Abstracts

Holography for bulk-boundary local topological order

David Penneys

(joint work with Corey Jones, Pieter Naaijkens, and Daniel Wallick)

This talk was based on two articles:

- *Local topological order and boundary algebras* by Corey Jones, Pieter Naaijkens, David Penneys, and Daniel Wallick [JNPW23], and
- *Holography for bulk-boundary local topological order* by Corey Jones, Pieter Naaijkens, and David Penneys [JNP25]

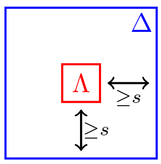
Local topological order axioms. We begin with an *abstract quantum system* on a lattice $\mathcal{L}$, typically assumed to be $\mathbb{Z}^d$. We have a single unital C*-algebra A called the *quasi-local algebra*, and to each d-dimensional rectangle $\Lambda \subset \mathcal{L}$, we have a unital C*-algebra $A(\Lambda) \subset A$ satisfying the following axioms:

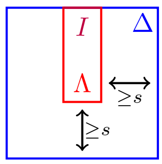
- $A(\emptyset) = \mathbb{C}1_A$,
- $\Lambda \subset \Delta$ implies $A(\Lambda) \subset A(\Delta)$,
- $\Lambda \cap \Delta = \emptyset$ implies $[A(\Lambda), A(\Delta)] = 0$ in A, and
- $\bigcup_\Lambda A(\Lambda)$ is norm-dense in A.

This net of algebras is equipped with a *net of projections* $p_\Lambda \in A(\Lambda)$ for each Λ satisfying $p_\Delta \leq p_\Lambda$ whenever $\Lambda \subset \Delta$; these projections are used instead of any local Hamiltonian.

We give a rough sketch of the *local topological order* (LTO) axioms for (A, p); for the precise version, see [JNPW23].

- Whenever $\Lambda \ll \Delta$ (Δ sufficiently completely surrounds Λ), $p_\Delta A(\Lambda) p_\Delta = \mathbb{C}p_\Delta$, and
- Whenever $\Lambda \Subset \Delta$ (Δ sufficiently completely surrounds Λ on all but one side), there is an algebra $B(I)$ where $I = \partial\Lambda \cap \partial\Delta$ which is supported on sites near I such that $p_\Delta A(\Lambda) p_\Delta = B(I) p_\Delta$. This algebra $B(I)$ is independent of $\Lambda \subset \Delta$ beyond that $I = \partial\Lambda \cap \partial\Delta$.

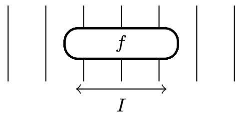
$\Lambda \ll \Delta$: 

$\Lambda \Subset \Delta$: 

One gets a canonical pure state on A by the formula $p_\Delta x p_\Delta = \psi(x) p_\Delta$ for $x \in A(\Lambda)$ and $\Lambda \ll \Delta$. Choosing a half-space $\mathbb{H} \subset \mathcal{L}$ and setting $A(\mathbb{H}) = \varinjlim_{\Lambda \subset \mathbb{H}} A(\Lambda)$, we get a net of boundary algebras $B = \varinjlim B(I)$ on $\partial\mathbb{H}$, together with a quantum channel $\mathbb{E} : A(\mathbb{H}) \to B$ defined by $p_\Delta x p_\Delta = \mathbb{E}(x) p_\Delta$ for $x \in A(\Lambda)$ and $\Lambda \Subset \Delta \subset \mathbb{H}$ with $\partial\Lambda \cap \partial\Delta = I$. One should think of this boundary algebra B as living on a *physical cut/boundary* of our abstract quantum spin system.

Examples of LTOs include almost all known topologically ordered commuting projector lattice models, including Kitaev's Toric Code and Quantum Double [Kit97, Kit03], the Levin-Wen model [LW05], and the Walker-Wang model [WW12]. In these cases, the boundary algebra can be described by a *fusion spin chain*, which is an abstract 1D spin chain built from a unitary fusion category (UFC) $\mathcal{C}$ and a strong tensor generator $X \in \mathcal{C}$. The local algebras are

$$B(I) \cong \mathrm{End}_{\mathcal{C}}(X^{\otimes I})$$

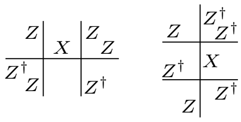


and inclusion is given by tensoring with copies of id_X as appropriate. One the recovers the bulk topological order by looking at C. Jones' *DHR bimodules* for this 1D fusion spin system [Jon24].

For the Walker-Wang model built from a unitary braided fusion category $\mathcal{B}$, the boundary algebra is a 2D net of algebras built from $\mathcal{B}$ and our strong tensor generator $X \in \mathcal{B}$ called a *braided fusion spin system.* Since $\mathcal{B}$ is braided, it makes sense to take a tensor product of objects at points in a 2D plane, so for each 2D rectangle I, we can define $\mathrm{End}_{\mathcal{B}}(X^{\otimes I})$. As an aside, we remark that this net of algebras has a canonical state corresponding to tensor powers of the map $1_{\mathcal{B}} \to X$, and in this state, the category of superselection sectors of this net of von Neumann algebras following [BBC+25] is equivalent to the completion (in the sense of [HNP24]) to $\mathcal{B}$ as a W*-category. We make the following conjectures, the second pointed out to us by C. Jones.

Conjecture 1. *The superselection sectors for this net is equivalent to the completion of $\mathcal{B}$ as a* W* *braided tensor category.*

Conjecture 2. *The braided fusion spin system for $SU(3)_1$ (or possibly its reverse) is bounded-spread isomorphic to a net of algebras from* [Haa23, Fig. 1] *discussed by Jeongwan Haah in his Oberwolfach talk given just prior to this talk, i.e., the net of algebras with local generators*

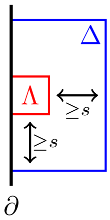


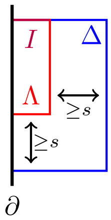
$$, \qquad X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \; \zeta = \exp(2\pi i/3)$$

on a 2D edge lattice with $\mathbb{C}^3$ spins on each edge.

Introducing a topological boundary. One can now introduce a *topological boundary* for our lattice; the term 'topological' here is used to distinguish this boundary from the 'physical' boundary cut obtained from choosing a half-space. The LTO axioms in the presence of a topological boundary are almost identical to the previous LTO axioms. Roughly speaking (for a precise version see [JNP25]),

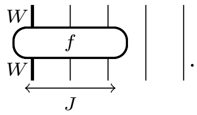
- Whenever $\Lambda \ll^{\partial} \Delta$, $p_\Delta A(\Lambda) p_\Delta = \mathbb{C} p_\Delta$, and
- Whenever $\Lambda \Subset^{\partial} \Delta$, there is an algebra $B^{\partial}(I)$ similar to $B(I)$ such that $p_\Delta A(\Lambda) p_\Delta = B^{\partial}(I) p_\Delta$.

$\Lambda \ll^{\partial} \Delta :$ 

$\Lambda \Subset^{\partial} \Delta :$ 

Again, one gets a net of algebras on the physical cut/boundary at the edge of a chosen half-space. Following [KK12], one gets topological boundaries for the 2D LTOs from our models above via module categories $\mathcal{M}$ for the corresponding UFCs $\mathcal{C}$, together with a choice of strong module generator $W \in \mathcal{M}$. For these topological boundaries, the boundary algebra is the a *fusion module spin chain*, where sites meeting the topological boundary give rise to the local algebras

$$B^{\partial}(J) \cong \operatorname{End}_{\mathcal{M}}(W \lhd X^{\otimes J-1})$$



We then define a notion of boundary DHR-bimodule meant to capture the topological boundary excitations. We use subfactor theory to prove that that for these fusion module spin chains, the boundary DHR bimodules give exactly the dual category $\operatorname{End}(\mathcal{M}_{\mathcal{C}})$.

For our 3D Walker-Wang model from a UBFC $\mathcal{B}$, we get a 2D topological boundary from a untiary module tensor category [HPT16], i.e., a UFC $\mathcal{C}$ equipped with a unitary braided central functor $\mathcal{B} \to Z(\mathcal{C})$. Indeed, the half-braiding for $\mathcal{B}$ with $\mathcal{C}$ is exactly the data needed to attach the 3D $\mathcal{B}$-Walker-Wang model to the 2D $\mathcal{C}$-Levin-Wen model [HBJP23, GHK+24]. For such a 2D topological boundary, the category of boundary DHR bimodules has a canonical braiding as in [Jon24]. We use a folding trick and our 2D result for fusion module spin chains to prove that the 2D topological boundary excitations is equivalent to the *enriched center/Müger centralizer* $Z^{\mathcal{B}}(\mathcal{C}) = \mathcal{B}' \subset Z(\mathcal{C})$ [KZ18]. Using a 3D folding trick for our original 3D Walker-Wang model for $\mathcal{B}$, we get a 2D topological boundary labeled by $\mathcal{B}$ for the 3D bulk labelled by $\mathcal{B} \boxtimes \mathcal{B}^{\text{rev}}$. Applying the above result, we see that the ordinary DHR bimodules for the 3D model is given by $Z^{\mathcal{B}\boxtimes\mathcal{B}^{\text{rev}}}(\mathcal{B}) \cong Z_2(\mathcal{B})$, the Müger center of $\mathcal{B}$.

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