \cdot, \cdot \rangle_n$ on $TL_n(d)$ is positive semi-definite, i.e., the $d \in \mathbb{C}$ for which there exists a positive-definite semi-simple quotient of $TL_n(d)$.

Looking at $TL_1(d)$, it is easy to see that we must have $d > 0$. We now make the following change of variables.

Definition 2.17. Consider $Q = \left\{ e^{i\theta} \middle| \theta \in (0, \frac{\pi}{2}) \right\} \cup [1, \infty)$



For $q \in Q$, we define quantum integer $[n]$ by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Quantum integers satisfy the following formulas:

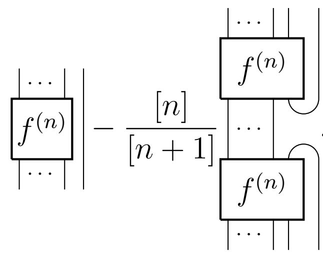
- $[1] = 1$
- $[2][1] = [2] = q + q^{-1}$
- $[2][n] = [n+1] + [n-1]$.

Exercise 2.18. For each $d > 0$, there is a unique $q \in Q$ such that $d = q + q^{-1} = [2]$.

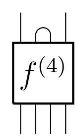
We write $TL_n(q)$ for $TL_n(d)$ when $d = q + q^{-1} = [2]$.

Definition 2.19. Let $f^{(0)} \in TL_0(q)$ be the empty diagram. Let $f^{(1)} \in TL_1(q)$ be the strand, i.e., $f^{(1)} =$ . If $[n+1] \neq 0$, we use Wenzl's recurrence relation to inductively define

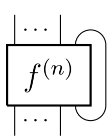
$$f^{(n+1)} = i_n(f^{(n)}) - \frac{[n]}{[n+1]} i_n(f^{(n)}) E_n i_n(f^{(n)}) = \quad \text{(Wenzl's relation)}$$



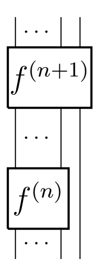
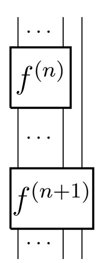
Exercise 2.20. Suppose $[1], \cdots, [n] \neq 0$ so that $f^{(0)}, f^{(1)}, \ldots, f^{(n)}$ are well-defined. Use induction on n to show that $f^{(n)}$ satisfies the following properties:

(1) Capping any two strands on top of $f^{(n)}$ gives zero, e.g.,  $= 0$,

(2) $f^{(n)}$ is an orthogonal projection, i.e., $f^{(n)} = (f^{(n)})^* = (f^{(n)})^2$,

(3) $\mathcal{E}_n(f^{(n)}) =$  $= \frac{[n+1]}{[n]} f^{(n-1)}$,

(4) $\mathrm{tr}_n(f^{(n)}) = [n+1]$, and

(5) $(i_n(f^{(n)}))f^{(n+1)} =$  $= f^{(n+1)}(i_n(f^{(n)}) =$  $= f^{(n+1)}$.

Hint: As the induction hypothesis, assume $f^{(n)}$ satisfies all of (1)-(5), and then show (1)-(5) for $f^{(n+1)}$.

Lemma 2.21. *Suppose $q = e^{i\theta}$ for some $\theta \in (0, \frac{\pi}{2})$, where $\theta \neq \frac{2\pi}{2n}$ for some $n \geq 3$, i.e., q is not a primitive $(2n)$-th root of unity for some $n \geq 3$. Let $k \geq 2$ be minimal such that $\theta > \frac{2\pi}{2(k+1)}$. Then $[1], [2] \ldots, [k] > 0$, but $[k+1] < 0$.*

Proof. Note that since $q = e^{i\theta}$,

$$[j] = \frac{e^{ij\theta} - e^{-ij\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(j\theta)}{\sin(\theta)}.$$

Since $\sin(\theta) > 0$, we only care about $\sin(j\theta)$. Since $\frac{\pi}{k} > \theta > \frac{\pi}{k+1}$, we have that $\sin(1\theta), \ldots, \sin(k\theta) > 0$, but $\sin((k+1)\theta) < 0$. $\square$

Remark 2.22. Note that $[n] = 0$ if and only if q is a $(2n)$-th root of unity. If $q = e^{i\theta}$, then $[2] = q + q^{-1} = 2\cos(\theta)$.

Theorem 2.23 (Jones' index rigidity)**.** *Suppose $\langle \cdot, \cdot \rangle_j$ is positive semidefinite for all $j \geq 0$. Then either $q \geq 1$, or q is a primitive $(2n)$-th root of unity for some $n \geq 3$. Hence*

$$[2] = q + q^{-1} \in \left\{ 2\cos\left(\frac{\pi}{n}\right) \middle| n \geq 3 \right\} \cup [2, \infty).$$

Proof. If q is not of this form, then let k be as in Lemma 2.21. We see that since $[1], [2], \ldots, [k] \neq 0$, $f^{(k)}$ is well-defined. However,

$$\langle f^{(k)}, f^{(k)} \rangle_k = \mathrm{tr}_k(f^{(k)}) = [k+1] < 0,$$

which is a contradiction. $\square$

Definition 2.24. The Temperley-Lieb subfactor planar algebra is the quotient of the Temperley-Lieb planar algebra by the length-zero vectors.

Corollary 2.25. *For a subfactor $A \subseteq B$, we must have*

$$[B : A] = d^2 \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \middle| n \geq 3 \right\} \cup [4, \infty].$$

2.4 Subfactors give planar algebras

The Temperley-Lieb planar algebra is the prototype for a subfactor planar algebra, which is exactly the type of planar algebra which comes from a subfactor.

Definition 2.26. A planar $*$-algebra $\mathcal{P}_\bullet$ is called a subfactor planar algebra if it is:

- finite dimensional: $\dim(\mathcal{P}_{n,\pm}) < \infty$ for all $n, \pm$,
- evaluable: $\mathcal{P}_{0,\pm} \cong \mathbb{C}1$ under the map sending the empty diagram to $1_{\mathbb{C}}$,
- unitary: $\langle x, y \rangle_n = \mathrm{tr}_n(y^* x)$ is positive definite for every $n \geq 0$, and
- spherical: spherical isotopy does not change the value of a closed diagram.

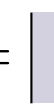
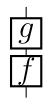
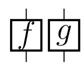
Fact 2.27. The closed loop in a subfactor planar algebra $\mathcal{P}_\bullet$ must count for a multiplicative factor of some $d \in \mathbb{C}$ called the loop parameter or modulus. In fact, $\mathcal{P}_\bullet$ has a canonical Temperley-Lieb planar subalgebra $\mathcal{TL}_\bullet$ with the same modulus, so we must have $d \in \left\{ 2\cos\left(\frac{\pi}{n}\right) \middle| n \geq 3 \right\} \cup [2, \infty)$.

Remark 2.28. A non-spherical subfactor planar algebra can have two loop parameters $d_\pm$ depending on the shading of the closed loop. However, every subfactor planar algebra with one of $d_\pm < 2$ is spherical.

The fact that a subfactor gives a subfactor planar algebra is a specific example of the following.

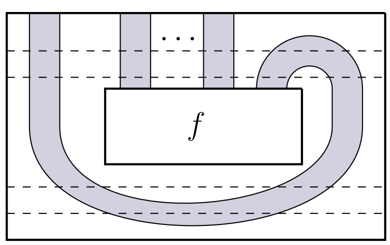
Theorem 2.29 (e.g., see [Gho11]). *Given a unitary, pivotal 2-category with 0-morphisms A and B and a 1-morphism ${}_AX_B$, we get a planar algebra by setting*

$$\mathcal{P}_{n,+} = \mathrm{Hom}(1, (X \otimes \overline{X})^{\otimes n}) \text{ and } \mathcal{P}_{n,-} = \mathrm{Hom}(1, (\overline{X} \otimes X)^{\otimes n}).$$

- *The strand is the identity 1-morphsim:* $\mathrm{id}_X =$  *and* $\mathrm{id}_{\overline{X}} =$
- *Caps are evaluation* $\mathrm{ev}_X =$  *and* $\mathrm{ev}_{\overline{X}} =$ 
- *Cups are coevaluation* $\mathrm{coev}_X =$  *and* $\mathrm{coev}_{\overline{X}} =$ 
- *Vertical join is composition* $g \circ f =$ 
- *Horizontal join is tensor product* $f \otimes g =$ 

Remark 2.30. In the subfactor case, we have $\mathcal{P}_{n,+} \cong A_0' \cap A_n$ and $\mathcal{P}_{n,-} \cong A_1' \cap A_{n+1}$.

Example 2.31. If $f \in \mathcal{P}_{n,+} = \mathrm{Hom}(1, (X \otimes \overline{X})^{\otimes n})$, the tangle below is a composite map, read from bottom to top:



$\mathrm{id}_{(X\otimes\overline{X})^{\otimes n}} \otimes \mathrm{ev}_{\overline{X}}$

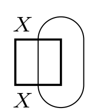
$\mathrm{id}_{(X\otimes\overline{X})^{\otimes n}\otimes X} \otimes \mathrm{ev}_X \otimes \mathrm{id}_{\overline{X}}$

$\mathrm{id}_{X\otimes\overline{X}} \otimes f \otimes \mathrm{id}_{X\otimes\overline{X}}$

$\mathrm{id}_X \otimes \mathrm{coev}_{\overline{X}} \otimes \mathrm{id}_{\overline{X}}$

coev_X

Fact 2.32. The Temperley-Lieb planar subalgebra $\mathcal{TL}_\bullet$ of the planar algebra built from choosing an $A-B$ object X in the unitary, pivotal 2-category $\mathcal{C}$ has parameter

$$d = \dim(X) = \mathrm{tr}(\mathrm{id}_X) =$$ 

(If $\mathcal{C}$ is not spherical, there may be two loop parameters – one for $\mathrm{tr}(\mathrm{id}_X)$ and one for $\mathrm{tr}(\mathrm{id}_{\overline{X}})$. Again, this never happens in the extremal case, and thus never for the finite depth case.)

Definition 2.33. The subfactor planar algebra of $A \subseteq B$, denoted by $\mathcal{P}_\bullet(A \subseteq B)$, is the planar algebra given by forming the unitary 2-category $\mathcal{C}(A \subseteq B)$ and choosing the 1-morphism $X = {}_AL^2(B)_B$.

By a deep theorem of Popa [Pop95], subfactor planar algebras give subfactors.

Theorem 2.34. *If $\mathcal{P}_\bullet$ is a subfactor planar algebra of modulus d, then there is a* II_1*-subfactor $A \subset B$ with index $[B : A] = d^2$ such that $\mathcal{P}_\bullet(A \subseteq B)$ is isomorphic to $\mathcal{P}_\bullet$.*

By results of Popa-Shlyakhtenko [PS03], one can find such a subfactor with $A \cong B \cong L(F_\infty)$. The following question is an important open question.

Question 2.35. *When can a subfactor planar algebra $\mathcal{P}_\bullet$ be realized as $\mathcal{P}_\bullet(A \subseteq B)$ with $A \cong B \cong R$, the hyperfinite* II_1*-factor?*

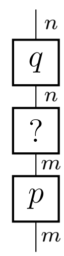
2.5 Principal graphs of subfactor planar algebras

The standard invariant of a subfactor is a complete invariant of the subfactor when it is finite depth. When a subfactor is not finite depth, the standard invariant may be much less information. Given a 1-morphism ${}_AX_B$ in the standard invariant $\mathcal{C}$, the planar algebra generated by X may be even less information.

However, the planar algebra still remembers the principal graph. This should not be surprising from the definition of $\mathcal{P}_\bullet$ and $\Gamma_\pm$, both which are built using the particular $A-B$ morphism $L^2(B)$ in $\mathcal{C}$.

We now give another planar algebraic definition of the principal graph.

Definition 2.36. Given projections $p \in \mathcal{P}_{m,+}$ and $q \in \mathcal{P}_{n,+}$, we define $\mathrm{Hom}_{\mathcal{P}_\bullet}(p, q)$ to be the vector space of all morphisms in $\mathcal{P}_{(m+n)/2,+}$ that can be fit between p and q in the following sense:

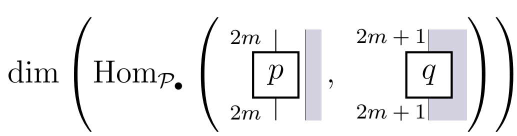


A projection is called simple or minimal if $\dim(\mathrm{Hom}_{\mathcal{P}_\bullet}(p, p)) = 1$.

Fact 2.37 ([Bis97])**.** Simple projections in $\mathcal{P}_{2n,\pm}$ correspond to the simple projections in $A_0' \cap A_{2n}$, which correspond to the simple $A-A$ bimodule summands of $L^2(A_n)$. Simple projections in $\mathcal{P}_{2n-1,\pm}$ correspond to the simple projections in $A_0' \cap A_{2n-1}$, which correspond to the simple $A-B$ bimodule summands of $L^2(A_n)$.

Definition 2.38. Two projections p, q are isomorphic if there is an element $u \in \mathrm{Hom}_{\mathcal{P}_\bullet}(p, q)$ such that $uu^* = p$ and $u^*u = q$. Such a u is called a partial isometry.

Fact 2.39. The even vertices of Γ_+ are the isomorphism classes of simple projections in the $\mathcal{P}_{2n,+}$, the odd vertices of Γ_+ are the isomorphism classes of simple projections in the $\mathcal{P}_{2n-1,+}$, and there are exactly



edges between an even $[p]$ and an odd $[q]$.

The dual principal graph is given similarly, but using the reverse shading corresponding to $\mathcal{P}_{n,-}$.

Example 2.40. By Wenzl's relation, the principal graph of the Temperley-Lieb subfactor planar algebra with index d^2 is the Coxeter-Dynkin diagram A_{n-1} if $d = 2\cos(\pi/n)$ for $n \geq 3$ and A_∞ otherwise.

3 Graph planar algebra embedding and evaluability

One advantage that the language of subfactors offers to people studying CFT and VOA is our abundance of examples. The third lecture will focus on a uniform method for constructing subfactor planar algebras, involving two difficult steps:

(1) finding generators in a graph planar algebra, and

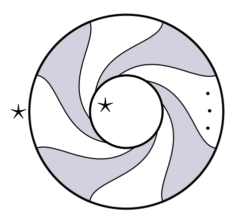
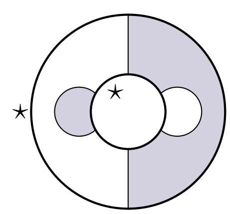
(2) finding relations to show they generate an evaluable planar subalgebra, which is necessarily a subfactor planar algebra.

We begin by describing how one goes about doing the first step.

3.1 Annular Temperley-Lieb modules

We have seen that a subfactor planar algebra $\mathcal{P}_\bullet$ has a canonical Temperley-Lieb planar subalgebra $\mathcal{TL}_\bullet$, which gave a strong restriction on the loop parameter from Jones' index rigidity theorem. There is also an action of the annular Temperley-Lieb category which reveals more structure of the planar algebra. The material in this section is from [Jon01].

Definition 3.1. An annular Temperley-Lieb tangle is a shaded planar tangle with exactly one input disk. The set of such tangles forms the involutive category ATL whose objects are $(n, \pm)$, and whose morphisms are annular Temperley-Lieb tangles. The composition is the usual tangle composition, and the involution $\dagger$ is turning the annular tangle inside out. (Note that this is not the same as the reflection!)

Example 3.2. Examples of annular tangles include  and .

We will now discuss representations of ATL, which consist of:

- A collection of Hilbert spaces $\mathcal{V}_\bullet = (\mathcal{V}_{n,\pm})$, and
- an action of annular Temperley-Lieb tangles. If a tangle T has inner boundary $(n, +)$ and outer boundary $(n', \pm')$, then we must get a map

$$Z(T) : \mathcal{V}_{n,+} \longrightarrow \mathcal{V}_{n',\pm'}.$$

Moreover, the assignment $T \mapsto Z(T)$ satisfies the isotopy invariance, identity, and naturality axioms as before (along with the reflection axiom if $\mathcal{V}_\bullet$ is a $*$-module). However, there is another axiom corresponding to the adjoint of linear maps:

(5) adjoint: The inside-out tangle $T^\dagger$ gives the adjoint of T: $Z(T^\dagger) = Z(T)^*$, the adjoint linear transformation.

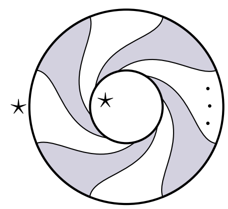
Example 3.3. A planar algebra $\mathcal{P}_\bullet$ is an ATL-module, and so is its canonical Temperley-Lieb planar subalgebra $\mathcal{TL}_\bullet$. Moreover, if $\mathcal{P}_\bullet$ is a subfactor planar algebra, the orthogonal complement of $\mathcal{TL}_\bullet$ given by $\mathcal{V}_{n,\pm} = \mathcal{TL}_{n,\pm}^{\perp} \subset \mathcal{P}_{n,\pm}$ is an ATL-module.

Definition 3.4. Suppose $\mathcal{P}_\bullet$ is a planar algebra. Given an element $S \in \mathcal{P}_{n,\pm}$, the ATL-module generated by S is the collection of vector spaces $\mathcal{V}_{k,\pm}$ obtained by applying all possible annular tangles to S.

We call $S \in \mathcal{P}_{n,\pm}$ uncappable if any annular tangle inner boundary $(n, \pm)$ and outer boundary $(n-1, \pm')$ applied to S gives zero.

There is an obvious notion of irreducibility and indecomposability of ATL-modules. These are equivalent due to the adjoint axiom.

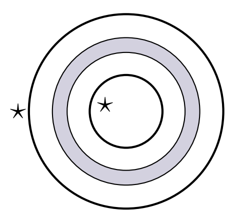
We now want to find out when a given ATL-module is irreducible. Suppose $\mathcal{V}_\bullet$ is an ATL-module. The least n for which $\mathcal{V}_{n,\pm} \neq (0)$ is called the weight of $\mathcal{V}_\bullet$. The annular tangles with inner and outer boundary $(n, \pm)$ form an algebra, so $\mathcal{V}_{n,\pm}$ is a module for this algebra. If the weight of $\mathcal{V}_\bullet$ is $n > 0$, then this action basically consists of the action by the 2-click rotation, which is a unitary.

$\mathcal{F} =$ 

Since a unitary is orthogonally diagonalizable, we get the following.

Fact 3.5. An ATL-module $\mathcal{V}_\bullet$ of weight $n > 0$ is irreducible if and only if $\mathcal{V}_{n,\pm}$ is spanned by a single self-adjoint uncappable rotational eigenvector S with eigenvalue ω_S an n^{th} root of unity.

It is possible that $\mathcal{V}_\bullet$ has weight zero, in which case it is irreducible if it is generated by a self-adjoint element which is an eigenvector for the double ring operator.



We no longer have that the eigenvalue must be a root of unity. In fact, the eigenvalue is in $[0, d^2]$, where d is the loop parameter. (Note that the double ring operator is a positive operator, as it is of the form $T^\dagger T$.)

Example 3.6. The Temperley-Lieb planar algebra $\mathcal{TL}_\bullet$ is an irreducible ATL-module generated by the empty diagram, which has eigenvalue d^2 for the double ring operator.

Fact 3.7. Every subfactor planar algebra decomposes as an orthogonal direct sum of irreducible ATL-modules.

One does this decomposition inductively, starting with the irreducible ATL-module coming from $\mathcal{TL}_\bullet$, taking the orthogonal complement, and looking for uncappable self-adjoint rotational eigenvectors.

Definition 3.8. We say a subfactor planar algebra $\mathcal{P}_\bullet$ is k-supertransitive if $\mathcal{P}_{k,\pm} = \mathcal{TL}_{k,\pm}$. This is the case if and only if the principal and dual principal graphs up to depth k are both just the Coxeter-Dynkin diagram A_{k+1}.

The orthogonal complement $\mathcal{TL}_{k,\pm}^{\perp}$ is an ATL-module, and the codimension of $\mathcal{TL}_{k,\pm}$ in $\mathcal{P}_{k,\pm}$ is called the k^{th} annular multiplicity. Since $\mathcal{P}_\bullet$ is a subfactor planar algebra, the zeroth annular multiplicity is always 1.

We call $\mathcal{P}_\bullet$ exactly k-supertransitive if k is minimal such that $\mathcal{P}_{k+1,\pm} \supsetneq \mathcal{TL}_{k+1,\pm}$. One can recognize this k immediately since it is the depth of the branch point of both the principal and dual principal graphs. In this case, the $(k+1)^{th}$ annular multiplicity is two less than the valence of the branch point.

When trying to find skein theories for exactly $(n-1)$-supertransitive subfactor planar algebras, we often look at the new uncappable self-adjoint rotational eigenvectors at depth n which generate the new irreducible ATL-modules orthogonal to $\mathcal{TL}_{n,\pm}$.

3.2 The planar algebra of a bipartite graph

Definition 3.9. Given a finite bipartite graph Γ, the bipartite graph planar algebra $\mathcal{G}_\bullet = \mathcal{G}_\bullet(\Gamma)$ is defined as follows. Let $\mathcal{L}_{n,\pm}(\Gamma)$ be the $\mathbb{C}$-vector space whose basis is the loops of length $2n$ on Γ which begin at an even/odd vertex corresponding to $+/-$ respectively. The space $\mathcal{G}_{n,\pm}$ is the dual space of $\mathcal{L}_{n,\pm}(\Gamma)$.

We now define the action of tangles. This means that given a tangle T with r input disks of type $(k_i, \pm_i)$ and an output disk of type $(k_0, \pm_0)$, along with functionals $f_i \in \mathcal{G}_{k_i,\pm_i}$ for $i = 1, \ldots, r$, and a loop $\ell \in \mathcal{L}_{k_0,\pm_0}$, we need to assign a number in $\mathbb{C}$. We do this by defining $T(f_1, \ldots, f_r)(\ell)$ as a "weighted state sum."

A state σ on T is an assignment of even/odd vertices of Γ to the unshaded/shaded regions of T respectively, and edges of Γ to the strings of T such that the edge assigned to a string must connect the vertices assigned to the adjacent regions. Note that the a σ assigns a loop $\partial_i(\sigma)$ to each input disk.

Now the action is given by

$$T(f_1, \ldots, f_k)(\ell) = \sum_{\sigma} c(T, \sigma) \cdot \prod_{i=1,\ldots,k} f_i(\partial_i(\sigma)),$$

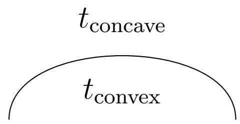
where $c(T, \sigma)$ is a constant obtained as follows. First, we isotope the tangle into a standard form where

- all disks are drawn as rectangles with the $\star$ on the left, and each has half their strings emanating up and half emanating down, and
- the strings are smooth with a finite number of local maxima and minima.

We let $E(T)$ be the set of local extrema of the strings of T drawn in a standard form. The constant is then given by

$$c(T, \sigma) = \prod_{t \in E(T)} \sqrt{\frac{\mathrm{FPdim}(\sigma(t_{\text{convex}}))}{\mathrm{FPdim}(\sigma(t_{\text{concave}}))}},$$

where $\mathrm{FPdim}(v)$ is the Frobenius-Perron dimension of the vertex v of Γ, and t_{convex} and t_{concave} are the on the convex and concave sides of t respectively.

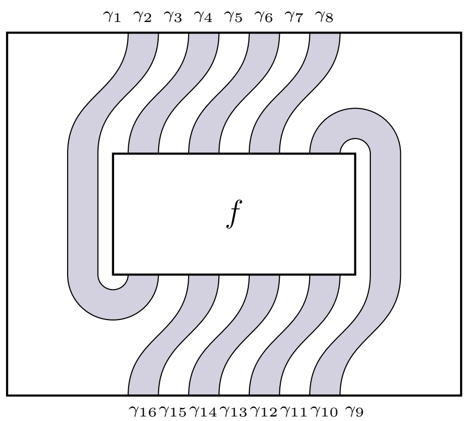


$$\longrightarrow \sqrt{\frac{\mathrm{FPdim}(\sigma(t_{\mathrm{convex}}))}{\mathrm{FPdim}(\sigma(t_{\mathrm{concave}}))}}.$$

It is straightforward to show that $c(\sigma, T)$ is independent of the standard form.

Example 3.10 ([BMPS12])**.**

$\rho(f)(\gamma) =$



$$= \sqrt{\frac{\mathrm{FPdim}(\gamma_3)\,\mathrm{FPdim}(\gamma_{11})}{\mathrm{FPdim}(\gamma_1)\,\mathrm{FPdim}(\gamma_9)}} f(\gamma_3 \ldots \gamma_{16}\gamma_1\gamma_2\gamma_3).$$

Fact 3.11. Since Γ is finite, $\mathcal{G}_\bullet$ is finite dimensional. There is a $*$-structure on $\mathcal{G}_\bullet$ by reversing loops: $f^*(\ell) = f(\ell^*)$ where ℓ^* is the reverse loop of ℓ. Moreover, it is possible to put a trace on $\mathcal{G}_{n,\pm}$ which is compatible with the inclusions. This gives a partition function which is spherical.

We see that $\mathcal{G}_\bullet$ has all the requisite characteristics of a subfactor planar algebra, except in general, $\dim(\mathcal{G}_{0,\pm}) > 1$.

3.3 The embedding theorem and evaluable planar subalgebras

Fact 3.12. Any planar $*$-subalgebra $\mathcal{P}_\bullet \subset \mathcal{G}_\bullet$ for which $\dim(\mathcal{P}_{0,\pm}) = 1$ is necessarily a subfactor planar algebra.

Example 3.13. Every graph planar algebra $\mathcal{G}_\bullet$ contains a canonical Temperley-Lieb planar subalgebra $\mathcal{TL}_\bullet$ with modulus $d = \|\Gamma\|$.

This gives one way to construct subfactors: find an evaluable planar $*$-subalgebra. In practice, this is difficult and computationally expensive, and there is no one method that will always work to find such planar subalgebras.

Remark 3.14. If $\mathcal{P}_\bullet$ is an evaluable planar $*$-subalgebra of $\mathcal{G}_\bullet(\Gamma)$, it is not necessarily the case that $\Gamma_\pm$ will be equal to Γ! One needs to do additional work to find the principal graphs $\Gamma_\pm$ of $\mathcal{P}_\bullet$.

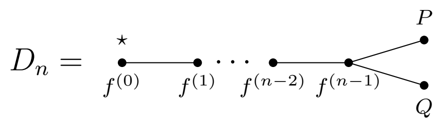
Because looking for evaluable planar subalgebras of $\mathcal{G}_\bullet$ can be difficult and computationally expensive, it would be great to know that it is possible to actually find such planar subalgebras! The next theorem tells us our computations are not done in vain, but note that finding evaluable planar subalgebras in no way depends on this theorem.

Theorem 3.15 ([JP11, MW]). *Suppose $\mathcal{P}_\bullet$ is a subfactor planar algebra. There is a planar $*$-algebra embedding $\mathcal{P}_\bullet \to \mathcal{G}_\bullet(\Gamma_+)$, where Γ_+ is the principal graph of $\mathcal{P}_\bullet$.*

It is interesting to note that this map does not use Γ_- at all. It is important to remember that just because we have an evaluable planar subalgebra $\mathcal{Q}_\bullet \subset \mathcal{G}_\bullet(\Gamma_+)$, it is not necessarily the case that $\mathcal{P}_\bullet = \mathcal{Q}_\bullet$. In general, there can be many evaluable planar subalgebras of $\mathcal{G}(\Gamma_+)$, and finding $\mathcal{P}_\bullet$ is an art. There also may be some parameters in the embedding map, so it is not canonical.

3.4 Generators and evaluation algorithms

Suppose we have a candidate principal graph and we want to determine if there is a subfactor with that principal graph. We will focus on the D_{n+2} Coxeter-Dynkin diagrams for $n \geq 2$, which have norm $2\cos(\pi/(2n+2))$:



Let $\mathcal{G}_\bullet$ be the graph planar algebra of D_{n+2}.

We'll first discuss how to look for a generator of a subfactor planar subalgebra of $\mathcal{P}_\bullet \subseteq \mathcal{G}_\bullet$ which could have principal graph D_{n+2}. Since D_{n+2} is exactly $n-1$ supertransitive with n^{th} annular multiplicity 1, the minimal projections at depths $0, \ldots, n-1$ are the Jones-Wenzl idempotents, and there is one uncappable self-adjoint rotational eigenvector at depth n.

The following fact is not at all obvious, but can be shown using a triple point obstruction [Pen13]. I won't say much else about triple point obstructions, but they are essential to the small index classification program, and they will certainly appear in Scott's lectures.

Fact 3.16. The eigenvalue must be -1, which implies n is even!

Remark 3.17. A similar application of this triple point obstruction shows E_7 does not exist.

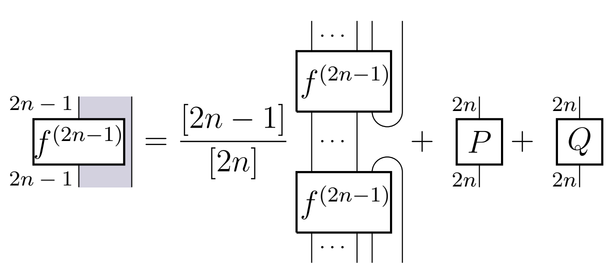
We want to find an evaluable planar subalgebra of the graph planar algebra which contains the canonical evaluable Temperley-Lieb planar subalgebra

$$\mathcal{TL}_\bullet \subset \mathcal{P}_\bullet \subset \mathcal{G}_\bullet,$$

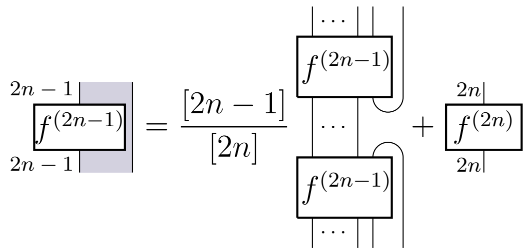
and we want $\Gamma_+ = D_{2n+2}$ (note we've replaced n with $2n$). This means we want to find a special element $S \in \mathcal{G}_{n,+}$ which is uncappable, self-adjoint, and satisfies $\mathcal{F}^2(S) = -\mathcal{S}$, where $\mathcal{F}^2$ is the two-click rotation.

Let $S \in \mathcal{G}_{2n,+}$ be a functional on $\mathcal{L}_{n,+}$ with variable values, i.e., $S(\ell) = s_\ell \in \mathbb{C}$ for $\ell \in \mathcal{L}_{2n,+}$. Requiring $S = S^*$, $\mathcal{F}^2(S) = -S$, and S uncappable imposes a set of linear and anti-linear equations on the s_ℓ's which can be easily solved. However, this does not solve uniquely for all the variables.

We now obtain a quadratic relation which has a finite solution space. First, since we want $\Gamma_+ = D_{2n+2}$, we must have



but, by Wenzl's relation, we know



This tells us that there should be elements $P, Q \in \mathcal{P}_{2n,+}$ such that $P + Q = f^{(2n)}$. Now since we want $S \perp \mathcal{TL}_{2n,+}$, we must have $S = P - Q$ up to a scalar multiple, since $P - Q$ is uncappable, self-adjoint, and perpendicular to $\mathcal{TL}_{2n,+}$. Since $\mathcal{TL}_{2n,+}$ is rotationally invariant and $\mathcal{F}^2$ is a unitary, its orthogonal complement must also be invariant, and thus $P - Q$ must be an eigenvector for $\mathcal{F}^2$!

Now the two equations $P + Q = f^{(2n)}$ and $P - Q = S$ give the quadratic relation $S^2 = f^{(2n)}$. We can solve this equation for the remaining s_ℓ's in $\mathcal{G}_{2n,+}$, and we see that there is a finite solution space, i.e., we can solve for all the variables.

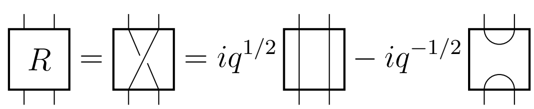
Exercise 3.18 (Extremely challenging!)**.** Actually do the calculation described above to get a formula for $S \in \mathcal{G}_{2n,+}$.

Note: No one has actually done this calculation for a general n, so there's no guarantee it actually works. But I would bet a beer that it's true. That said, I will buy a beer for the first person who actually does this or shows it does not actually work.

So we now have our hands on an explicit generator $S \in \mathcal{G}_{n,+}$ which generates some planar subalgebra $\mathcal{P}_\bullet \subset \mathcal{G}_\bullet$. We hope that this $\mathcal{P}_\bullet$ is evaluable and has principal graph D_{2n+2}. This means we need to find a set of relations to evaluate all closed diagrams in S to give a number.

We will use the braiding on Temperley-Lieb coming from the next exercise:

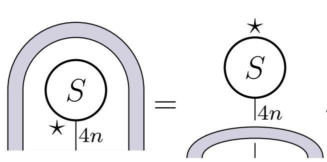
Exercise 3.19 ([Jon85, Kau87])**.** For every $q \in Q$ from Definition 2.17 (where $d = q + q^{-1} = [2]$), the element



is a braiding that satisfies the Reidemeister II and III relations, and the Reidemeister I relation up to a writhe factor $iq^{3/2}$.

Using this braiding, one can show the following facts.

Fact 3.20 ([MPS10])**.**

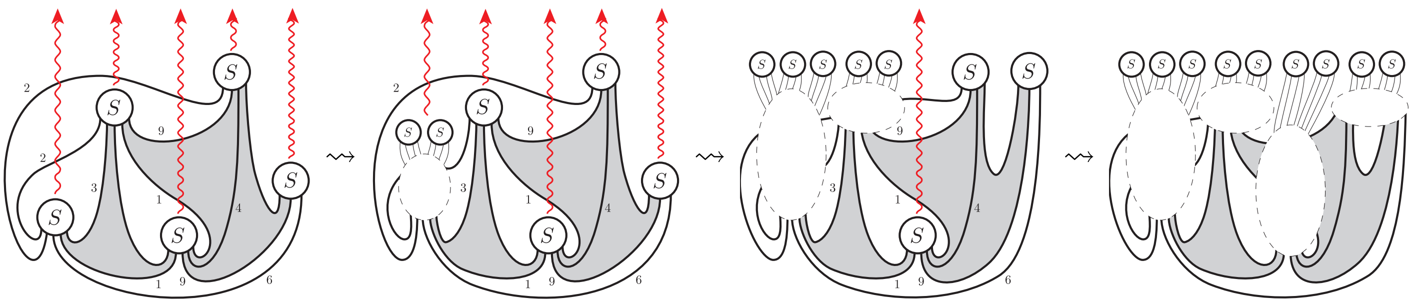


. (There is also an underbraiding, but with a -1.)

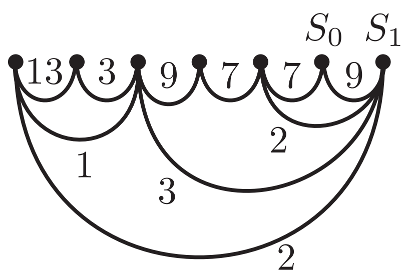
Using this overbraiding, we can now evaluate all closed diagrams in S as follows using the jellyfish algorithm [BMPS12].

Algorithm 3.21 (Jellyfish)**.**

(1) If there are any S's, use the overbraiding to pull the S's to the outside to get a linear combination of closed diagrams. Such a diagram is said to be in jellyfish form.



(2) Given a diagram in jellyfish form, we get a triangulation (actually a polygonization) of a polygon. Looking for an outer-most triangle (or polygon), the Pigeonhole Principle implies that there is an S connected by at least $2n$ strings to one of its neighbors.



Two S's connected by $2n$ strings is really S^2, which we can replace with $f^{(2n)}$, which gives a linear combination of diagrams in jellyfish form with 2 fewer generators. Repeat this step until there are no generators, and then evaluate the resulting Temperley-Lieb digrams at the appropriate d.

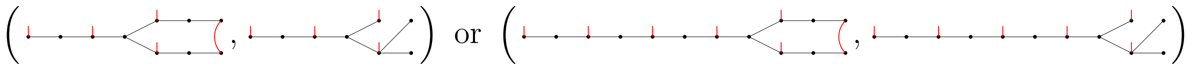
Theorem 3.22. *The above algorithm is consistent, i.e., no matter how we pull the S's to the outside or cancel the pairs of S's, we always get the same number. Thus $\mathcal{P}_\bullet$ is evaluable.*

In fact, by the next (challenging) exercise, $\mathcal{P}_\bullet$ must have $\Gamma_\pm = D_{2n+2}$; there is no other possibility by calculating $\dim(\mathcal{P}_{2n+1,\pm}$.

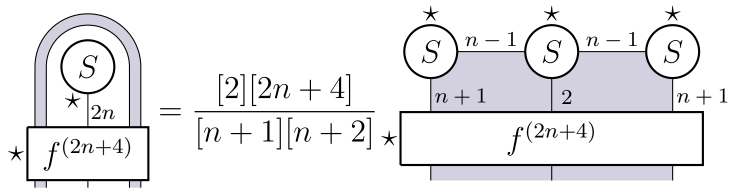
Exercise 3.23. Use the classification of bipartite graphs with norm less than 2 $(A_k, D_k, E_6, E_7, E_8)$ to show that if $\mathcal{P}_\bullet$ is evaluable, then Γ_+ must be D_{2n+2}. Then use the 1-click rotation $\mathcal{F} : \mathcal{P}_{2n,+} \to \mathcal{P}_{2n,-}$ to show that Γ_- is also D_{2n+2}.

3.5 Stability and the jellyfish algorithm

It turns out that the jellyfish algorithm is a universal skein theory for finite depth subfactors [BP14]. In general, subfactor planar algebras do not have overbraiding relations, so general jellyfish relations are much more complex. While the overbraiding gives a linear relation, most jellyfish relations are quadratic or cubic in nature, so applying the first step of the jellyfish algorithm leads to an increase in the number of generators! For example, the relations for the (extended) Haagerup subfactor with principal graphs



are given by $S^2 = f^{(n)}$ and



where $n = 4, 8$.

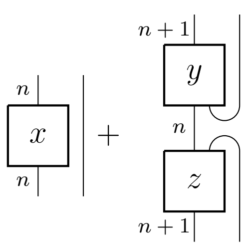
This makes the jellyfish algorithm quite unique amongst known evaluation algorithms, which usually try to reduce the number of generators at every step. We now give a brief outline of the stability theorem and its application to the jellyfish algorithm.

Definition 3.24 (Principal graph stability [Pop95])**.** The (dual) principal graph $\Gamma_\pm$ of $\mathcal{P}_\pm$ is said to be stable at depth n if every vertex at depth n connects to at most one vertex at depth $n+1$, no two vertices at depth n connect to the same vertex at depth $n+1$, and all edges between depths n and $n+1$ are simple. We say (Γ_+, Γ_-) is stable at depth n if both $\Gamma_\pm$ are stable at depth n.

Definition 3.25 (Popa's Stability Criterion [Pop95])**.** We say $\mathcal{P}_+$ is stable at depth n if

$$P_{n+1,+} = P_{n,+} + P_{n,+}e_{n,+}P_{n,+},$$

where we identify $P_{n,\pm}$ with its image in $P_{n+1,\pm}$ under the right inclusion. This means that $\mathcal{P}_{n+1,+}$ is spanned by elements of the form



where $x, y, z \in \mathcal{P}_{n,+}$.

There is a similar definition for stability of $\mathcal{P}_-$ at depth n. We say $\mathcal{P}_\bullet$ is stable at depth n if both $\mathcal{P}_+$ and $\mathcal{P}_-$ are stable at depth n.

Theorem 3.26. *The following are equivalent:*

(1) $\Gamma_\pm$ *is stable at depth* $n, n+1, \ldots, k-1$.

(2) $\mathcal{P}_+$ *is stable at depth* $n, n+1, \ldots, k-1$.

(3) Diagrams in jellyfish form from $\mathcal{P}_{n,\pm}$ *span* $\mathcal{P}_{k,\pm}$.

Definition 3.27. Let $\Gamma_\pm(n)$ be the truncation of $\Gamma_\pm$ to depth n, i.e., the bipartite graph obtained from $\Gamma_\pm$ by taking all vertices with depth at most n and all edges connecting them.

Theorem 3.28 (Principal graph stability [Pop95, BP14])**.**

(1) If (Γ_+, Γ_-) is stable at depth n, the truncation $\Gamma_\pm(n+1) \neq A_{n+2}$, and $\delta > 2$, then (Γ_+, Γ_-) is stable at depth k for all $k \geq n$, and Γ_+, Γ_- are finite.

(2) If Γ_+ is stable at depths n and $n+1$, the truncation $\Gamma_+(n+1) \neq A_{n+2}$, and $\delta > 2$, then (Γ_+, Γ_-) is stable at depth k for all $k \geq n+1$, and Γ_+, Γ_- are finite.

By this theorem, we could hope to do the following procedure to construct a finite depth subfactor planar algebra $\mathcal{P}_\bullet$ with principal graph Γ_+. First, we find the minimal n so that Γ_+ is stable at all depths higher than n. We then find a sufficiently large generating set $\mathcal{S} = \{S_i\}_{i=1}^N \subset \mathcal{G}_{n,+}(\Gamma_+)$ so that diagrams on $\mathcal{S}$ in jellyfish form would span $\mathcal{P}_{k,\pm}$ for every k. We then calculate 2-strand jellyfish relations for the generators in $\mathcal{S}$, so the planar subalgebra $\mathcal{P}_\bullet(\mathcal{S}) \subseteq \mathcal{G}_\bullet$ is evaluable and thus a subfactor planar algebra. We then use additional arguments to show the principal graph is correct.

We note that in practice, this is very difficult and computationally expensive. As of now, there is no one way to do this that works for all subfactors, and each implementation of this procedure requires many tricks to get the job done. Implementations of the above procedure include [BMPS12, MP12, PP13, Liu13].

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