1

Intro to Subfactors:

Overview:

Subfactors $\longleftrightarrow$ **Standard Invariants** $\longrightarrow$ **Principal graphs**

① $A \subseteq B$ — Planar Algebra $P_\bullet$ ② $\dashleftarrow ? \cdots$ (Γ_+, Γ_-)

- in nice cases, 1:1
- classification
- construction

(Γ_+, Γ_-) $\rightsquigarrow$ Graph Planar Alg.

$P_\bullet \overset{③}{\underset{\text{Embedding Thm}}{\hookrightarrow}} G_\bullet(\Gamma)$

- construction of examples from candidate principal graphs

II_1-Subfactors + Standard invariants

Def: A von Neumann algebra is a unital $*$-closed subalg. $A \subseteq B(H)$ s.t. $A = A''$ $\quad (S' = \{x \in B(H) \mid xs = sx \ \forall s\})$

- vNa's come in pairs A, A'
- center of A is $Z(A) = A' \cap A$.

A factor is a vNa w/ trivial center $Z(A) = \mathbb{C}1$.

3 types (of factors)

① type I_n if $A \cong B(H)$ where H (separable) is n-dim'l, $n \in \mathbb{N} \cup \{\infty\}$

② type II_1 if $\exists$ tracial state $tr: A \to \mathbb{C}$

- if tr exists, and is normalized $(tr(1_A) = 1_{\mathbb{C}})$, it is unique + faithful.

type II_∞ if $A \cong B \otimes B(H)$ for B type II_1

③ type III if not type I or II.

Facts on II_1-factors. A is a II_1-factor.

Define $\langle a,b\rangle = tr(b^*a)$, a positive-definite sesqui-linear form ($\langle a,a\rangle = tr(a^*a) = 0 \iff a=0$)

<u>Def:</u> $L^2(A)$ is completion of A in $\|\cdot\|_2$ ($\|a\|_2 = tr(a^*a)^{1/2}$)

- denote image of $A \subseteq L^2(A)$ by $\hat{A}$.
- $A \curvearrowright \hat{A}$ by $a\hat{b} = \widehat{ab}$, bdd for $\|\cdot\|_2$:

$$\|a\hat{b}\|_2^2 = tr(b^*a^*ab) \leq \|a^*a\|_\infty tr(b^*b) = \|a\|_\infty^2 \|b\|_2^2.$$

- get action $A \curvearrowright L^2(A)$ <u>left regular rep.</u>
- have map $J: \hat{a} \mapsto \widehat{a^*}$ isometric, so it extends to an anti-linear unitary on $L^2(A)$.

$$Ja^*J\hat{b} = J\widehat{a^*b^*} = \widehat{ba}, \text{ right mult.}$$

<u>Standard form:</u> $A \longrightarrow L^2(A) \longleftarrow JAJ = A'$

<u>Example:</u> G countable, i.c.c., $G \curvearrowright \ell^2(G)$, $L(G) = G''$. $tr = \langle \cdot\delta_e, \delta_e\rangle$

<u>Subfactors:</u> A II_1-subfactor is an inclusion of II_1-factors $A \subseteq B$. By uniqueness of tr, $tr_B|_A = tr_A$.

<u>Def:</u> $[B:A] = \dim_A L^2(B)$ von Neumann dimension $[\in \mathbb{R}_{\geq 1}]$

- finite $\iff {}_AB$ is a finitely generated projective module

$$\hookrightarrow \quad {}_AB \cong \bigoplus_{i=1}^{n} {}_AA \oplus {}_AAp \quad \text{some } p \in A \text{ proj.}$$

$$[B:A] = n + tr(p)$$

what structure is available? ① $tr_B|_A = tr_A$, ② $L^2(B)$, J ③
$B' = JBJ$.

① $L^2(A) \hookrightarrow L^2(B)$ ⇝ $e_A \in B(L^2(B))$ proj. w/ range $L^2(A)$
Jones projection.

②
$$\begin{tikzcd} \boxed{?} \arrow[dr] & & A' \arrow[dl] \\ B \arrow[r] & L^2(B) & JBJ = B' \arrow[l] \\ A \arrow[ur] & & JAJ \arrow[ul] \end{tikzcd}$$

$A \subseteq B$
$B' \subseteq A'$] subfactors come in pairs.

Fact: $JA'J = (JAJ)' = \langle B, e_A \rangle$ Jones basic construction

- $\langle B, e_A \rangle$ is a factor since JAJ is, type II.
- type $II_1 \iff [B:A] < \infty$. In this case,

$[\langle B, e_A \rangle : B] = [B:A]$

Example: $G \curvearrowright B$, $A = B^G \subseteq B$. Then $\langle B, e_A \rangle \cong \underbrace{B \rtimes G}_{\langle B, G \rangle}$, like a semi-direct product. $\sum b_g u_g$.

Iterate: $A_0 = A \subseteq B = A_1$ ⇝ Jones tower

$$A_0 \subseteq A_1 \overset{e_1}{\subseteq} A_2 \overset{e_2}{\subseteq} A_3 \overset{e_3}{\subseteq} A_4 \subseteq \cdots$$

Prop: The Jones projections e_i satisfy the Jones–Temperley–Lieb relations:

① $e_i^2 = e_i = e_i^*$

② $e_i e_j = e_j e_i \quad |i-j| > 1$

③ $e_i e_{i\pm 1} e_i = \underbrace{\frac{1}{[B:A]}}_{d^2} e_i$

Look at the tower more closely.

Fact: $\mathrm{End}_{A-A}(A_n) \cong A_0' \cap A_{2n}$
$\mathrm{End}_{A-B}(A_n) \cong A_0' \cap A_{2n-1}$
$\mathrm{End}_{B-A}(A_n) \cong A_1' \cap A_{2n}$
$\mathrm{End}_{B-B}(A_n) \cong A_1' \cap A_{2n-1}$

Finite dim'l C^*-alg's, so they are semi-simple.

Def: collection of these centralizer alg's is called the Standard invariant of $A \subseteq B$.

- axiomatizations as Popa's λ-lattices / commuting squares. Jones' planar algebras. (2) Ocneanu's paragroups / connections

Alternate description:

projections $p \in A_i' \cap A_n$, $i=0,1$ $\longleftrightarrow$ pA_n bimodule A-A, A-B, B-A, B-B

finite dim'l $A_i' \cap A_n \Rightarrow A_n$ splits as a finite $\oplus$ of irreducibles.

Fact: $A_n \cong \bigotimes_A^n B$ algebraically

- Look at A-A, A-B, B-A, B-B bimod's gen. by ${}_A B_B$, ${}_B B_A$ under $\otimes$, decomposing into irreducibles
- bimod's + intertwiners $\rightsquigarrow$ unitary pivotal 2-category.
  - objects are A, B
  - 1-morphisms are ${}_A H_B$, composition $\otimes_A$ or $\otimes_B$
  - 2-morphism are bimod. maps.

Extra structure:

(1) Since A_n is a $*$-alg, have dual bimodules $\overline{H} = \{\overline{\xi} \mid \xi \in H\}$, $c\overline{\xi}d = \overline{d^*\xi c^*}$

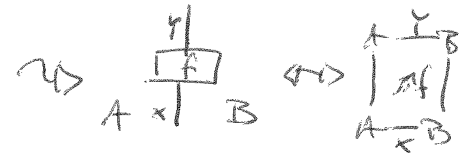
- have canonical evaluation ${}_B\overline{H} \otimes_A H_B \to {}_B B_B$ + coevaluation ${}_A A_A \to {}_A H \otimes_B \overline{H}_A$

<u>1-category</u>: $\underset{A}{\circ} \xrightarrow{X} \underset{B}{\circ} \xrightarrow{Y} \underset{C}{\circ}$ 1 dim'l

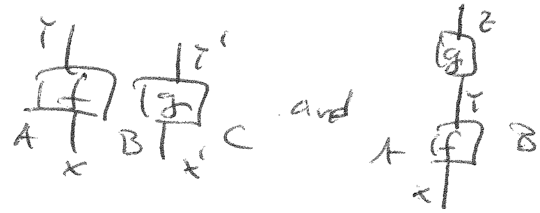
<u>2-category</u>:

$$\begin{array}{ccccc} A & \overline{Z} & B & \overline{Z'} & C \\ | & \Uparrow g & | & \Uparrow g' & | \\ A & \overline{Y} & B & \overline{Y'} & C \\ | & \Uparrow f & | & \Uparrow f' & | \\ A & \overline{X} & B & \overline{X'} & C \end{array}$$

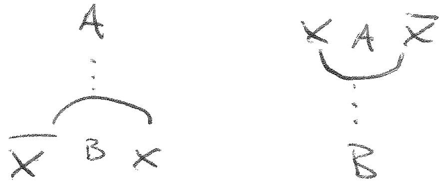
2 dim'l

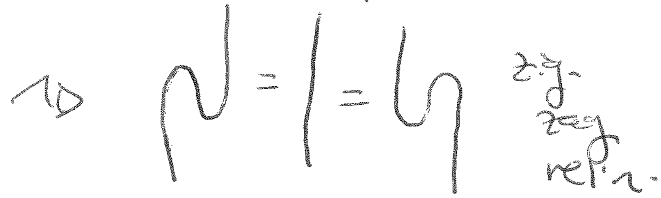


2 compositions for new notation:

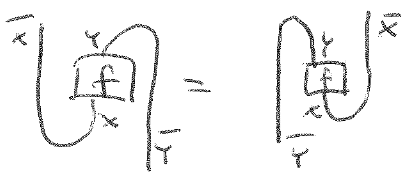


Draw evaluation + coevaluation as caps + cups:

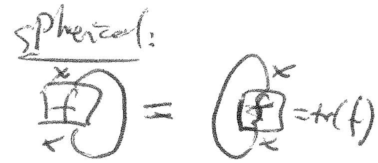




<u>Pivotal</u>:



$\longrightarrow$ planar isotopy



<u>Unitary</u>: have adjoint $X \xrightarrow{f} Y \rightsquigarrow Y \xrightarrow{f^*} X$
intertwiner space ${}_A X_B \to {}_A Y_B$ is a f.dim'l
~~[illegible]~~ Hilbert space w/ $\langle f, g \rangle = tr(g^* \circ f)$

<u>Note</u>: Spherical is extra structure $\Longleftrightarrow$ extremality of $A \subseteq B$.
— always case for finite depth or $A' \cap B = \mathbb{C}1$.

<u>Principal graphs</u>: pair (Γ_+, Γ_-) of bipartite graphs

— is an invariant of the std. inv. + generating bimod ${}_A B_B$.

Even vertices of Γ_+: iso classes simple A-A bimod's. | Γ_- B-B
Odd $\quad$ " : $\quad$ A-B | $\quad$ B-A

${}_A X_A$ — ${}_A Y_B$ connected by $\dim(\mathrm{Hom}(X \otimes_A B_B, Y))$ edges.

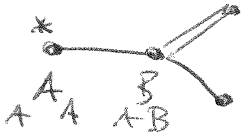
Example: G finite $G \curvearrowright B$. $A = B^G \subset B$.

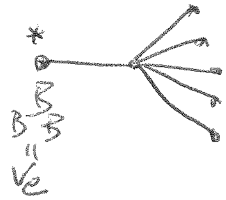
A-A irred. bimods $\longleftrightarrow$ irreps. of G.

B-B $\longleftrightarrow$ G-graded $\vec{v}$ spaces. $V_g \otimes V_h = V_{gh}$

A-B irred. $= {}_A B_B$

$G = S_3$





V_g

Note: Γ_- is also principal graph of dual SF $B' \subseteq A'$.

Depth: distance to $* = {}_A A_A$ or ${}_B B_B$

finite depth $\longleftrightarrow$ $\Gamma_\pm$ finite, in which case $|\text{depth}_+ - \text{depth}_-| \leq 1$
($\Rightarrow$ extremal)

Fact: $A \subseteq B$ finite depth hyperfinite. Then its standard invariant is a complete invariant of $A \subseteq B$.

Other properties + facts:

① $\dim(A_0' \cap A_n) = \#$ loops of length $2n$ on Γ_+ starting at $*$.
$\dim(A_1' \cap A_{n+1}) =$ - - - - Γ_-
— are equal!

② recall ${}_A X_B \longleftrightarrow p_X \in A_0' \cap A_{2n-1}$. $\overset{\dim(X)}{\overset{\shortparallel}{\mathrm{tr}}}(p_X)$ gives a dimension fct. on vertices, satisfying

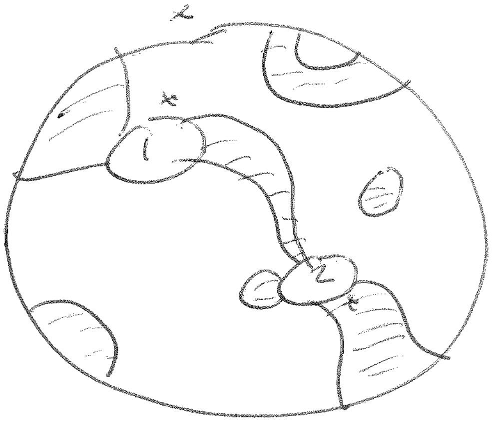
$$\dim(X)\dim(B) = \sum N_{X,B}^{Y} \dim(Y)$$

— eigenvector for adjacency matrix of Γ_+.
— if $\Gamma_\pm$ finite, this is normalized Frobenius-Perron eigenvector.

③ $[B:A] \geq \|\Gamma_\pm\|^2$. If $\Gamma_\pm$ finite, $[B:A] = \|\Gamma_\pm\|^2$.

Planar Algebras + TL:

Def: A shaded planar tangle consists of:



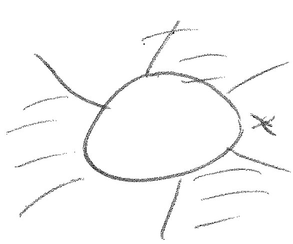
1. output disk
2. finite # of disjoint inner disks.
3. even # of pts on disks
4. strings, non-intersecting
5. distinguished (*) intervals on disks
6. checkerboard shading.

— up to isotopy.

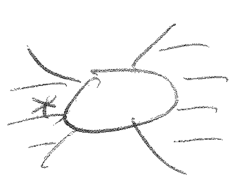
Def: Shaded planar operad $\mathbb{P}$ is all tangles w/ composition:

Def: A shaded planar algebra is an algebra over the shaded planar operad $\mathbb{P}$

1. A collection $P_\bullet = (P_{n,\pm})$ of $\mathbb{C}$ v spaces.
2. An action of planar tangles as multilinear maps.



input disk eats vectors in $P_{3,+}$



- - - - - - - - $P_{2,-}$

Action must satisfy:

(1) isotopy invariance — isotopic tangles give same map.

(2) naturality — gluing $\longleftrightarrow$ composition

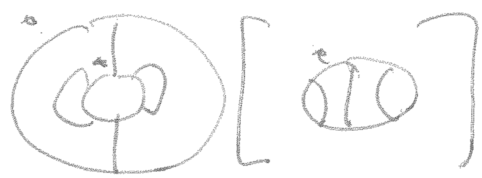
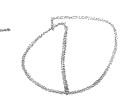
(3) identity  $= id_{P_{n,\pm}}$

Call $P_\bullet$ a $*$-PA if $\exists$ involution $*$ on $P_{n,\pm}$ s.t.

(4) reflection — T^* reflection of T, then
$$[T(\xi_1, \dots, \xi_n)]^* = T^*(\xi_1^*, \dots, \xi_n^*)$$

Example: Temperley-Lieb $TL_\bullet(d)$ $d \in \mathbb{C}$.

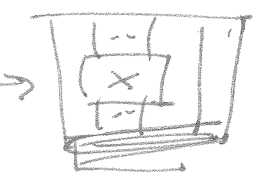
$TL_{3,+} = \mathbb{C}$-span , $TL_{3,-}$ reverse shading.

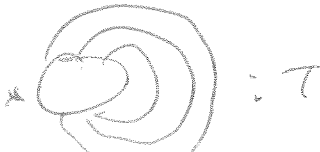
Action:  $=$  $= \delta^2$ 

extra structure:

(1) multiplication  $: TL_3 \times TL_3 \longrightarrow TL_3$

(2) adjoint  $=$

(3) inclusion $TL_{n,\pm} \hookrightarrow TL_{n+1,\pm}$ by $x \longmapsto$ 

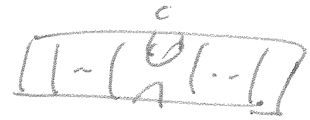
(4) trace  $: TL_{3,+} \to TL_{0,+} \cong \mathbb{C}$ via  $\mapsto 1_{\mathbb{C}}$

alg-iso.

$TL_{n,+}$ forms a unital $*$-alg under mult.,

$TL_{n,+} \hookrightarrow TL_{n+1,+}$, ~~trace satisfies~~

~~Trace satisfies tr_{n+1}~~

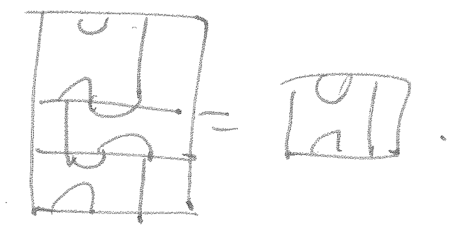
Distinguished elts: $E_i =$  for $i=1,\dots,n-1$.

Fact: E_i's satisfy:

① $E_i^2 = dE_i = dE_i^*$

② $E_iE_j = E_jE_i$ $\quad |i-j|>1$

③ $E_iE_{i\pm1}E_i = E_i$

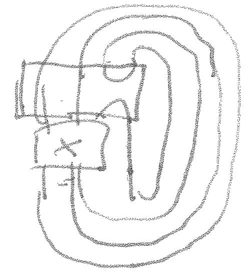


Ex: $TL_n \cong$ abstract $\mathbb{C}$-$*$-alg gen by $1, E_1, \dots, E_{n-1}$ subject to above relations.

Cor: If $d^2 = [B:A]$, get a $\mathbb{C}$ $*$-alg rep.

$TL_n \xrightarrow{\pi_n} A_0' \cap A_n$ by $E_i \mapsto de_i$

Ex: $\forall x \in TL_{n-1}$, $tr_n(xE_n) = tr_{n-1}(x)$



"Markov property"

$\Rightarrow \pi_n$ preserves trace up to d^n

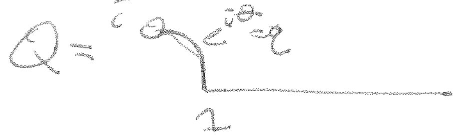
Q: When is $\langle x,y\rangle_n = tr_n(y^*x)$ pos. semi-def. on TL_n?

Thm (Jones '83) index rigidity: $\langle\, ,\rangle_n$ pos. semi-def. $\forall n$

$\Longleftrightarrow d \in \{2\cos\frac{\pi}{k} \mid k\geq 3\} \cup [2,\infty)$.

Cor: $[B:A] = d^2 \in \{4\cos^2\frac{\pi}{k} \mid k \geq 3\} \cup [4,\infty]$

<u>Sketch of pf:</u> 1^{st}: $\langle\cdot,\cdot\rangle$, $\langle \boxed{\,}, \boxed{\,} \rangle = \ominus = d > 0.$
$([B:A] \geq 1)$

- Distinguished elts of TL_n
- change of variables: $Q =$ 

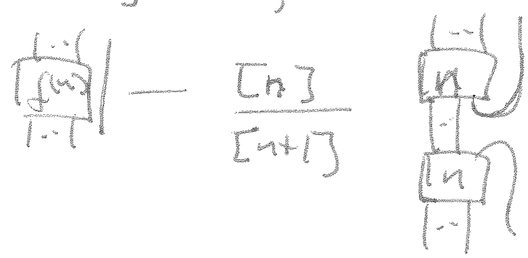
for $d>0$, $\exists!\ q \in Q$ s.t. $d = q + q^{-1}$.

<u>define:</u> $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$ so $[2] = \dfrac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1} = d.$

<u>Fact:</u> $[2][n] = [n-1] + [n+1]$. $\quad [0] = 0$, $[1] = 1$

<u>Define:</u> $f^{(0)} = \phi$ $\qquad f^{(1)} = \boxed{\,|\,}$.

If $[2], \dots, [n+1] \neq 0$, inductively define

$f^{(n+1)} =$ 

Wenzl's recurrence formula.

<u>Fact:</u> $f^{(n)}$ is a proj. ~~[illegible]~~ satisfying

$f^{(n)} E_i = E_i f^{(n)} = 0 \qquad i = 1, \dots, n-1$

$tr(f^{(n)}) = [n+1].$

all contexts.

<u>Fact:</u> $\pi_n(f^{(n)}) = 1 - \bigvee_{i=1}^{n-1} e_i$ in $A_0' \cap A_n$

<u>Lemma:</u> If $q = e^{i\theta}$, $\theta \in (0, \frac{\pi}{2})$ and $\theta \neq \frac{2\pi}{2k}$ $k \geq 3$,

pick ~~n~~ n minimal s.t. $\theta, \dots, n\theta < \pi$, $(n+1)\theta > \pi$.

Then $[1], \dots, [n] > 0$, but $[n+1] < 0$. Pf: $[j] = \dfrac{q^j - q^{-j}}{q - q^{-1}} = \dfrac{\sin(j\theta)}{\sin(\theta)}$

TL_{α} is a subfactor PA : $*$-PA $P_\bullet$ s.t.

(1) $\dim(P_{n,\pm}) < \infty$ $\forall n$ (f. dim.)

(2) Evaluable $P_{0,\pm} \cong \mathbb{C}$ via $\bigcirc \mapsto 1_{\mathbb{C}}$

(3) Unitary $\langle x, y \rangle_n = tr_n(y^* x)$ pos. def. $\forall n$.

(4) Spherical $\left(\boxed{x} = \boxed{x} \right)$ $\forall x \in P_{1,\pm}$.

Thm: Standard invariant of $A \subseteq B$ is a subfactor PA

w/ $P_{n,+} = A_0' \cap A_n$

$P_{n,-} = A_1' \cap A_{n+1}$ $\quad d^2 = [B:A]$

$E_i \leftrightarrow d e_i$

Thm: Every subfactor PA is std. inv. of some subfactor.

- if $A \subseteq B$ finite depth, then $P_\bullet$ is a complete invariant.

and ~~B~~ hyperfinite $\left(B = \left[\bigcup_{n \geq 0} F_n \right]'' \right)$

$\subset$ f. dim.

Rk: $\Gamma_\pm$ principal graphs are invariant of $P_\bullet$

~~[illegible]~~: proj's p, q in $P_\bullet$ are equivalent if

$\exists u$ s.t. $uu^* = p$, $u^* u = q$  same parity.

~~$p \sim q \Leftrightarrow [illegible]$~~

$\Gamma_\pm$: even's : Iso classes of simple proj's in $P_{2n,\pm}$

odd's : Iso classes of simple - - - $P_{2n+1,\pm}$

edges:  dim Hom. Watt's Formula $\Rightarrow$

Γ_+ for TL is A_n

GPA Embedding:

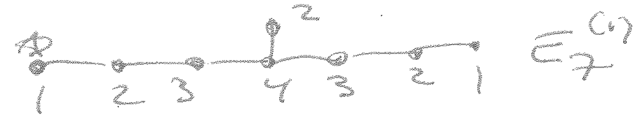
$A \subseteq B \longleftrightarrow P_\bullet \underset{?}{\longrightarrow} (\Gamma_+, \Gamma_-)$ (1)

Q: Given finite bipartite Γ, $\exists P_\bullet$ s.t. $\Gamma_+ = \Gamma$?

Thm (Jones-P., Morrison-Walker) $P_\bullet \hookrightarrow G_\bullet = GPA(\Gamma_+)$

$G_\bullet$ is a combinatorial PA built from Γ, where to look for $P_\bullet$

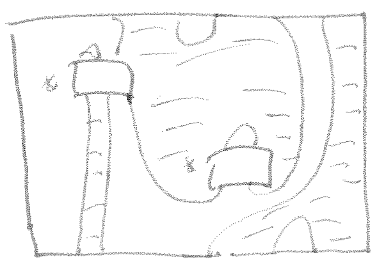
Def: Given Γ finite bipartite, dim is a positive dim fct on $V(\Gamma)$ s.t. $\dim(*) = 1$



satisfying $d \cdot \dim(v) = \sum_{w \sim v} \dim(w)$

$G_{n,\pm}$ is $\mathbb{C}$-span of loops of length $2n$ starting at $\pm$ vertex

ℓ^* is reverse loop. $\leadsto$ $G_\bullet$ a $*$-PA.

Action: $T =$  $(\ell_1, \ell_2) = \sum_{\text{State } \sigma} c(\sigma, T)\, \ell_\sigma$

State σ: assignment vertices $\longleftrightarrow$ regions, edges $\longleftarrow$ strings. v/w ε $v \overset{\varepsilon}{—} w$ compatible

- σ ~~gives~~ loops $\longleftarrow$ disks.
- State must be consistent w/ ℓ_1, ℓ_2
- Output disk labelled by ℓ_σ.

$$c(\sigma, T) = \prod_{\text{extrema}(T)} \left(\frac{\dim(\text{cnvx})}{\dim(\text{cncv})} \right)^{1/2}$$

cnvx concave cncv

Facts about $G_\bullet$: (1) $\dim(G_{0,\pm}) > 1$ (usually)

- $\dim(P_{n,\pm})$ = # loops (length $2n$ starting at $*$.

(2) $\dim(G_{n,\pm}) < \infty$ $\forall n$

(3) $\exists$ trace on $G_\bullet$ s.t. $\langle\cdot,\cdot\rangle$ pos. def.
spherical

Thy $G_\bullet$ is almost a subfactor PA.

Fact: Any evaluable $P_\bullet \subseteq G_\bullet$ is a subfactor PA.

ex: $TL_\bullet \subseteq G_\bullet$, know this embedding exactly?

Note: Γ_+ of $P_\bullet \subseteq G_\bullet$ is not nec. Γ!

GPA embedding program:
Step (1): find gen's.
Step (2): show generated PA evaluable
Step (3): compute Γ_+.

Step (1): tool: $ATL_\bullet$

tangles w/ 1 input disk form a category $ATL_\bullet$

Obj: $(n,\pm)$ Mor:  $: (3,+) \to (1,-)$

- A PA $P_\bullet$ is a rep'n of $ATL_\bullet$
- when $\delta > 2$, irreps of $ATL_\bullet$ easy to describe.

  • (n,ω) $n>0$ $\omega^n = \mathrm{id}$ $V_\bullet^{n,\omega}$

  • $(0,\lambda)$ $0 \le \lambda \le \delta^2$ $V_\bullet^{0,\lambda}$

- realized by a cyclic vector $V_\bullet^{n,\cdot} = ATL(S)$ weight n
- S satisfies: (1)  uncappable $V^{n,\cdot}_{k,\pm} = (0)$ $k<n$.

  (2) $S = S^*$ (3)  $= \omega S$ $n>0$ or  $= \lambda S$ $n=0$

Example: $TL = ATL(\text{⸰}\bigcirc)$, $\lambda = d^2$.

- irred $\iff$ indecomposable, $P_\bullet$ semi-simple $P_\bullet = \bigoplus_n \bigoplus_\omega a_{n,\omega} V_\bullet^{n,\omega}$ ← finite

Def: Annular multiplicities of $P_\bullet$ are $a_n = \sum_\omega a_{n,\omega}$

Subfactor PA $\iff$ $a_n < \infty$ $\forall n$, $a_0 = 1$.

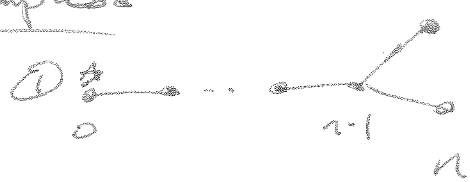
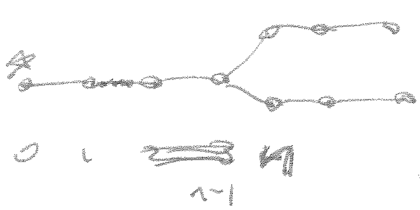
$TL_\bullet$ has $a_0 = 1$, $a_1 = 0$, ... $(1, 0, 0, \ldots)$

Def: $P_\bullet$ is k-supertransitive if $(a_0, a_1, \ldots, a_k) = (1, 0, \ldots, 0)$.

$\iff$ $P_{k,\pm} = TL_{k,\pm}$

$\iff$ up to depth k, $\Gamma_\pm =$  A_{k+1}

Examples:

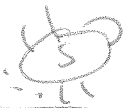
①  D_{n+2} ②  H_n

$(n-1)$-ST 3-ST

Look at depth 1 past branch pt.

Since $\dim(P_{n,\pm}) = \#$loops length $2n$ starting at ☆,

we see $a_n =$ valence of branch pt $-$ **2**.

- $\exists$ 1 new ATL module of weight n
- ~~[illegible]~~ $\exists\, S \in P_{n,+}$ s.t. ①  $= 0$, ② $S = S^*$, ③  $= \omega S$.

$S \perp TL$.

Fact: for these graphs, $\omega = -1 \Rightarrow$ [n even $\omega^n = 1$] [for H_n, $4 | n$]

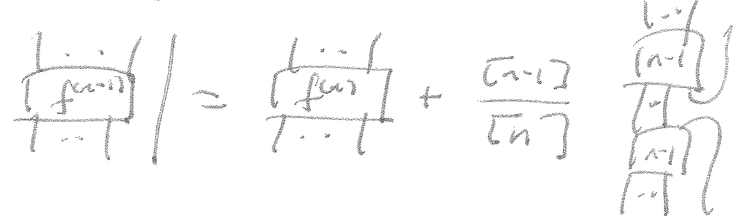
Do: look in $\mathcal{G}_{n,+}$ for such an S.

①-③ gives system of (anti-) linear eqs. to solve.

Fact: In P_0, $p - q \perp TL_{n,+}$, $TL_{n,+}^{\perp} = \mathbb{C} S$.

$\Rightarrow$ can take $S = p - q$.

By Wenzl's formula:



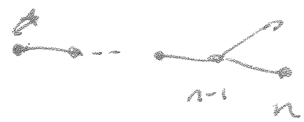
By looking at Γ: $= p + q + \text{(above)}$

$\Rightarrow p + q = f^{(n)}$, which we know in $G_{n,\pm}$.

Quadratic eq'n: $S^2 = (p-q)^2 = p+q = f^{(n)}$.

Fact: $\exists!$ sol'n up to a sign. $\rightsquigarrow S \in G_{n,\pm}$ generator.
(for D_{n+2}, n even, H_4, H_8)

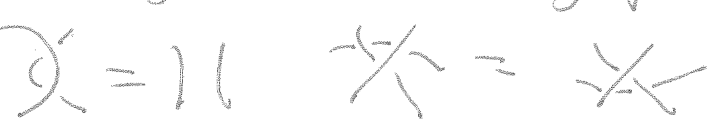
Step (2): Find relations.

(1) $\Gamma =$ 

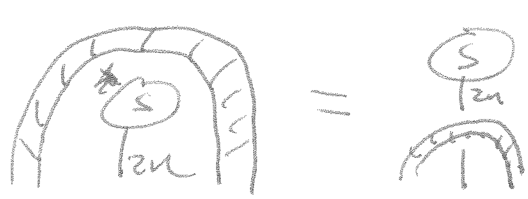
D_{n+2}, n even.

Ex: $\forall q \in \mathbb{Q}$, $d = q + q^{-1}$, let $\boxed{R} = \times = i q^{1/2} \, || - i q^{-1/2} \cup\cap$

is a braiding in TL. Satisfying R2 + R3 rel'ns:



(and R1 up to $i q^{3/2}$ writhe)

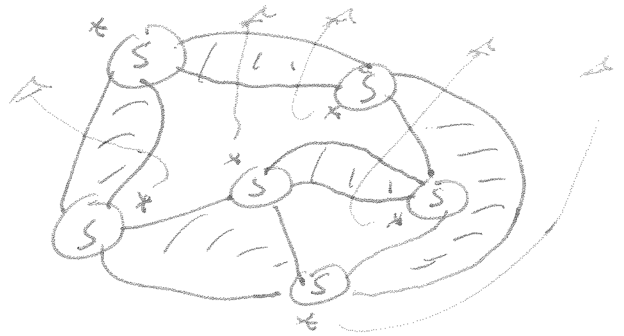
Fact: 

"overbraiding" (Morrison-Peters-Snyder)

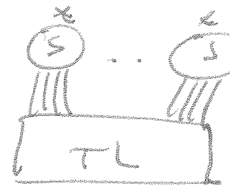
Claim: PA(S) is evaluable.

Jellyfish Algorithm: (Bigelow)

Step ①: pull S's to outside using overbraiding

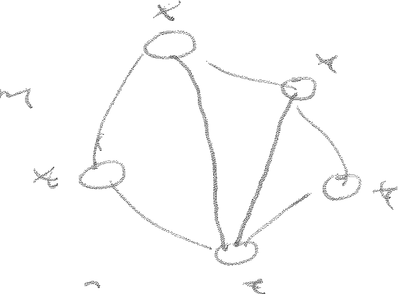


$\rightsquigarrow \sum$

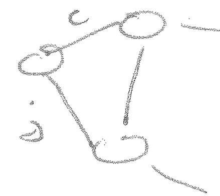


jellyfish form.

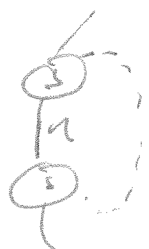
Step 2: take one diagram in j.f. form



- strings give some "triangulation" of polygon.
- look for outermost triangle then i or $j \geq n$



$\rightsquigarrow$



$S^2 = f^{(n)} \in TL$

- reduces # of gen's.

Note: Since $S \in G_0$, this evaluation algorithm is consistent, i.e., independent of any choices.

For H_n: no overbraiding. Have

Step ① of j.f. increases # of gen's!

Open Q: Show above rel'n is inconsistent for $n = 12, 16, \ldots$