Subfactors: past, present, and future

Subfactor theory in mathematics and physics
In honor of Vaughan Jones' 60th birthday
In memory of Uffe Haagerup

David Penneys
UCLA

July 17, 2015

Quantum symmetries

A *quantum symmetry* is a non-commutative analog of the representation category of a finite group.

Where do quantum symmetries come from?

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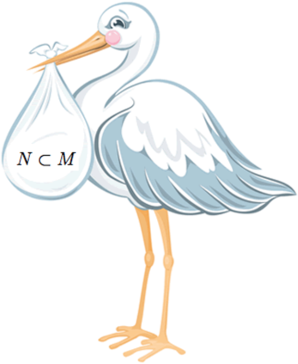

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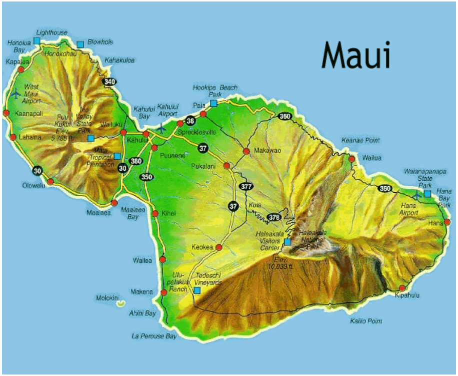

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Where do quantum symmetries come from?

- Finite groups
- Quantum groups
- Conformal field theory
- Subfactors!
  - There are examples only coming from subfactors which remain mysterious. These have only been constructed by brute force methods.

Quantum symmetries

Slogan:

Subfactors are universal hosts for quantum symmetries.

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Subfactors are universal hosts for quantum symmetries.

- The standard invariants of finite index subfactors are quantum symmetries.
- All quantum symmetries can be realized by subfactors.
  - Given a unitary fusion category $\mathcal{C}$, there is an essentially unique way to realize $\mathcal{C} \subset \mathrm{Bim}(R)$.
  - The subfactor $R \subset R \rtimes \mathcal{C}$ remembers $\mathcal{C}$.
  - Popa showed all standard invariants are realized by subfactors, but they may not be hyperfinite.

Index quantization

Jones' index rigidity theorem [Jon83]

For a II_1-subfactor $A \subset B$,

$$[B\colon A] \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \middle| n = 3, 4, \ldots \right\} \cup [4, \infty].$$

Moreover, there exists a subfactor at each index.

The basic construction

To prove the index restriction, Jones used the *basic construction*. Starting with a subfactor $A_0 \subset A_1$, we take the Jones projection $e_1 \in B(L^2(A_1))$, which is the orthogonal projection with range $L^2(A_0)$. The basic construction is $A_2 = \langle A_1, e_1 \rangle$.

- If $[A_1 : A_0] < \infty$, it is equal to $[A_2 : A_1]$.
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- Can iterate the basic construction to obtain the Jones tower:

$$A_0 \subset A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \overset{e_3}{\subset} \cdots$$

  The first sign something remarkable is happening is that the Jones projections satisfy the Temperley-Lieb-Jones relations:

  1. $e_i = e_i^* = e_i^2$
  2. $e_i e_j = e_j e_i$ for $|i - j| > 1$
  3. $e_i e_{i\pm1} e_i = [A_1 : A_0]^{-1} e_i$

Principal graphs

Let $\rho = {}_A B_B$. We look at the tensor products $\bigotimes_A^n B$, and decompose into irreducibles.

Definition

The principal graph Γ_+ has one vertex for each isomorphism class of simple ${}_A\alpha_A$ and ${}_A\beta_B$. There are

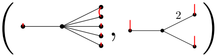
$$\dim(\mathrm{Hom}_{A-B}(\alpha\rho, \beta))$$

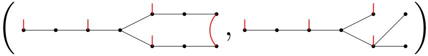
edges from α to β.
The dual principal graph Γ_- is defined similarly using $B - B$ and $B - A$ bimodules.

- $\Gamma_\pm$ has base point the trivial bimodule.
- The depth of a vertex is the distance to the base point.

Examples of principal graphs

- index < 4: A_n, D_{2n}, E_6, E_8. No D_{odd} or E_7.
- Graphs for $R \subset R \rtimes G$ obtained from G and Rep(G).



- Haagerup 333 

The standard invariant: two towers of centralizer algebras

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \cup & & \cup & & \\ P_{3,+} & = & A_0' \cap A_3 & \supset & A_1' \cap A_3 & = & P_{2,-} \\ & & \cup & & \cup & & \\ P_{2,+} & = & A_0' \cap A_2 & \supset & A_1' \cap A_2 & = & P_{1,-} \\ & & \cup & & \cup & & \\ P_{1,+} & = & A_0' \cap A_1 & \supset & A_1' \cap A_1 & = & P_{0,-} \\ & & \cup & & & & \\ P_{0,+} & = & A_0' \cap A_0 & & & & \end{array}$$

These centralizer algebras are finite dimensional [Jon83], and they form a planar algebra [Jon99].

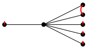
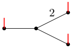
Finite depth

Definition

If the principal graph is finite, then the subfactor and standard invariant are called finite depth.

Example: $R \subset R \rtimes G$ for finite G

For $G = S_3$:

- Principal graph: 
- Dual principal graph: 

Theorem (Ocneanu Rigidity)

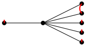
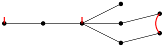
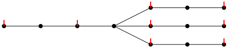
There are only finitely many standard invariants with the same finite principal graphs.

Supertransitivity

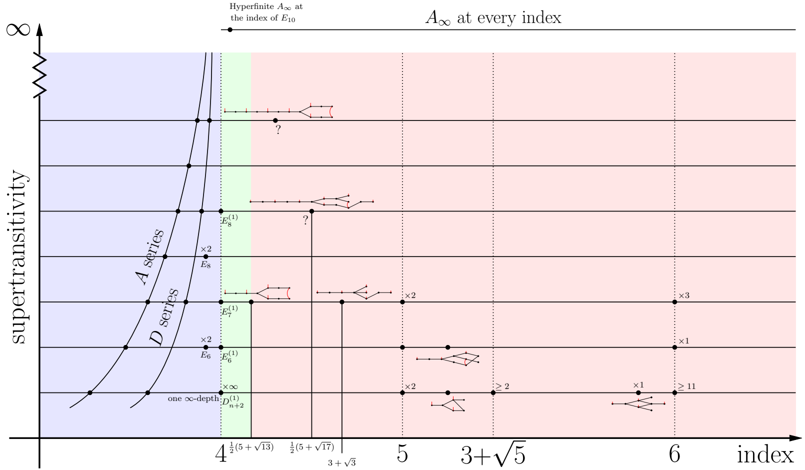
Definition

We say a principal graph is <u>n-supertransitive</u> if it begins with an initial segment consisting of the Coxeter-Dynkin diagram A_{n+1}, i.e., an initial segment with n edges.

Examples

-  is 1-supertransitive
-  is 2-supertransitive
-  is 3-supertransitive

Known small index subfactors, 1994



- Haagerup's partial classification to $3+\sqrt{3}$
- Popa's A_∞ at all indices
- Wenzl's quantum group subfactors

Small index subfactor classification program

Steps of subfactor classifications:

1. Enumerate graph pairs which survive obstructions.
2. Construct examples when graphs survive.

Fact (Popa [Pop94])

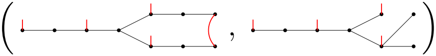
For a subfactor $A \subset B$, $[B : A] \geq \|\Gamma_+\|^2 = \|\Gamma_-\|^2$.

If we enumerate all graph pairs with norm at most r, we have found all principal graphs with index at most r^2.

Haagerup's enumeration

Theorem (Haagerup [Haa94])

Any non A_∞-standard invariant in the index range $(4, 3+\sqrt{2})$ must have principal graphs a translation of one of

- 
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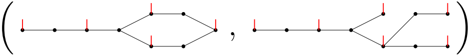
- 

Translation means raising the supertransitivity of both graphs by the same even amount.

Definition (Morrison-Snyder [MS12])

A vine is a graph pair which represents an infinite family of graph pairs obtained by translation.

Main tool for Haagerup's enumeration

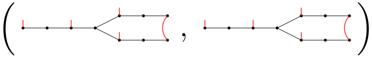
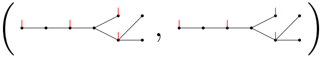
Play associativity against Ocneanu's triple point obstruction.

- Associativity: graphs must be similar
- Ocneanu's triple point obstruction: graphs must be different!

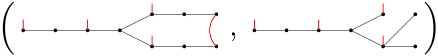
The consequence is a strong constraint.

Example

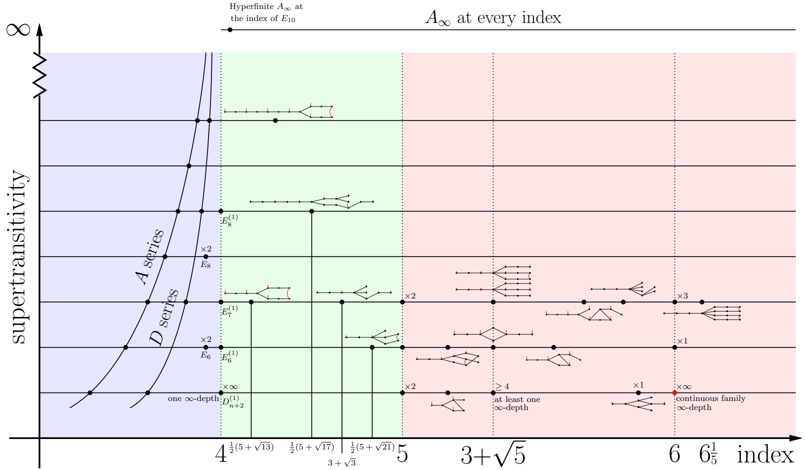
The following pairs are not allowed:

 and 

They must be paired with each other:



Known small index subfactors, 2011



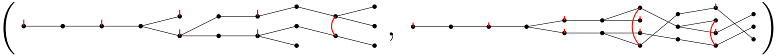
- Classification to $3 + \sqrt{3}$, Extended Haagerup
- Classification to index 5 (Izumi, Jones, Morrison, P, Peters, Snyder, Tener)

Weeds and vines

The classification to index 5 introduced weeds and vines.

Definition

A <u>weed</u> is a graph pair which represents an infinite family of graph pairs obtained by translation and extension.
An <u>extension</u> of a graph pair adds new vertices and edges at strictly greater depths than the maximum depth of any vertex in the original pair.

$\mathcal{F} =$ 

Using weeds allows us to bundle hard cases together. By carefully choosing weeds we can deal with later, we ensure the enumerator terminates.

Eliminating vines with number theory

We can uniformly treat vines using number theory, based on the following theorem inspired by Asaeda-Yasuda [AY09]:

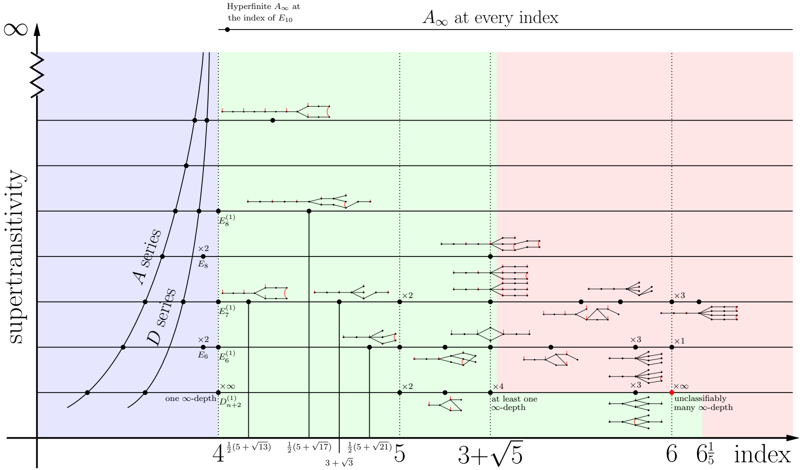
Theorem (Calegari-Morrison-Snyder [CMS11])

For a fixed vine $\mathcal{V}$, there is an effective (computable) constant $\mathcal{R}(\mathcal{V})$ such that any n-translate with $n > \mathcal{R}(\mathcal{V})$ has norm squared which is not a cyclotomic integer.

Theorem [CG94, ENO05]

The index of a finite depth subfactor (which is equal to the norm squared of the principal graph) must be a cyclotomic integer.

Known small index subfactors, today



Theorem (Afzaly-Morrison-P)

We know all subfactor standard invariants with index at most $5\frac{1}{4} > 3 + \sqrt{5}$.

Why do we care about index $3 + \sqrt{5}$?

$3 + \sqrt{5} = 2 \cdot \tau^2$ is the first interesting composite index.

- Standard invariants at index $4 = 2 \cdot 2$ are classified.
  - $\mathbb{Z}/2 * \mathbb{Z}/2 = D_\infty$ is amenable
- Standard invariants at index $6 = 2 \cdot 3$ are wild.
  - There is (at least) one standard invariant for every normal subgroup of the modular group $\mathbb{Z}/2 * \mathbb{Z}/3 = PSL(2, \mathbb{Z})$
  - There are unclassifiably many distinct hyperfinite subfactors with standard invariant $A_3 * D_4$ (Brothier-Vaes [BV13])
- Possibly there would be an profusion of subfactors at $3 + \sqrt{5}$!

New ideas to extend the classification

Enumeration:

- 1-supertransitive classification to $6\frac{1}{5}$ [LMP15], based on Liu's virtual normalizers [Liu13], and Liu's classification of composites of A_3 and A_4 [Liu15]
- New high-tech graph pair enumerator, based on Brendan McKay's isomorph free enumeration by canonical construction paths [McK98]. Two independent implementations, same results. (Afzaly and Morrison-P)
- Popa's principal graph stability [Pop95, BP14]

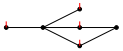
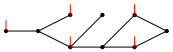
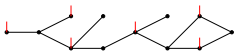
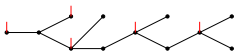
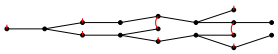
Obstructions:

- Number theory for stable weeds (Calegari-Guo) [CG15], adapted for periodic weeds!
- Morrison's hexagon obstruction [Mor14]
- Souped up triple point obstruction [Pen15]

1-supertransitive subfactors at index $3 + \sqrt{5}$

Theorem [Liu15]

There are exactly seven 1-supertransitive standard invariants with index $3 + \sqrt{5}$:

- ( , ) self-dual
- ( , ) and its dual
- ( , ) and its dual
- ( $\cdots$,  $\cdots$) and its dual ($A_3 * A_4$)

These are all the standard invariants of composed inclusions of A_3 and A_4 subfactors.

Open question

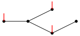
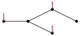
How many hyperfinite subfactors have Bisch-Jones' Fuss-Catalan $A_3 * A_4$ standard invariant at index $3 + \sqrt{5}$?

- $A_3 * A_4$ and $A_2 * T_2$ are not amenable [Pop94, HI98].

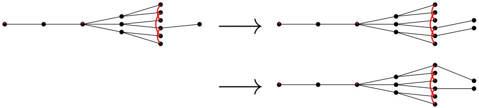
Why better combinatorics are needed

Three ways we produce redundant isomorphism classes of graphs:

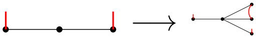
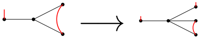
(1) Equivalent generating steps from same object give isomorphic results.

 and 

(2) Two inequivalent generating steps applied to the same object can yield isomorphic objects.



(3) Starting with two non-isomorphic objects and applying a generating step can result in isomorphic objects.

 and 

Problems fixed by McKay's isomorph-free enumeration [McK98]!

Popa's principal graph stability

Definition

We say $\Gamma_\pm$ is stable at depth n if every vertex at depth n connects to at most one vertex at depth $n+1$, no two vertices at depth n connect to the same vertex at depth $n+1$, and all edges between depths n and $n+1$ are simple.

Theorem (Popa [Pop95], Bigelow-P [BP14])

Suppose $A \subset B$ (finite index) has principal graphs (Γ_+, Γ_-). Suppose that the truncation $\Gamma_\pm(n+1) \neq A_{n+2}$ and $\delta > 2$.

(1) If $\Gamma_\pm$ are stable at depth n, then $\Gamma_\pm$ are stable at depth k for all $k \geq n$, and $\Gamma_\pm$ are finite.

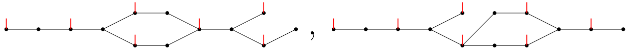
(2) If Γ_+ is stable at depths n and $n+1$, then $\Gamma_\pm$ are stable at depth $n+1$.

Part (2) uses the 1-click rotation in the planar algebra.

Stable weeds

Definition

A stable weed represents an infinite family of graph pairs obtained by translation and finite stable extension.

$\mathcal{C} = \Big($  $\Big)$

Theorem (Guo)

Let $\mathcal{S}_M$ be the class of finite graphs satisfying:

1. all vertices have valence at most M, and
2. at most M vertices have valence > 2.

Then ignoring A_n, D_n, $A_n^{(1)}$, and $D_n^{(1)}$, only finitely many graphs in $\mathcal{S}_M$ have norm squared which is a cyclotomic integer.

- Result is effective for a given fixed stable weed [CG15].
- Calegari-Guo eliminate our troublesome cylinder $\mathcal{C}$ by hand.

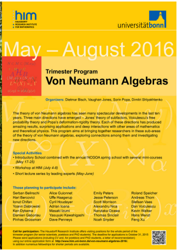
Lessons from classification thus far

1. Small index subfactors are much rarer than expected!
2. Even with the new combinatorics, computational complexity grows quickly as the index increases:
   - 1 Haagerup to index $3 + \sqrt{3}$
   - 4-5 Haagerups to index 5
   - 69 Haagerups to index $5\frac{1}{4}$
3. Many new ideas needed just to get from 5 to $5\frac{1}{4}$

Other ways to search for quantum symmetries

Other ways to search for quantum symmetries

▶ 



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HAUSDORFF RESEARCH INSTITUTE FOR MATHEMATICS

universitätbonn

May – August 2016

Trimester Program

Von Neumann Algebras

Organizers: Dietmar Bisch, Vaughan Jones, Sorin Popa, Dimitri Shlyakhtenko

The theory of von Neumann algebras has seen many spectacular developments in the last ten years. Three main directions have emerged -- Jones' theory of subfactors, Voiculescu's free probability theory and Popa's deformation-rigidity theory. Each of these directions has produced amazing results, surprising applications and deep interactions with other areas of mathematics and theoretical physics. This program aims at bringing together researchers in these sub-areas of the theory of von Neumann algebras, exploring connections among them and investigating new directions.

Special Activities

- Introductory School combined with the annual NCGOA spring school with several mini-courses *(May 17-25)*
- Workshop at HIM *(July 4-8)*
- Short lecture series by leading experts *(May-June)*

Those planning to participate include:

Serban Belinschi, Hari Bercovici, Ionut Chifan, Yoann Dabrowski, Ken Dykema, Damien Gaboriau, Pinhas Grossman, Alice Guionnet, Uffe Haagerup, Cyril Houdayer, Adrian Ioana, Masaki Izumi, Yasuyuki Kawahigashi, Dave Penneys, Emily Peters, Jesse Peterson, Scott Morrison, Alexandru Nica, Narutaka Ozawa, Thomas Sinclair, Noah Snyder, Roland Speicher, Andreas Thom, Stefaan Vaes, Dan Voiculescu, Kevin Walker, Hans Wenzl, Feng Xu

Call for participation: The Hausdorff Research Institute offers visiting positions for the whole period of the trimester program (for senior scientists, postdocs and PhD students). The deadline for applications is October 31, 2015. Please send applications (including CV and, for postdocs and PhD students, a letter of recommendation) using our online application form at **http://www.him.uni-bonn.de/von-neumann-algebras-2016/.** In addition numerous fellowships for shorter periods are available.

Other ways to search for quantum symmetries

- 

- planar algebras generated by small elements (Bisch-Jones, Liu)
- Restrict to small global dimension
- Look at fusion categories

Open questions in subfactor theory

1. How many hyperfinite $A_3 * A_4$ subfactors are there?

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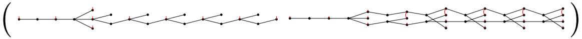
2. Can we eliminate stubborn weeds at index just above 5.27?



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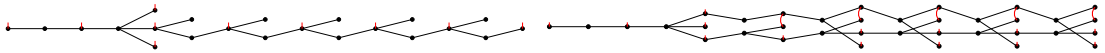
   ()

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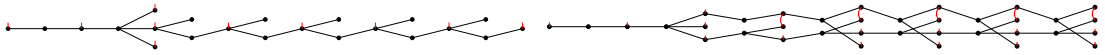
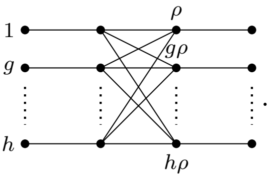
()

3. Is there a global bound on supertranstitivty?
4. Where do the quadratic categories come from?
   The 3^G subfactors for $|G|$ odd have a reduced subfactor which has Yang-Baxter relations (Liu-P., uploaded to `arXiv` today). Could these come from quantum groups?



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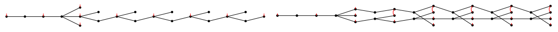
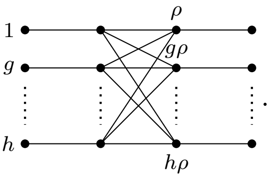
()

3. Is there a global bound on supertranstitivty?
4. Where do the quadratic categories come from?
   The 3^G subfactors for $|G|$ odd have a reduced subfactor which has Yang-Baxter relations (Liu-P., uploaded to `arXiv` today). Could these come from quantum groups?



5. Where does extended Haagerup come from?

Open questions in subfactor theory

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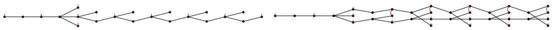
()

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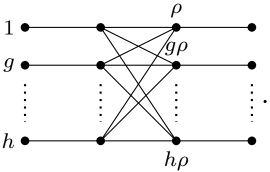

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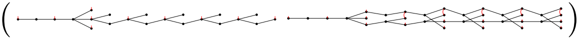

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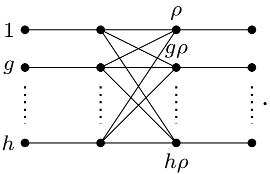

5. Where does extended Haagerup come from?
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7. Do all subfactors come from CFT?

Thank you for listening!

Slides available at

`http://www.math.ucla.edu/~dpenneys/PenneysQinhuangdao2015.pdf`

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