Planar algebras in modular tensor categories

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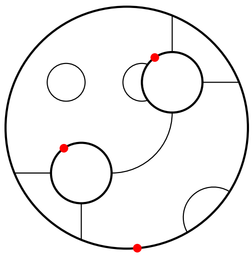
(joint work with André Henriques and James E. Tener)

The motivation for this project is a common generalization of genus zero Segal conformal field theory (CFT) and Jones' planar algebras. In our study of genus zero Segal CFT with topological defect lines, we came across an algebraic structure which generalizes the usual notion of Jones' planar algebras to planar algebras in a modular tensor category.

1. Planar algebras in Vec

Jones' planar algebras [Jon] have proven to be useful in the construction [Pet10, BMPS12] and classification [JMS14] of subfactors. We give a brief definition following [MPS10, BHP12].

Definition 1. A planar algebra is a sequence of vector spaces $\mathcal{P}_\bullet = (\mathcal{P}_n)_{n\geq 0}$ together with an action of the planar operad, i.e., every planar tangle with $k_1, \ldots, k_r$ points on the input disks and k_0 points on the output disk corresponds to a linear map from the unordered tensor product $\mathcal{P}_{k_1} \otimes \cdots \otimes \mathcal{P}_{k_r} \to \mathcal{P}_{k_0}$. For example,

 $: \mathcal{P}_3 \otimes \mathcal{P}_5 \to \mathcal{P}_6.$

This data must satisfy the following axioms:

- isotopy invariance: isotopic tangles produce the same multilinear maps,
- identity: the identity tangle (which only has radial strings and no rotation between marked points) acts as the identity transformation, and
- naturality: gluing tangles corresponds to composing maps. When we glue tangles, we match up the points along the boundary disks making sure the distinguished intervals marked by the distinguished dots align.

The following folklore theorem (needing additional adjectives) has made appearances in various forms in [MPS10, Yam12, BHP12] (see also [Kup96, Jon]).

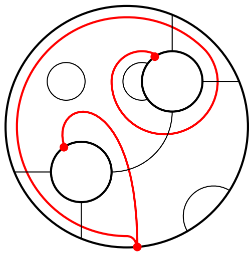
Theorem 2. *Starting with a pair $(\mathcal{C}, X)$ with $\mathcal{C}$ a pivotal category and X a distinguished symmetrically self-dual object, we can construct a canonical planar algebra in* Vec*. Conversely, given a planar algebra in* Vec*, we can construct its pivotal category of projections, where the strand is the distinguished projection.*

These constructions are mutually inverse in the sense that going from pairs to planar algebras back to pairs produces an equivalent pair, and going from planar algebras to pairs back to planar algebras is the identity.

2. Planar algebras in a balanced fusion category $\mathcal{C}$

We now want to relax the condition of working in Vec to working in a given balanced (braided with twists) fusion category $\mathcal{C}$. To define a planar algebra in $\mathcal{C}$, we need additional structure for our planar tangles.

Definition 3. An anchored planar algebra in $\mathcal{C}$ is a sequence of objects $\mathcal{P}_\bullet = (\mathcal{P}_n)_{n\geq 0}$ in $\mathcal{C}$ together with an action of the anchored planar operad, i.e., every anchored planar tangle with $k_1, \dots, k_r$ points on the input disks and k_0 points on the output disk corresponds to a morphism $\mathcal{P}_{k_1} \otimes \cdots \otimes \mathcal{P}_{k_r} \to \mathcal{P}_{k_0}$. For example,



$: \mathcal{P}_5 \otimes \mathcal{P}_3 \to \mathcal{P}_6.$

There is one anchor line for each input disk, which is a homotopy class of paths from each internal marked point to each external marked point. They are transparent to the ordinary strings of the tangle, but they cannot cross each other.

When $\mathcal{C}$ was Vec, our tensor products were unordered, but in $\mathcal{C}$, order matters. The domain of the corresponding morphism is obtained by ordering the input disks counterclockwise according to anchor line entry. In addition to the previous axioms, we also require:

- twist anchor dependence: the n-string 2π rotation gives the map $\theta_{\mathcal{P}_n}$.
- swap anchor dependence: swapping anchor lines induces a braiding in $\mathcal{C}$.

Similar to the classical classification theorem for planar algebras $\mathcal{P}_\bullet$ and pairs $(\mathcal{C}, X)$ of pivotal categories with distinguished objects, we have the following classification result (again, with additional adjectives).

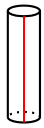
Theorem 4 (Henriques-Penneys-Tener)**.** *Given a balanced fusion category $\mathcal{C}$, a pivotal category $\mathcal{M}$, a braided tensor functor $G : \mathcal{C} \to \mathcal{Z}(\mathcal{M})$, and $m \in \mathcal{M}$ which generates $\mathcal{M}$ as a $\mathcal{C}$-module, there exists a canonical anchored planar algebra $\mathcal{P}_\bullet$ in $\mathcal{C}$. Conversely, we can produce a tuple $(\mathcal{M}, G, m)$ from such an anchored planar algebra in $\mathcal{C}$.*

These constructions are mutually inverse in the sense that going from tuples to anchored planar algebras back to tuples gives an equivalent tuple, and going from anchored planar algebras to tuples back to anchored planar algebras is the identity.

Example 5. One can get examples with $\mathcal{C} \neq$ Vec as follows. Take $\mathcal{C}$ to be a modular category, and choose a commutative algebra object $a \in \mathcal{C}$. Let $\mathcal{M}$ to be the left a-modules in $\mathcal{C}$, and choose an $m \in \mathcal{M}$ which generates $\mathcal{M}$ as a $\mathcal{C}$-module. When $\mathcal{C} = \mathsf{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))$, the commutative algebras correspond to the A_n, D_{2n}, and E_6 and E_8 Coxeter-Dynkin diagrams [KO02]. Then there are two canonical braided, balanced tensor functors $G_\pm : \mathcal{C} \to \mathcal{Z}(\mathcal{M})$ given by α-induction [BEK01].

3. Ingredients of the proof

We begin by considering Walker's unpublished work studying module 2-categories for a braided tensor category $\mathcal{C}$ as functors $G : \mathcal{C} \to \mathcal{Z}(\mathcal{M})$ as a heuristic. Similar to Ostrik's internal hom for fusion categories [Ost03], we found an internal trace functor $\mathrm{Tr}_{\mathcal{C}} : \mathcal{M} \to \mathcal{C}$ as the composite functor $\Phi = G^T \circ I$, where G^T is the left adjoint of G (which exists by semi-simplicity and finiteness conditions), and $I : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ is the induction functor [Müg03]. Diagrammatically, we denote $x \in \mathcal{M}$ as a strand on the plane, and we represent $\mathrm{Tr}_{\mathcal{C}}(x)$ as a strand on a cylinder.

$x =$ $\longmapsto \mathrm{Tr}_{\mathcal{C}}(x) =$ 

We call $\mathrm{Tr}_{\mathcal{C}}$ an internal trace because we have two natural isomorphisms called the 'traciators' $\tau_{\pm} : \mathrm{Tr}_{\mathcal{C}}(x \otimes y) \cong \mathrm{Tr}_{\mathcal{C}}(y \otimes x)$ which categorify the notion of a trace (see the left hand side of Figure 1). The idea behind the traciator is that we may lift a one-click rotation from $\mathcal{M}$ into $\mathcal{C}$ to get an isomorphism. However, if we lift the full 2π rotation, we get a twist rather than the identity.

Another important ingredient is a natural multiplication map $\mu : \mathrm{Tr}_{\mathcal{C}}(x) \otimes \mathrm{Tr}_{\mathcal{C}}(y) \to \mathrm{Tr}_{\mathcal{C}}(x \otimes y)$, which has the properties of an associative multiplication.

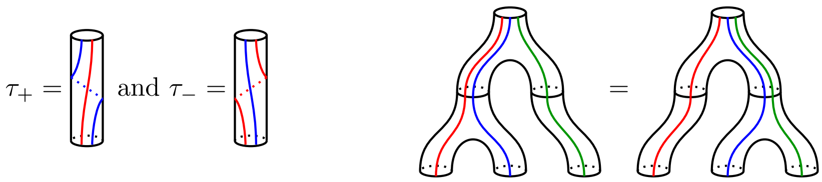
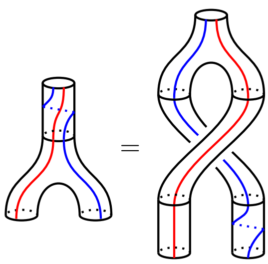


Figure 1. The traciators $\tau_{\pm}$ (left) and associativity of the multiplication map μ (right)

The traciator $\tau_{\pm}$ is compatible with the multiplication μ, and the braiding $\beta_{\pm}$ and twists θ in $\mathcal{C}$. We use our diagrams as heuristics to prove many relations. We give one example below, corresponding to the commutative diagram on the right:



$\longleftrightarrow$

$$\begin{array}{ccc} \mathrm{Tr}_{\mathcal{C}}(x) \otimes \mathrm{Tr}_{\mathcal{C}}(y) & \xrightarrow{\mu} & \mathrm{Tr}_{\mathcal{C}}(x \otimes y) \\ {\scriptstyle \beta_+ \circ (\mathrm{id}_{\mathrm{Tr}_{\mathcal{C}}(x)} \otimes \theta_{\mathrm{Tr}_{\mathcal{C}}(y)})} \downarrow & & \downarrow {\scriptstyle \tau_+} \\ \mathrm{Tr}_{\mathcal{C}}(y) \otimes \mathrm{Tr}_{\mathcal{C}}(x) & \xrightarrow{\mu} & \mathrm{Tr}_{\mathcal{C}}(y \otimes x). \end{array}$$

As we developed anchored planar algebras in balanced fusion categories with a view toward a common generalization of genus zero Segal CFT and Jones' planar algebras, we state another result which will play an important role in the theory.

Proposition 6. *The object* $A = \mathrm{Tr}_{\mathcal{C}}(1_{\mathcal{M}})$ *is a symmetrically self-dual commutative Frobenius algebra object in* $\mathcal{C}$ *with* $\theta_A = 1$.

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