Discrete subfactors, realization of algebra objects, and Q-system completion

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QGS: Quantum Groups Seminar

January 11, 2021

The Ohio State University

IPAM: Actions of tensor categories on C^*-algebras

Today, C^*-algebras stand at an analogous stage to vNAs in the '80s when Jones pioneered subfactor theory. This virtual workshop will bring together researchers at the interface of structure and classification of C^*-algebras and subfactor theory/tensor categories to set foundations for actions of tensor categories on C^*-algebras.

- Thurs 21 (8am - 11:30 am), Fri 22 Jan (8am - 10:30am): Expository overview talks by Courtney, Carrion, Szabo, Vaes, Yamashita on classification of simple nuclear C^*-algebras; group actions on the hyperfinite II_1 factor and on classifiable C^*-algebras; tensor categories associated to subfactors.
- Mon 25 Jan - Thurs 28 Jan (8am - 11am): Research talks, discussion of overview talks, ask expert sessions.

Participation is open to all. Current speakers and registration information can be found here:
`http://www.ipam.ucla.edu/programs/workshops/actions-of-tensor-categories-on-c-algebras/?tab=overview`

Overview

- Unitary tensor categories (UTCs) encode quantum symmetry and act on operator algebras via unitary tensor functors

$$\mathbf{H} : \mathcal{C} \to \mathsf{Bim}(A) = \mathrm{End}(\mathsf{Mod}(A))$$

- A (...) subfactor $N \subset M$ can be viewed a triple $(\mathcal{C}, \mathbf{H}, \mathbf{A})$ where $\mathcal{C}$ is a UTC, $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}(N)$ is an action, and $A \in \mathcal{C}$ is an (...) algebra object.

$$N \subset N \rtimes_{\mathbf{H}} A = M$$

- Q-systems in UTCs are particularly nice algebra objects where the above construction is easy. They are *higher idempotents*, and we can take a *higher idempotent completion*.

$$\begin{tikzcd} & \mathsf{QSys}(\mathcal{C}) \arrow[d, "\exists !"] \\ \mathcal{C} \arrow[ur, hook] \arrow[r] & \mathcal{D} \end{tikzcd}$$

Unitary (multi)tensor categories

A *unitary multitensor category* is a semisimple tensor C^* category

- (linear) hom spaces $\mathcal{C}(a \to b)$ finite dimensional vector spaces
- (Cauchy complete) admits finite direct sums, and all idempotents split
- (C^*) For all $a, b \in \mathcal{C}$, $\dagger : \mathcal{C}(a \to b) \to \mathcal{C}(b \to a)$ such that $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ and $f^{\dagger\dagger} = f$, and all endomorphism algebras are C^* algebras under $\dagger$.
- (tensor) $\dagger$-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ ($(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$) with *unitary* ($u^{-1} = u^\dagger$) coherence isomorphisms α, λ, ρ
- (rigid) every object admits left and right duals

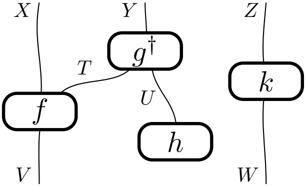
We call $\mathcal{C}$ a *unitary tensor category* if the unit is simple.
We call $\mathcal{C}$ a *unitary (multi)fusion category* if there are only finitely many isomorphism classes of simple objects.

Fact

Every UMC is *semisimple*, i.e., every object is a finite direct sum of simples ($\mathrm{End}_{\mathcal{C}}(c) = \mathbb{C}$)

2D graphical calculus for UMCs

0. Objects denoted by labelled strands, oriented bottom to top.
1. 1-morphisms denoted by coupons



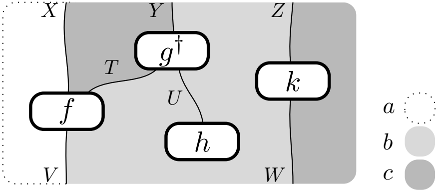
$$\begin{aligned} f &: V \to X \otimes T \\ g &: Y \to T \otimes U \\ h &: 1_{\mathcal{C}} \to U \\ k &: W \to Z \end{aligned}$$

- vertical stacking is composition
- horizontal juxtaposition is $\otimes$
- vertical reflection is $\dagger$
- suppress unit $1_{\mathcal{C}}$ and all coheretors α, λ, ρ

2D graphical calculus for $\mathrm{C}^*/\mathrm{W}^*$ 2-categories

A tensor category is a 2-category with one object. For 2-categories, we have a dimension shift.

0. shadings for regions to denote objects
1. 1-morphisms denoted by strands
2. 2-morphisms denoted by coupons



$$
\begin{aligned}
T &: c \to b \\
U &: b \to b \\
V &: a \to b \\
W &: b \to c \\
X &: a \to c \\
Y &: c \to b \\
Z &: b \to c
\end{aligned}
\qquad
\begin{aligned}
f &: V \Rightarrow X \otimes T \\
g &: Y \Rightarrow T \otimes U \\
h &: 1_c \Rightarrow U \\
k &: W \Rightarrow Z
\end{aligned}
$$

Where do UTCs come from?

1. Subfactor standard invariants $A \subset B \rightsquigarrow \mathcal{C}(A \subset B)$
2. Compact groups $G \rightsquigarrow \mathsf{Rep}(G)$
3. Discrete/compact quantum groups (Tannaka-Krein duality)

$$\mathbb{G} \qquad \rightsquigarrow \qquad (\mathsf{Rep}(\mathbb{G}), \mathbf{F} : \mathsf{Rep}(\mathbb{G}) \to \mathsf{Hilb})$$

4. Generators and relations [VV19]
5. Constructions of new UTCs from existing UTCs

Many people care about UTCs because of physics

- conformal field theory ($\mathsf{Rep}(\mathcal{A})$ of a conformal net)
- unitary fusion categories give Turaev-Viro TQFTs
- unitary modular categories give Reshetikhin-Turaev TQFTs
- topological phases of matter (UMTCs)

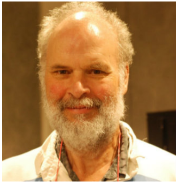
Subfactors

- A II_1 *factor* is an infinite dimensional von Neumann algebra with trivial center and a trace. (Eg: $L\Gamma := \mathbb{C}[\Gamma]'' \subset B(\ell^2\Gamma)$)
- A II_1 *subfactor* is a unital inclusion of type II_1 factors.

Jones' Index Rigidity Theorem [Jon83]

The index $[B : A] := \dim({}_A L^2 B)$ of a II_1 subfactor $A \subset B$ takes values in:

$$[B : A] \in \left\{4\cos^2(\pi/n)\middle| n \geq 3\right\} \cup [4, \infty].$$



Example

Given a finite index II_1 subfactor $A \subset B$, the UTC ${}_A\mathcal{C}_A$ is the category of $A - A$ bimodules generated by L^2B under

- $\oplus$ direct sum
- $\boxtimes$ Connes' fusion relative tensor product over A
- $\subseteq$ sub-bimodules
- $\overline{\cdot}$ conjugates

The standard invariant

Definition

The *standard invariant* of $A \subset B$ is the collection of all $A - A$, $A - B$, $B - B$, and $B - A$ bimodules generated by $L^2 B$ under

- $\oplus$ direct sum
- $\boxtimes$ Connes' fusion relative tensor product (over A or B)
- $\subseteq$ sub-bimodules
- $\overline{\cdot}$ conjugates.

We can think of this as a 2×2 UMC of bimodules of $A \oplus B$

$$\mathcal{C} = \mathcal{C}(A \subset B) := \begin{pmatrix} {}_A\mathcal{C}_A & {}_A\mathcal{C}_B \\ {}_B\mathcal{C}_A & {}_B\mathcal{C}_B \end{pmatrix} \subset \mathsf{Bim}(A \oplus B)$$

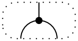
with the *generator* ${}_A L^2 B_B$.

- If there are only finitely many isomorphism classes of simple bimodules, we call $A \subset B$ and $\mathcal{C}$ *finite depth*.

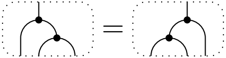
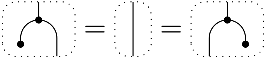
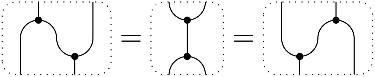
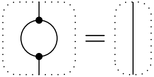
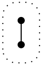
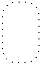
Alternate definition via Q-systems

Alternative Definition

Alternatively, we can define the standard invariant as the UTC ${}_A\mathcal{C}_A$ of $A - A$ bimodules generated by $L^2 B$ with the *Q-system* ${}_A L^2 B_A$.

 $: L^2B \boxtimes L^2B \xrightarrow{\text{multiplication}} L^2B$

 $: L^2A \xrightarrow{\text{unit}} L^2B$

- (associative) 
- (unital) 
- (Frobenius) 
- (separable) 
- (minimal/standard)  $= \dim_{\min}(Q)$ 

Classification of subfactors/UTCs

Example

The subfactor $R \subset R \rtimes G$ for a finite group G 'remembers' G. So classifying hyperfinite subfactors is hopeless. We must restrict to some notion of 'smallness.'

Strategy for small index classification:

1. Classify possible standard invariants with $\dim({}_A L^2 B_B)$ small
2. Determine how many subfactors give each standard invariant.

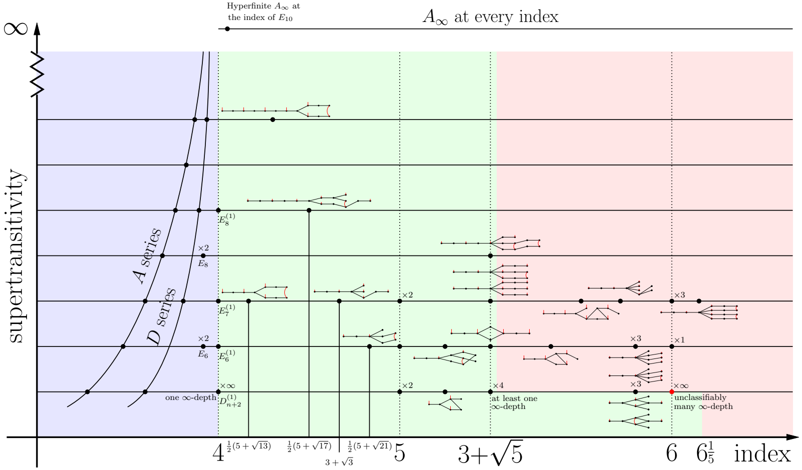
Popa's Subfactor Reconstruction Theorem [Pop90, Pop95]

Every standard invariant comes from a subfactor. If the standard invariant is *strongly amenable* (eg: finite depth), the subfactor can be taken to be hyperfinite.

Theorem [BHP12], cf. [PS03]

Every UTC admits a fully faithful embedding into $\mathsf{Bim}_{\mathsf{ext}}(L\mathbb{F}_\infty)$.

Known small index standard invariants



Theorem [AMP15, Liu15]

We know all standard invariants up to index $5\frac{1}{4} > 3+\sqrt{5}$, the first interesting composite index.

Amenability

Amenability arises in *two places* when subfactors can be classified:

1. We restrict to subfactors of the amenable II_1 factor R
2. We embed amenable unitary tensor categories into $\mathrm{Bim}(R)$.

Question

How many ways can $\mathrm{Ad}(A_3 * A_4)$ embed into $\mathrm{Bim}(R)$?

Question

How many ways can $TLJ(d)$ embed into $\mathrm{Bim}(R)$ for $d > 2$?

Beyond small index classification

What new directions can we go in?

- Infinite index
- Horizontal categorification
- Vertical categorification
- Ask higher categorical questions in this context
- Actions of unitary tensor categories on C^*-algebras

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- Ask higher categorical questions in this context
  - Q-system completion is a higher idempotent completion
- Actions of unitary tensor categories on C^*-algebras
  - Use Q-system completion to induce new actions from existing actions

Discrete subfactors

With Corey Jones [JP19], we characterize the class of extremal irreducible discrete subfactors $(N \subset M, E)$ with N type II_1 with trace τ and $E : M \to N$ a f.n. conditional expectation.

- (discrete) Setting $\phi := \tau \circ E$, ${}_N L^2(M, \phi)_N$ decomposes as a direct sum of dualizable $N - N$ bimodules (generates a UTC!)
- (irreducible) $N' \cap M = \mathrm{End}_{N-M}(L^2(M, \phi)) = \mathbb{C}$
- (extremal) For every $N - N$ sub-bimodule ${}_N K_N \subset {}_N L^2(M, \phi)_N$, $\dim({}_N K) = \dim(K_N)$.

Examples

- Any finite depth, finite index irreducible II_1 subfactor is automatically extremal and discrete.
- If $\alpha : \Gamma \curvearrowright N$ is an outer action of a discrete countable group, then $N \subset N \rtimes_\alpha \Gamma$ is an extremal irreducible discrete subfactor.

Characterization of discrete subfactors

Such a subfactor $(N \subset M, E)$ can be viewed as a triple $(\mathcal{C}, \mathbf{A}, \mathbf{H})$:

1. Unitary tensor category $\mathcal{C}$,
2. Connected W^* algebra object $\mathbf{A} \in \mathsf{Vec}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}^{\mathsf{op}} \to \mathsf{Vec})$ ($\mathsf{Vec}(\mathcal{C})$ is a model for $\mathrm{ind}(\mathcal{C}^\natural)$, where $\natural$ means forget $\dagger$),
3. Fully faithful unitary tensor functor $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}_{\mathsf{ext}}(N)$ which lands in extremal $N - N$ bimodules.

The *standard invariant* of $(N \subset M, E)$ is the pair $(\mathcal{C}, \mathbf{A})$.

W^* algebra objects

Definition

A connected W^* algebra object $\mathbf{A} = \underline{\mathrm{End}}_{\mathcal{C}}(m)$ for some simple object m in some $\mathcal{C}$-module $\mathrm{C}^*/\mathrm{W}^*$-category ${}_{\mathcal{C}}\mathcal{M}$.

$$\mathbf{A}(c) := \mathcal{M}(c \triangleright m \to m) \in \mathsf{Vec}$$

Example

For an irreducible extremal discrete subfactor $(N \subset M, E)$ and $K \in \mathcal{C} = \langle {}_N L^2(M, \phi)_N \rangle$,

$$\begin{aligned}\mathbf{A}(K) :&= \mathrm{Hom}_{N-N}(K \to L^2(M, \phi)) \\ &\cong \mathrm{Hom}_{N-M}(K \boxtimes_N L^2(M, \phi) \to L^2(M, \phi)).\end{aligned}$$

Theorem [JP19]

Fix a unitary tensor category $\mathcal{C}$ and a fully faithful unitary tensor functor $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}_{\mathsf{ext}}(N)$ where N is a II_1 factor. There is an equivalence of categories

$$\left\{ \begin{array}{l} \text{Connected W}^* \text{ algebra} \\ \text{objects } \mathbf{A} \in \mathsf{Vec}(\mathcal{C}) \\ \text{with ucp morphisms} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Extremal irreducible discrete inclu-} \\ \text{sions } (N \subseteq M, E) \text{ supported on} \\ \mathbf{H}(\mathcal{C}) \text{ with normal } N-N \text{ bilinear} \\ \text{ucp maps preserving } \tau \circ E \end{array} \right\}$$

- This effectively splits subfactor classification into 2 parts:
  1. Classify embeddings of unitary tensor categories $\mathbf{H} : \mathcal{C} \to \mathsf{Bim}(N)$
  2. Classify connected W^* algebra objects in $\mathcal{C}$.

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- Generalizes all known Galois correspondences for intermediate subfactors. (finite groups: [NT60], discrete groups: [ILP98], compact quantum groups: [Tom09])

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- Generalizes all known Galois correspondences for intermediate subfactors. (finite groups: [NT60], discrete groups: [ILP98], compact quantum groups: [Tom09])
- Gives well-behaved notion of standard invariant for a large class of infinite index subfactors.

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- Gives new examples of subfactors from an embedding of $\mathcal{C}$, a $\mathcal{C}$-module $\mathrm{C}^*/\mathrm{W}^*$-category $\mathcal{M}$, and a simple object $m \in \mathcal{M}$.

Example

$\mathbf{F} : \mathcal{C} \to \mathsf{Hilb}$ a fiber functor (discrete quantum group) and $m = \mathbb{C}$. M is type II_1 iff $(\mathcal{C}, \mathbf{F})$ is Kac-type; otherwise M is type III!

Example: quantum homogeneous spaces

Recall Tannaka-Krein duality for compact quantum groups:

$$\mathbb{G} \qquad \longleftrightarrow \qquad (\mathrm{Rep}(\mathbb{G}), \mathbf{F} : \mathrm{Rep}(\mathbb{G}) \to \mathrm{Hilb}_{\mathrm{fd}})$$

The articles [DCY13, DCY15] give a Tannaka-Krein duality for *quantum homogeneous spaces*, which is a coaction

$$\alpha : C(\mathbb{X}) \to C(\mathbb{X}) \otimes C(\mathbb{G})$$

satisfying certain properties.

[DCY13, Thm. 6.4]

There is a one-to-one correspondence between

1. Morita classes of ergodic quantum homogeneous spaces for $\mathbb{G}$
2. Connected module C^* categories for $\mathrm{Rep}(\mathbb{G})$

Quantum homogeneous spaces give C^* algebra objects

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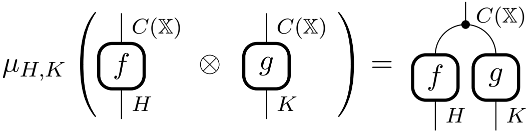
Quantum homogeneous spaces give lax monoidal functors [PR08].

$$C(\mathbb{X}) \qquad \longmapsto \qquad \big(\mathrm{Hom}_{\mathbb{G}}(- \to C(\mathbb{X})) : \mathsf{Rep}(\mathbb{G}) \to \mathsf{Vec}_{\mathsf{fd}}\big)$$



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A lax monoidal functor gives an algebra object in $\mathsf{Vec}(\mathsf{Rep}(\mathbb{G}))$.
It is C^* by [DCY13, Lem. 6.1]:

$$\mathbf{A}(K) := \mathrm{Hom}_{\mathbb{G}}(K \to C(\mathbb{X})) \cong \mathrm{Hom}_{\mathbb{G}-C(\mathbb{X})}(K \otimes C(\mathbb{X}) \to C(\mathbb{X}))$$

so $\mathbf{A} \cong \underline{\mathrm{End}}_{\mathsf{Rep}(\mathbb{G})}(C(\mathbb{X}))$.

(If $A \in \mathcal{C}$ and $m \in \mathcal{C}(A \otimes A \to A)$, $\mathcal{C}(c \to A) \cong \mathcal{C}_A(c \otimes A \to A)$)

Realization [JP19]

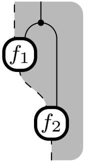
The main tool we provide is *realization*. Given $(\mathcal{C}, \mathbf{A}, \mathbf{H})$, we reconstruct a subfactor

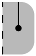
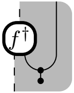
$$N = \underbrace{A(1_{\mathcal{C}})}_{\mathbb{C}} \otimes \underbrace{\mathbf{H}^\circ(1_{\mathcal{C}})}_{N} \subset \overline{\bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} \mathbf{A}(c) \otimes \underbrace{\mathbf{H}^\circ(c)}_{\text{bdd. vects.}}}^{\mathrm{W}^*} =: \begin{cases} N \rtimes_{\mathbf{H}} \mathbf{A} \\ |\mathbf{A}|_{\mathbf{H}} \end{cases}$$

This is much easier when $\mathbf{A}$ is a Q-system in $\mathcal{C}$ rather than a W^*-algebra object in $\mathsf{Vec}(\mathcal{C})$. In this case,

$$|\mathbf{A}|_{\mathbf{H}} = \mathbf{H}(\mathbf{A})^\circ := \mathrm{Hom}_{-N}(L^2N \to \mathbf{H}(\mathbf{A}))$$

is easily equipped with the structure of a unital C^*-algebra which has a predual and is thus a von Neumann algebra:

$f_1 \cdot f_2 :=$ 

, $\qquad 1_{|\mathbf{A}|_{\mathbf{H}}} :=$ 
, $\qquad$ and $\qquad f^* :=$ 

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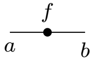
Realization is a $\dagger$-2-functor

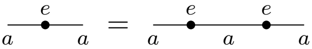
With Quan Chen, Roberto Hernandez Palomares, and Corey Jones,

we extend realization to a $\dagger$-2-functor in the C^* setting (proof also works in W^* setting).

- Given a $\mathrm{C}^*/\mathrm{W}^*$ 2-category $\mathcal{C}$, Q-systems, separable bimodules, and intertwiners in $\mathcal{C}$ form a $\mathrm{C}^*/\mathrm{W}^*$ 2-category $\mathsf{QSys}(\mathcal{C})$.
- Have canonical inclusion $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{QSys}(\mathcal{C})$. $\mathcal{C}$ is *Q-system complete* if $\iota_{\mathcal{C}}$ is a $\dagger$-2-equivalence.
- Realization inverse $\dagger$-2-functor $|\cdot| : \mathsf{QSys}(\mathsf{C^*Corr}) \to \mathsf{C^*Corr}$. $\mathsf{C^*Corr}$ is Q-system complete (as is $\mathsf{W^*Corr} \simeq \mathsf{vNA}$).

Idempotents and condensation

- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

- Starting with $\mathbb{C}$, we can *deloop* to get the category with one object with endomorphisms $\mathbb{C}$. We then Cauchy complete ($\oplus$ and idempotent) to obtain the category $\mathsf{Vec}_{\mathsf{fd}}$
- We can do this process again; starting with $\mathsf{Vec}_{\mathsf{fd}}$, we can deloop to get $\mathsf{Vec}_{\mathsf{fd}}$ as a *tensor* category. We then 'higher' idempotent complete (unital separable algebra object completion) to obtain $2\mathsf{Vec}_{\mathsf{fd}}$, the 2-category of finite dim algebras, finite dim bimodules, and intertwiners.
- The next step yields $3\mathsf{Vec}$, the 3-category of multifusion categories! [GJF19, JF20]

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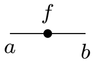
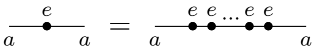
- 1-morphisms in a category $\mathcal{C}$ live on a line. 

- idempotents can replicate freely. 

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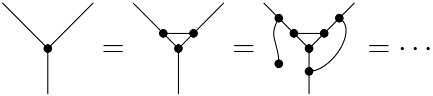
Q-systems are higher categorical idempotents

A Q-system is a unitary (co)unital *higher categorical idempotent*.



Now strands and tri/univalent vertices can replicate freely.

Warning

Unitary condensation (in progress with Reutter and Steinebrunner) is extremely nuanced, and you don't want to use Q-systems!

Definition based on [Yam04, EGNO15, BKLR15, CR16, NY16, DR18, GY20]

The *Q-system completion* $\mathrm{QSys}(\mathcal{C})$ of a $\mathrm{C}^*/\mathrm{W}^*$ 2-category $\mathcal{C}$ has

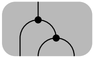
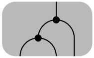
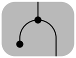
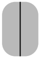
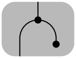
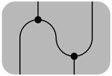
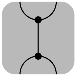
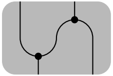
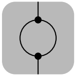
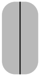
- objects are Q-systems,
- 1-morphisms are unitarily separable bimodules, and
- 2-morphisms are intertwiners.

Q-systems

Recall that a Q-system in a $\mathrm{C}^*/\mathrm{W}^*$ 2-category $\mathcal{C}$ is a 1-morphism $Q \in \mathcal{C}(b \to b)$ together with

 $: Q \otimes_b Q \xrightarrow{\text{multiplication}} Q$  $: 1_b \xrightarrow{\text{unit}} Q$

such that the following relations hold:

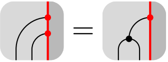
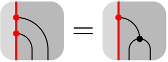
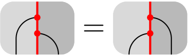
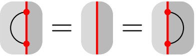
- (associative)  $=$ 
- (unital)  $=$  $=$ 
- (Frobenius)  $=$  $=$ 
- (unitarily separable)  $=$ 
- (standard)  $\in \mathrm{End}_{\mathcal{C}}(1_b)^{\times}$

Frobenius actually follows from associative, unital, and unitarily separable by [LR97]; see [BKLR15, Lem. 3.7].

Unitarily separable bimodules

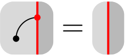
Suppose $P \in \mathcal{C}(a \to a)$, $Q \in \mathcal{C}(b \to b)$ are Q-systems and $X \in \mathcal{C}(a \to b)$.

 $: P \otimes_a X \xrightarrow{\text{left action}} X$  $: X \otimes_b Q \xrightarrow{\text{right action}} X$

- (bimod) , , and 
- (unitarily separable) 

[BKLR15, Lem. 3.23]

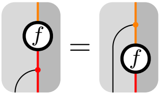
A unitarily separable $P - Q$ bimodule ${}_P X_Q$ over Q-systems P, Q is automatically unital and Frobenius:

- (unital)  and 
- (Frobenius)  and .

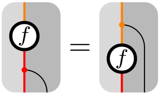
Intertwiners

Definition

If $P \in \mathcal{C}(a \to a)$ and $Q \in \mathcal{C}(b \to b)$ are Q-systems and ${}_aX_b, {}_aY_b \in \mathcal{C}(a \to b)$ are $P - Q$ bimodules, we define QSys$(\mathcal{C})({}_PX_Q \Rightarrow {}_PY_Q)$ as the set of $f \in \mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$ such that



and

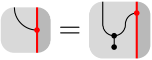


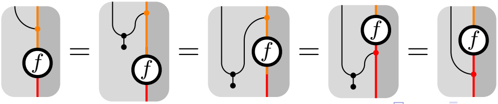
.

Lemma

$f^\dagger \in \mathcal{C}({}_aY_b \Rightarrow {}_aX_b)$ is also a $P - Q$ bimodule map.

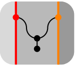
Proof.

Step 1: 

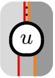
Step 2: Apply $\dagger$ to  . $\square$

Composition of 1-morphisms

To compose the $P - Q$ bimodule ${}_aX_b$ and the $Q - R$ bimodule ${}_bY_c$, we unitarily split the separability projector

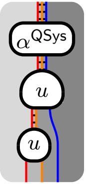
$p_{X.Y} :=$  $:=$  $=$  $= u^{\dagger}_{X,Y} u_{X,Y}$

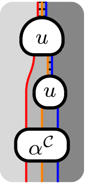
for a coisometry $u_{X,Y}$, unique up to unique unitary.

 $= X \otimes_Q Y$ 

$= u_{X,Y}$.

As in [NY16, Rem. 2.6], associator $\alpha^{\mathsf{QSys}(\mathcal{C})}$ uniquely determined by



$=$ 

$: (X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z \to X \otimes_Q (Y \otimes_R Z).$

Main results

Theorem* cf. [DR18]

QSys is a 3-functor on $\mathrm{C}^*/\mathrm{W}^*$ 2-categories.

Universal property for Q-system completion cf. [DR18]

$$\begin{tikzcd} & \mathsf{QSys}(\mathcal{C}) \arrow[d, "\exists!"] \\ \mathcal{C} \arrow[ur, hook, "\iota_{\mathcal{C}}"] \arrow[r] & \mathcal{D} \end{tikzcd}$$

for every †-2-functor from $\mathcal{C}$ to a Q-system complete $\mathcal{D}$.

Theorem cf. [GY20]

$\mathsf{C^*Corr}, \mathsf{W^*Corr}, \mathsf{vNA}$ are Q-system complete.

Corollary cf. [GY20]

Can induce action $\mathcal{C} \to \mathsf{Bim}(A) \subset \mathsf{R} \in \{\mathsf{C^*Corr}, \mathsf{W^*Corr}, \mathsf{vNA}\}$

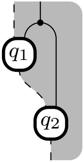
$$\mathsf{QSys}(\mathcal{C}) \to \mathsf{QSys}(\mathsf{Bim}(A)) \to \mathsf{QSys}(\mathsf{R}) \xrightarrow{\cong} \mathsf{R}$$

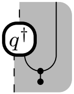
Main idea for C*Corr Q-system complete

Realization $|\cdot| : \mathsf{QSys}(\mathsf{C^*Corr}) \to \mathsf{C^*Corr}$ is inverse to natural inclusion $\iota : \mathsf{C^*Corr} \to \mathsf{QSys}(\mathsf{C^*Corr})$.

Definition

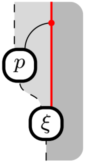
Q-system $Q \in \mathsf{C^*Corr}(B \to B)$ maps to $|Q| := \mathrm{Hom}_{\mathbb{C}-B}(B \to Q)$

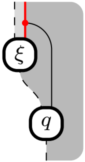
$q_1 \cdot q_2 :=$ 

, $1_{|Q|} :=$ 
, and $q^* :=$ 

.

For $P - Q$ bimod ${}_AX_B$, define $|X| := \mathrm{Hom}_{\mathbb{C}-B}(B \to A \boxtimes_A X)$.

$p \rhd \xi :=$ 

$\xi \lhd q :=$ 

$\forall f \in |P|$,
$\forall \eta \in |M|$, and
$\forall g \in |Q|$.

Induced actions on C^*-algebras

Theorem [Jon20]

Every pointed unitary fusion category $\mathsf{Hilb}_{\mathsf{fd}}(G, \omega)$ admits an action on $C(X)$ where X is some 'nice' compact Hausdorff space (e.g. closed connected n-manifold for $n \geq 3$).

- Can use our results to get actions of group-theoretical unitary fusion categories on continuous trace C^*-algebras.

Thank you for listening!

Slides available at:
`https://people.math.osu.edu/penneys.2/PenneysQGS2021.pdf`

Article in preparation!

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