Abstracts

Modular distortion for II_1 multifactor bimodules

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This project started at the 2018 AMS Mathematics Research communities program on Quantum Symmetries: Subfactors and Fusion Categories.

Bimodules over factors and unitary fusion categories. Let A, B be II_1 factors and ${}_AH_B$ an $A-B$ bimodule. We call H *dualizable* if there are maps $\mathrm{ev}_H \in \mathrm{Hom}_{B-B}(\overline{H} \boxtimes_A H \to L^2B)$ and $\mathrm{coev}_H \in \mathrm{Hom}_{A-A}(L^2A \to H \boxtimes_B \overline{H})$ satisfying the zig-zag equations. By [Bis97] (see also [EK98, BDH14]), dualizability is equivalent to H being *bifinite*: $\dim({}_AH)\cdot\dim(H_B) < \infty$, in which case H breaks up as a finite direct sum of simple bimodules. As an example, given a finite index II_1 subfactor, the *state independent* Haagerup L^2 space L^2B [Haa75] is an $A-B$ bimodule. Below, we assume all bimodules are dualizable.

We call ${}_AH_B$ *finite depth* if the unitary multitensor category (semisimple rigid tensor C* category)

$$\mathcal{C} = \mathcal{C}(H) := \begin{pmatrix} {}_A\mathcal{C}_A & {}_A\mathcal{C}_B \\ {}_B\mathcal{C}_A & {}_B\mathcal{C}_B \end{pmatrix} \subset \mathsf{Bim}(A \oplus B)$$

generated by H under $\boxtimes, \oplus, \subseteq, \overline{\cdot}$ is *multifusion* in the sense of [EGNO15].

Definition 1. The *modular distortion* of ${}_AH_B$ is

$$\delta = \delta(H) := \left(\frac{\dim({}_AH)}{\dim(H_B)}\right)^{1/2} \in \mathbb{R}_{>0}.$$

We say ${}_AH_B$ has *constant distortion* if for all sub-bimodules ${}_AK_B \subseteq {}_AH_B$, $\delta(K) = \delta(H)$. We call ${}_AH_B$ *extremal* if ${}_AH_B$ has constant distortion $\delta = 1$.

One can view the modular distortion as an analog of the modular function on a locally compact group, i.e., the ratio of left to right Haar measure.

Remark 2. The set of modular distortions of invertible $A-A$ bimodules is the *fundamental group* of A.

Given a unitary tensor category $\mathcal{C}$ and a group G, a G-*grading* on $\mathcal{C}$ is a decomposition $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ such that $\otimes : \mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh}$. There is a finest grading called the universal grading group $\mathcal{U}_\mathcal{C}$ [EGNO15]. For a II_1 factor, we denote the universal grading group of the dualizable bimodules $\mathsf{Bim}_\mathsf{d}(A)$ by $\mathcal{U}_A$.

Question 3. *What is $\mathcal{U}_R$ where R is the hyperfinite II_1 factor?*

Observe that δ gives a multiplicative map from the simple dualizable $A-A$ bimodules to $\mathbb{R}_{>0}$, which gives a group homomorphism $\delta : \mathcal{U}_A \to \mathbb{R}_{>0}$. Using this, we have an extremely quick proof of the following folklore result.

Proposition 4 (Folklore, [EK98]). *If ${}_AH_A$ is finite depth, then ${}_AH_A$ is extremal.*

Proof. Since $\mathcal{C}(H)$ is fusion, $\mathcal{U}_A$ is finite. Hence $\delta(\mathcal{U}_A) \subset \mathbb{R}_{>0}$ is a compact group, so it must be $\{1\}$. □

By [Pop90], a finite depth hyperfinite II_1 subfactor $A \subset B$ is completely determined by its standard invariant $\mathcal{C}({}_AL^2B_B)$. As a corollary, every unitary fusion category $\mathcal{C}$ admits an essentially unique embedding $\mathcal{C} \hookrightarrow \mathsf{Bim}(R)$, and every embedding is realized by a II_1 subfactor. [FR13, Izu17].

Bimodules over multifactors and unitary multifusion categories. Inspired by our investigation of *bicommutant categories* [HP17], we would like to extend this result to $n \times n$ unitary multifusion categories $\mathcal{C}$. Here, $n \times n$ means $\mathcal{C}$ is indecomposable and $\dim(\mathrm{End}(1_{\mathcal{C}})) = n$, so we can orthogonally decompose $1_{\mathcal{C}} = \bigoplus_{i=1}^n 1_i$ into n simples, and $\mathcal{C} = (\mathcal{C}_{ij})_{i,j=1}^n$ where $\mathcal{C}_{i,j} = 1_i \otimes \mathcal{C} \otimes 1_j$.

We observe that an $n \times n$ multifusion category is faithfully graded by the *groupoid* $\mathcal{G}_n$ with n objects and a unique isomorphism between any two objects. Only thinking about the arrows of the groupoid, an operator algebraist may prefer to think of $\mathcal{G}_n$ as a system of matrix units for $M_n(\mathbb{C})$.

One can already see there will be a slight difference for embeddings of 2×2 unitary multifusion categories.

Proposition 5. *Any 2×2 unitary multifusion category admits an essentially unique embedding $\mathcal{C} \hookrightarrow \mathsf{Bim}(R^{\oplus 2})$ up to the modular distortion on $\mathcal{C}_{12}$.*

All distortions can arise from embeddings. However, not all embeddings arise from subfactors $A \subseteq B$ where $\mathcal{C} \hookrightarrow \mathsf{Bim}(A \oplus B)$, as we *always* have $\delta({}_AL^2B_B) = [B : A]^{1/2}$, and the indices of possible subfactors realizing a 2×2 unitary multifusion category will be a discrete subset of $\mathbb{R}_{>0}$ in some interval above 1.

Example 6. Given any projection $p \in P(R)$ with $\mathrm{tr}(p) \in (0, 1]$, we have an embedding

$$\mathsf{Mat}_2(\mathsf{Hilb}_{\mathsf{fd}}) \hookrightarrow \mathsf{Bim}(R \oplus pRp) \qquad \begin{pmatrix} L^2R & L^2Rp \\ pL^2R & pL^2Rp \end{pmatrix}$$

Observe that $\delta(L^2Rp) = \mathrm{tr}(p)^{-1}$ which can take any value in $[1, \infty)$.

In order to embed multifusion categories, we must use II_1 *multifactors*, which are finite direct sums of II_1 factors. Below, A and B will denote multifactors where $A = \bigoplus_{i=1}^a A_i$ and $B = \bigoplus_{j=1}^b B_j$, where $Z(A) = \mathrm{span}_{\mathbb{C}}\{p_i\}_{i=1}^a$ with $A_i = p_iA$ and $Z(B) = \mathrm{span}_{\mathbb{C}}\{q_j\}_{j=1}^b$ with $B_j = q_jB$.

A II_1 multifactor bimodule ${}_AH_B$ is dualizable if and only if $H_{ij} := p_iHq_j$ is bifinite for all i, j. Again, we will only consider dualizable bimodules. We will also restrict our attention to *connected* bimodules, i.e., those which satisfy $Z(A) \cap Z(B) \cap B(H) = \mathbb{C}1_H$. The definition of finite depth is the same as above for multifactor bimodules.

Definition 7. The *modular distortion* of ${}_AH_B$ is a partially defined matrix $\delta = \delta(H) \in M_{a\times b}(\mathbb{R}_{>0})$ where $\delta_{ij} = \delta(H_{ij})$ when $H_{ij} \neq 0$. We say ${}_AH_B$ is *extremal* if every $A_i - A_i$ bimodule generated by H in $\mathcal{C}(H)$ is extremal.

Using the fact that a unitary multitensor category has a *universal grading groupoid* $\mathcal{U}_\mathcal{C}$ [Pen18], a similar proof as in Proposition 4 above shows that finite depth implies extremal for multifactor bimodules.

Theorem 8. *The following are equivalent for a multifactor bimodule* ${}_AH_B$*.*

- H *is extremal.*
- H_{ij} *has constant distortion for each* i, j*, and* (δ_{ij}) *extends to a well-defined groupoid homomorphism* $\mathcal{G}_{a+b} \to \mathbb{R}_{>0}$*, i.e.,*

$$\delta_{ij}\delta_{i'j'} = \delta_{ij'}\delta_{i'j} \qquad \forall\, 1 \leq i \leq a \text{ and } \forall\, 1 \leq j \leq b.$$

The analog of Popa's uniqueness theorem for finite depth connected II_1 multifactor inclusions only holds under the additional assumption that the two inclusions have identical distortions.

Example 9 ([Pop95b])**.** Consider the inclusion $P = \mathbb{C} \oplus \mathbb{C} \subset M_2(\mathbb{C}) \oplus \mathbb{C} = Q$ whose bipartite adjacency matrix is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

where the rows are indexed by i and the columns by j. The inclusion $A = P \otimes R \subset Q \otimes R = B$ does not admit any downward Jones basic construction [Jon83]. Taking the next two steps in the Jones tower $A_0 \subset A_1 \subset A_2 \subset A_3$, we get a Morita equivalent inclusion $A_2 \subset A_3$ with the same standard invariant which manifestly admits two downward basic constructions. One quickly observes these inclusions have different distortions:

$$\delta({}_{A_0}L^2A_{1A_1}) = \begin{pmatrix} 1 & 3/2 \\ 2 & 3 \end{pmatrix} \qquad\qquad \delta({}_{A_2}L^2A_{3A_3}) = \begin{pmatrix} 5/2 & 3/2 \\ 5/3 & 1 \end{pmatrix}.$$

One calculates that

$$\delta({}_{A_{2n}}L^2A_{2n+1A_{2n+1}}) \xrightarrow{n\to\infty} \begin{pmatrix} \phi^2 & \phi \\ \phi & 1 \end{pmatrix}$$

where ϕ is the golden ratio.

We calculate general formulas for the behavior of the distortion under Morita equivalence and taking basic constructions using some results from [GdlHJ89]. An inclusion $A \subset B$ admits an infinite Jone tunnel if and only if the distortion is *standard*. This condition is calculated from matrix (D_{ij}) of statistical dimensions of $(L^2B)_{ij}$. We show this is equivalent to Popa's *homogeneity* criterion [Pop95b] when we endow B with the unique Markov trace, and with Giorgetti-Longo's notion of *super-extremality* [GL19]. Using techniques from [Ocn88] and [Pop90], we prove the following.

Theorem 10. *An* $n \times n$ *unitary multifusion category admits an essentially unique embedding* $\mathcal{C} \hookrightarrow \mathsf{Bim}(R^{\oplus n})$ *up to the modular distortion.*

Again, not all embeddings are realized from multifactor inclusions, and we have explicit formulas to determine which distortions arise from inclusions.

Remark 11. At this workshop, we learned of the result [Tom18] which could also be used to prove the uniqueness part of the above results.

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