Applications of subfactors and fusion categories to mathematical physics

UC Davis Mathematical Physics & Probability Seminar

David Penneys
UCLA

October 29, 2014

What is a subfactor?

Definition

A factor is a von Neumann algebra with trivial center.
A subfactor is an inclusion $A \subset B$ of factors.

- Our factors are type II_1, which means they are infinite dimensional with a trace.

Remark

Von Neumann algebras come in pairs (M, M').
Subfactors do too: $(A \subset B, B' \subset A')$.

Where do subfactors come from?

Some examples include:

- Groups – from $G \curvearrowright R$, we get $R^G \subset R$ and $R \subset R \rtimes_\alpha G$.
- finite dimensional unitary Hopf/Kac algebras
- Quantum groups – Rep$(\mathcal{U}_q(\mathfrak{g}))$
- Conformal field theory
- endomorphisms of Cuntz C*-algebras
- tinkering with known subfactors (orbifolds, composites, ...)

However, there are certain possible infinite families without uniform constructions.

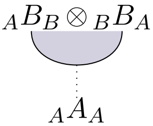
Finite index and the standard representation

Definition

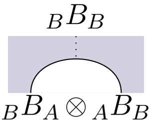
$A \subset B$ has finite index iff B is a finitely generated projective A-module.

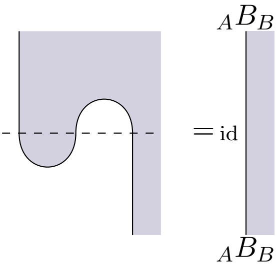
The bimodule ${}_AB_B$ is the standard representation of $A \subset B$.
A finite index subfactor $A \subset B$ comes with canonical maps:

Inclusion:



Evaluation:



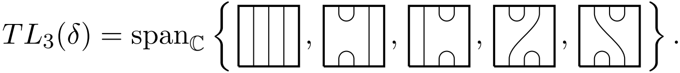


Since A, B are analytical objects, these maps also have adjoints.

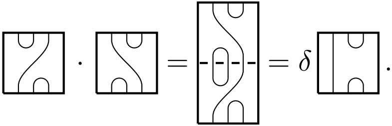
The Temperley-Lieb algebras

Definition

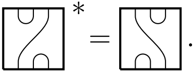
The Temperley-Lieb algebra $TL_n(\delta)$ is the complex $*$-algebra spanned by diagrams with n upper and lower boundary points, connected by non-crossing strings.



- Multiplication is stacking of diagrams, but we trade closed loops for multiplicative factors of δ:



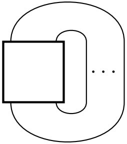
- The involution $*$ is given by vertical reflection:



Jones' index rigidity theorem

- The trace is given by capping off on the right

$$\mathrm{Tr}_n =$$



$$: TL_n(\delta) \to TL_0(\delta) \cong \mathbb{C}$$

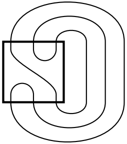
- There is a sesquilinear form given by $\langle x, y \rangle_n = \mathrm{Tr}_n(y^* x)$.

Theorem (Jones)

A finite index subfactor gives a positive-definite $*$-representation of the Temperley-Lieb algebra $TL_n(\delta)$ for $\delta^2 = [B : A]$ and all $n \geq 0$. This is possible iff $\delta \in \{2\cos(\pi/k) | k \geq 3\} \cup [2, \infty)$.

Jones' index rigidity theorem

- The trace is given by capping off on the right

$\mathrm{Tr}_n =$  $: TL_n(\delta) \to TL_0(\delta) \cong \mathbb{C}$

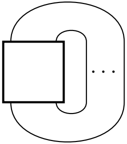
- There is a sesquilinear form given by $\langle x, y \rangle_n = \mathrm{Tr}_n(y^* x)$.

Theorem (Jones)

A finite index subfactor gives a positive-definite $*$-representation of the Temperley-Lieb algebra $TL_n(\delta)$ for $\delta^2 = [B : A]$ and all $n \geq 0$. This is possible iff $\delta \in \{2\cos(\pi/k) | k \geq 3\} \cup [2, \infty)$.

Jones' index rigidity theorem

- The trace is given by capping off on the right

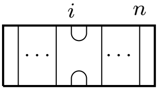
$\mathrm{Tr}_n =$  $: TL_n(\delta) \to TL_0(\delta) \cong \mathbb{C}$

- There is a sesquilinear form given by $\langle x, y \rangle_n = \mathrm{Tr}_n(y^* x)$.

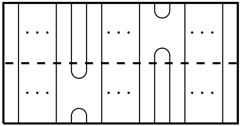
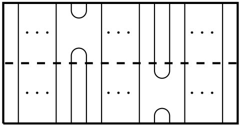
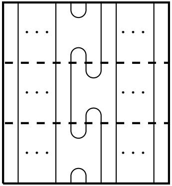
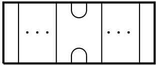
Theorem (Jones)

A finite index subfactor gives a positive-definite $*$-representation of the Temperley-Lieb algebra $TL_n(\delta)$ for $\delta^2 = [B : A]$ and all $n \geq 0$. This is possible iff $\delta \in \{2\cos(\pi/k) | k \geq 3\} \cup [2, \infty)$.

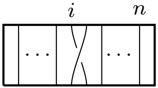
Temperley-Lieb and braid groups, part 1

$TL_n(\delta)$ has generators $E_i =$ 

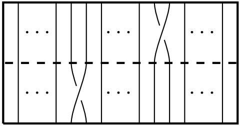
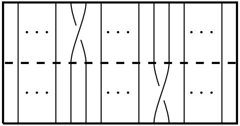
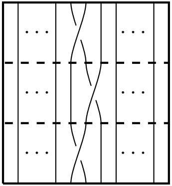
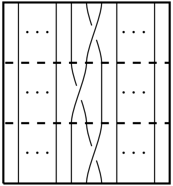
for $1 \leq i \leq n-1$, and relations

- $E_i^2 = \delta E_i = \delta E_i^*$,
- $E_i E_j =$  $=$  $= E_j E_i$ if $|i-j| > 1$
- $E_i E_{i\pm1} E_i =$  $=$  $= E_i$.

Temperley-Lieb and braid groups, part 2

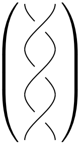
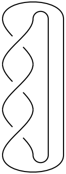
The braid group B_n has generators $\sigma_i =$ 

for $1 \leq i \leq n-1$, and relations

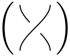
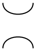
- $\sigma_i \sigma_j =$  $=$  $= \sigma_j \sigma_i$ for $|i-j| > 1$, and
- $\sigma_i \sigma_{i+1} \sigma_i =$  $=$  $= \sigma_{i+1} \sigma_i \sigma_{i+1}$

Knots and braids

Given a link, we can always write it as the closure of a braid.

Tr  $=$  , a trefoil knot.

We have an algebra homomorphism $\Phi : \mathbb{C}[B_n] \to TL_n(\delta)$ by

Φ  $= iq^{1/2}$  $- iq^{-1/2}$ 

where $\delta = q + q^{-1}$.

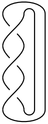
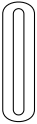
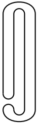
The Jones polynomial/Kauffman bracket

To get the framed Jones polynomial or Kauffman bracket of a link ℓ, first write $\ell = \mathrm{Tr}(b)$ for a braid b. Then

$$\langle \ell \rangle = \frac{1}{\delta} \mathrm{Tr} \circ \Phi(b)$$

is independent of the choice of braid representing the knot.

Example

$\Big\langle$  $\Big\rangle = \dfrac{1}{q+q^{-1}} \Big((iq^{1/2})^3$  $+ 3(iq^{1/2})^2(-iq^{-1/2})$ 

$+3(iq^{1/2})(-iq^{-1/2})^2$  $+ (-iq^{-1/2})^3$  $\Big)$

$$= i(q^{-7/2} - q^{-3/2} - q^{5/2})$$

Rep($A \subset B$)

Definition

The representation 2-category of $A \subset B$ is given by

(0) 0-morphisms: $\{A, B\}$

(1) 1-morphisms: bimodule summands of $\bigotimes_A^k B$ for some $k \geq 0$

(2) 2-morphisms: bimodule intertwiners

- This 2-category is semi-simple, unitary, rigid (duals are well behaved), pivotal, sometimes spherical (iff $A \subset B$ extremal).
- The $A - A$ bimodules form a rigid C^*-tensor category called the 'principal even part'.
- The $B - B$ bimodules form the 'dual even part'.
- The principal even and dual even parts are Morita equivalent via the $A - B$ bimodules.

Subfactor/representation 2-category correspondence

Theorem (Popa [Pop94])

There is a Tannaka-Krein like duality between (strongly) amenable subfactors and their representation 2-categories.

Theorem (many authors)

Subfactors correspond to Frobenius algebra objects in rigid C*-tensor categories.

Fusion categories

If there are only finitely many isomorphism classes of simple $A - A$ bimodules, the principal even part is a unitary fusion category.

- Subfactors are a vital source of interesting fusion categories.

Definition

A fusion category is a semisimple, rigid tensor category with finitely many isomorphism classes of simple objects.

Fact

An $X \in \mathcal{C}$ with quantum dimension δ gives a representation

$$TL_\bullet(\delta) \to \mathrm{End}(\underbrace{X \otimes \overline{X} \otimes \cdots}_{n \text{ alternating copies}}).$$

If $\mathcal{C}$ is unitary, the representation is positive definite.

Examples

Let G be a finite group.

Example

$\mathrm{Rep}(G)$, category of finite dimensional $\mathbb{C}$-representations.

Example

$\mathrm{Vec}(G, \omega)$, G-graded vector spaces, $\omega \in H^3(G, \mathbb{C}^\times)$.

- Simple objects $V_g \cong \mathbb{C}$ for each $g \in G$.
- $V_g \otimes V_h = V_{gh}$
- The 3-cocycle gives the associator natural isomorphism:

$$\alpha_{g,h,k} : (V_g \otimes V_h) \otimes V_k \xrightarrow{\omega_{g,h,k}} V_g \otimes (V_h \otimes V_k).$$

The pentagon axiom is exactly the 3-cocycle condition.

$\mathrm{Rep}(R \subset R \rtimes G)$

Let G be a finite group. Build the subfactor $R \subset R \rtimes G$.

Example

The representation 2-category $\mathrm{Rep}(R \subset R \rtimes G)$ has:

- principal even part $\mathrm{Vec}(G, 1)$ ($R - R$ bimodules)
- dual even part $\mathrm{Rep}(G)$ ($R \rtimes G - R \rtimes G$ bimodules)
- only one simple $R - R \times G$ bimodule: $R \rtimes G$.

We see $\mathrm{Vec}(G, 1)$ and $\mathrm{Rep}(G)$ are Morita equivalent.

Fact

The subfactor $R \subset R \rtimes G$ corresponds to the algebra object $\mathbb{C}[G] \in \mathrm{Vec}(G)$.

The Haagerup: an 'exotic' example

The Haagerup fusion category $\mathcal{H}$ has 6 simple objects $1, g, g^2, X, gX, g^2X$ satisfying the following fusion rules:

- $\langle g \rangle \cong \mathbb{Z}/3$,
- $Xg \cong g^{-1}X$, and
- $X^2 \cong 1 \oplus X \oplus gX \oplus g^2X$ (the quadratic relation).

($\mathrm{Vec}(\mathbb{Z}/3) \subset \mathcal{H}$ has trivial associator.)

The algebra object $1 \oplus X$ gives an 'exotic' subfactor with index

$$\frac{5 + \sqrt{13}}{2} \approx 4.30278.$$

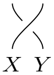
$\mathcal{H}$ has only been constructed by brute force.

- It appears $\mathcal{H}$ belongs to an infinite family, but only examples up to $\mathbb{Z}/19$ have been constructed [EG11].

Braided fusion categories

Definition

A fusion category is <u>braided</u> if it has natural isomorphisms



$$= c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying the braid relations and a compatibility requirement.

Example

Vec is a symmetric braided fusion category, i.e., $c_{b,a} \circ c_{a,b} = \mathrm{id}_{a \otimes b}$ for all $a, b \in \mathrm{Vec}$.

Facts

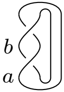
If $\mathcal{C}$ is braided, an $X \in \mathcal{C}$ gives a representation $B_n \to \mathrm{End}(X^{\otimes n})$.
If $\mathcal{C}$ is unitary, the representation is also.
If $\mathcal{C}$ is symmetric, the representation factors through S_n.

Modular tensor categories

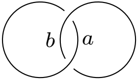
Definition

A modular tensor category is a braided spherical fusion category (and more axioms...) such that the S matrix $(S_{a,b})$ is invertible.

$$S_{a,b} = \mathrm{Tr}(c_{b,a} \circ c_{a,b}) =$$



$$=$$



Example

If $\mathcal{C}$ is a spherical fusion category over $\mathbb{C}$, then the quantum double $\mathcal{Z}(\mathcal{C})$ is a modular tensor category. If $\mathcal{C}$ is unitary, then so is $\mathcal{Z}(\mathcal{C})$.

Theorem (Bruillard-Ng-Rowell-Wang [BNRW13])

For a fixed n, there are only finitely many modular tensor categories with rank n.

- Rank finiteness not yet known for fusion categories.

Classification of fusion categories

Question (Hard!)

Can we classify all fusion categories with n objects for n small?

Examples

- Rank 2 was classified by Ostrik [Ost03]:
  - $\mathrm{Vec}(\mathbb{Z}/2, \omega)$ for $\omega \in H^3(\mathbb{Z}/2, \mathbb{C}^\times)$
  - $\mathrm{Fib} = \langle 1, \tau | \tau \otimes \tau \cong 1 \oplus \tau \rangle$ and Galois conjugate
- Rank 3 (pivotal) was classified by Ostrik [Ost13]:
  - $\mathrm{Vec}(\mathbb{Z}/3, \omega)$ for $\omega \in H^3(\mathbb{Z}/3, \mathbb{C}^\times)$
  - $\mathrm{Rep}(S_3)$ and twisted versions
  - Ising category (even part of $\mathfrak{sl}_2$ at 6th root of unity) and conjugates
  - even part of $\mathfrak{sl}_2$ at 7th root of unity and conjugates
  - even part of E_6 subfactor and conjugate
- Rank 4 (pseudo unitary) with a dual pair of objects $(1, X, Y, \overline{Y})$ was classified by Larson [Lar14].
  - New examples of Liu-Morrison-P [LMP14]

Topological quantum field theories (TQFTs)

Definition (Atiyah)

An n-dimensional TQFT is a symmetric monoidal functor

$$\left(\binom{n}{n-1}\mathrm{Bord}, \amalg\right) \longrightarrow (\mathrm{Vec}, \otimes)$$

Each $n-1$ manifold is assigned a vector space, and each bordism is assigned a linear operator.

Examples for $n = 3$

- Turaev-Viro associated to a spherical fusion category
- Reshetikhin-Turaev associated to a modular tensor category

In fact, $TV(\mathcal{C}) \cong RT(\mathcal{Z}(\mathcal{C}))$.

Extended topological field theories

Definition

An $(n, n-1, \ldots, d)$-TFT is a symmetric monoidal functor

$$\begin{pmatrix} n \\ \vdots \\ d \end{pmatrix} \mathrm{Bord} \longrightarrow (n-d) - \mathrm{Vec}$$

for an appropriate choice of $n-d$ category $(n-d) - \mathrm{Vec}$.

Examples

- Turaev-Viro is a $(3, 2, 1, 0)$-TFT (fully extended)
- Reshetikhin-Turaev is a $(3, 2, 1)$-TFT

The double construction relates these two.

Extended topological field theories

Definition

An $(n, n-1, \ldots, d)$-TFT is a symmetric monoidal functor

$$\begin{pmatrix} n \\ \vdots \\ d \end{pmatrix} \mathrm{Bord} \longrightarrow (n-d) - \mathrm{Vec}$$

for an appropriate choice of $n-d$ category $(n-d) - \mathrm{Vec}$.

Examples

- $\binom{1}{0}$-TFTs $\longleftrightarrow$ a dualizable object in a symmetric $\otimes$-category
- $\begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$-TFTs $\longleftrightarrow$ fusion categories in 3-category of $\otimes$-categories (recent work of Douglas-Schommer-Pries-Snyder [DSPS13])

Segal conformal field theory (CFT)

Definition (Segal)

A 2d-conformal field theory is a symmetric monoidal functor

$$\binom{2}{1}\mathrm{ConfBord} \longrightarrow \mathsf{Hilb}$$

This consists of:

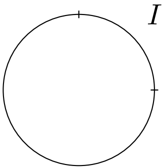
- a Hilbert space H_S assigned to each compact, connected oriented 1-manifold S
- a unitary $u_f : H_{S_1} \to H_{S_2}$ to every orientation preserving diffeomorphism $f : S_1 \to S_2$
  (an anti-unitary for an orientation reversing diffeomorphism)
- a map $g_\Sigma : \bigotimes H_{S_{\text{in}}} \to \bigotimes H_{S_{\text{out}}'}$ to each cobordism Σ with a complex structure, where orientation is reversed for each S_{out}.

Conformal welding allows for gluing along diffeomorphisms.

Conformal nets (algebraic quantum field theory)

Definition

A conformal net is a functor from intervals $I \subset S^1$ to von Neumann algebras in $B(H)$,



$$\longmapsto \mathcal{A}(I) \subset B(H),$$

satisfying axioms, like

- $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$
- locality: $I \cap J = \emptyset \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = 0.$

The net is irreducible if each $\mathcal{A}(I)$ is a factor.

- Disjoint intervals give subfactors: $I \cap J = \emptyset \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)'$.

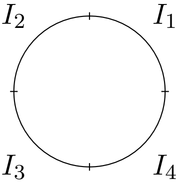
Modular tensor categories from conformal nets

Definition

A representation of the net $\mathcal{A}$ is a family of representations $\pi_I : \mathcal{A}(I) \to B(K)$ for a fixed Hilbert space K, such that if $I \subset J$, then $\pi_J|_{\mathcal{A}(I)} = \pi_I$.

Theorem (Kawahigashi-Longo-Müger [KLM01])

Consider the partition of S^1 into 4 disjoint intervals:



If $\mathcal{A}(I_1 \cup I_3) \subset \mathcal{A}(I_2 \cup I_4)'$ has finite index, then $\mathrm{Rep}(\mathcal{A})$ is a unitary modular tensor category.

Modular categories $\overset{?}{\longleftrightarrow}$ CFT

Conjecture (Kawahigashi)

The quantum double of every unitary fusion category arises as the representation category of some conformal net.

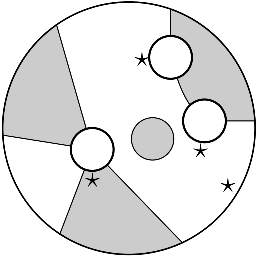
Conjecture (Evans-Gannon [EG11])

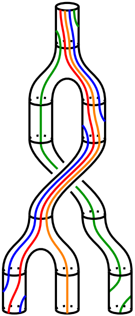
There should be a CFT realizing the double of the Haagerup fusion category. In particular, there should be a conformal subalgebra of the central charge $c = 8$ vertex operator algebra corresponding to the root lattice $E_6 \oplus A_2$.

- The modular data of the double of Haagerup is 'graft' of the double of S_3 and $\mathfrak{so}(13)_2$.
- They compute possible character vectors for the VOA, and show they have non-negative integral Fourier coefficients.

Work in progress: conformal planar algebras

- Subfactors and CFT are related via conformal nets.
- Tannaka-Krein duality $A \subset B \leftrightarrow \mathrm{Rep}(A \subset B)$ (Popa)
- $\mathrm{Rep}(A \subset B)$ axiomatized as a planar algebra (Jones)





- In joint work with Henriques and Tener, we expect a connection between genus zero Segal CFT (many-to-one genus zero Riemann surfaces) with topological defect strings and planar algebras.

Classifying small index subfactors

- To each finite group G, there is a dual pair of subfactors $R \subset R \rtimes G$ and $R^G \subset R$.

Thus, one cannot hope to classify all subfactors. We need to restrict our search space. One way to do this is to look at small index subfactors.

Recall:
The representation 2-category of $A \subset B$ is given by

(0) 0-morphisms: $\{A, B\}$

(1) 1-morphisms: bimodule summands of $\bigotimes_A^k B$ for some $k \geq 0$

(2) 2-morphisms: bimodule intertwiners

Principal graphs

Definition

The principal (induction) graph Γ_+ has one vertex for each isomorphism class of simple ${}_A P_A$ and ${}_A Q_B$. There are

$$\dim(\mathrm{Hom}_{A-B}(P \otimes_A B, Q))$$

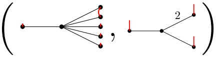
edges from P to Q.

The dual principal (restriction) graph Γ_- is defined similarly using $B - B$ and $B - A$ bimodules.

- $\Gamma_\pm$ is pointed, where the base point is ${}_A A_A$, ${}_B B_B$ respectively.
- Duality is given by contragredient, which is always at the same depth, since B is a $*$-algebra. However, duals at odd depths of $\Gamma_\pm$ are on $\Gamma_\mp$.

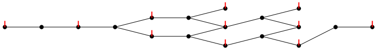
Examples of principal graphs

- index < 4: A_n, D_{2n}, E_6, E_8. No D_{odd} or E_7.
- index $= 4$: $A^{(1)}_{2n-1}, D^{(1)}_{n+2}, E^{(1)}_6, E^{(1)}_7, E^{(1)}_8, A_\infty, A^{(1)}_\infty, D_\infty$
- Graphs for $R \subset R \rtimes G$ obtained from G and $\text{Rep}(G)$.



$G = S_3$

- Principal graph for $R^G \subset R^H$ is the induction-restriction graph for $H \subset G$:



$S_5 \subset S_6$

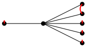
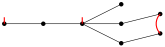
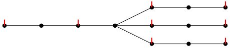
- First graph is principal, second is dual principal.
- Leftmost vertex corresponds to base points ${}_A A_A$, ${}_B B_B$.
- Red tags for duality of even vertices (${}_A P_A \mapsto \overline{{}_A P_A}$).
- Duality of odd vertices by depth and height

Supertransitivity

Definition

A principal graph is <u>n-supertransitive</u> if has an initial segment with n edges before branching.

Examples

-  is 1-supertransitive
-  is 2-supertransitive
-  is 3-supertransitive

Small index subfactor classification program

Steps of subfactor classifications:

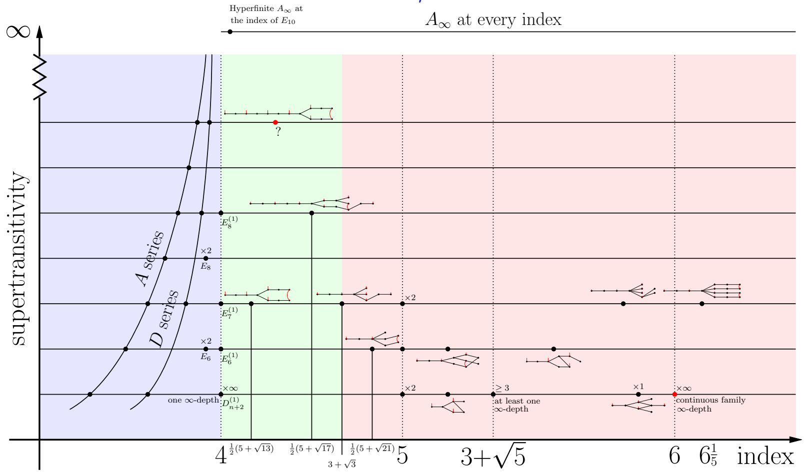
1. Enumerate graph pairs which survive obstructions.
2. Construct examples when graphs survive.

Fact (Popa [Pop94])

For a subfactor $A \subset B$, $[B : A] \geq \|\Gamma_+\|^2 = \|\Gamma_-\|^2$.

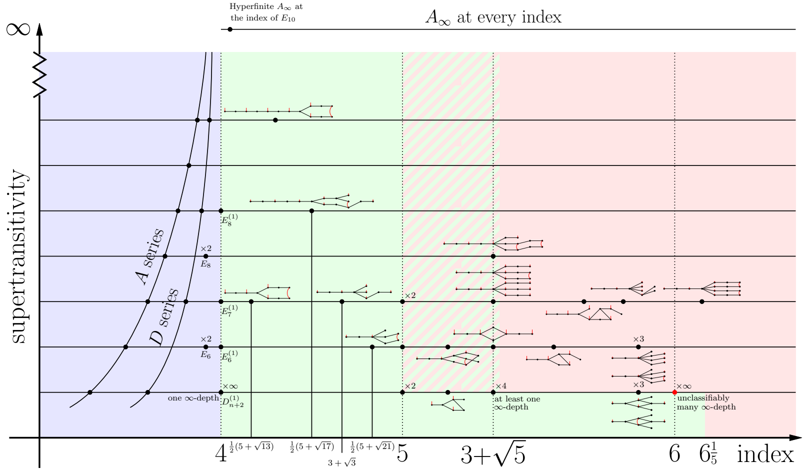
If we enumerate all graph pairs with norm at most r, we have found all principal graphs of subfactors with index at most r^2.

Known small index subfactors, 2009



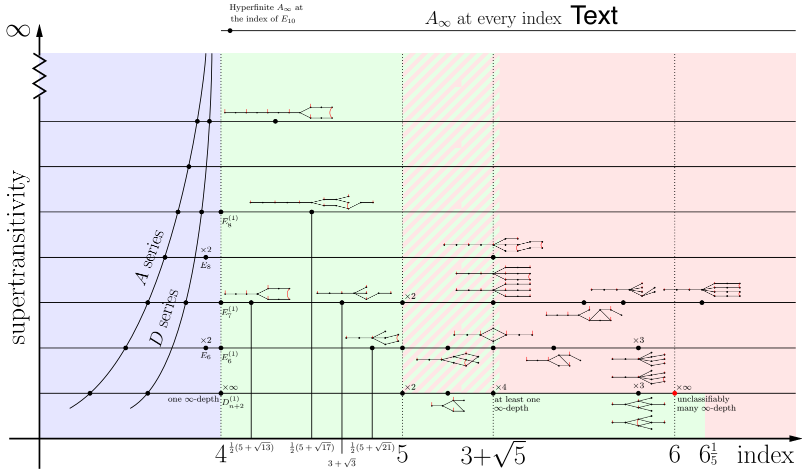
- Quantum groups and their quantum subgroups
- Composites
- Haagerup's exotic subfactor and classification to $3+\sqrt{3}$
- Izumi's Cuntz algebra examples (2221, 3^n)

Known small index subfactors, today



- Classification to 5 [MS12, MPPS12, IJMS12, PT12, IMP+14]
- Examples at $3+\sqrt{5}$ [MP13, PP13, IMP13, MP14]
- 1-supertransitive to $6\frac{1}{5}$ and examples at $3+2\sqrt{2}$ [LMP14]

Known small index subfactors, today



Theorem (Afzaly-Morrison-P)

We know all subfactor standard invariants up to index $5\frac{1}{4}$ (with at most finitely many exceptions).

Thank you for listening!

Slides available at
http:
//www.math.ucla.edu/~dpenneys/PenneysUCDavis2014.pdf

Paul Bruillard, Siu-Hung Ng, Eric C. Rowell, and Zhenghan Wang, *On modular categories*, 2013, arXiv:1310.7050.

Chris Douglas, Chris Schommer-Pries, and Noah Snyder, *Dualizable tensor categories*, 2013, arXiv:1312.7188.

David E. Evans and Terry Gannon, *The exoticness and realisability of twisted Haagerup-Izumi modular data*, Comm. Math. Phys. **307** (2011), no. 2, 463–512, arXiv:1006.1326 MR2837122 DOI:10.1007/s00220-011-1329-3.

Masaki Izumi, Vaughan F. R. Jones, Scott Morrison, and Noah Snyder, *Subfactors of index less than 5, Part 3: Quadruple points*, Comm. Math. Phys. **316** (2012), no. 2, 531–554, MR2993924, arXiv:1109.3190, DOI:10.1007/s00220-012-1472-5.

Masaki Izumi, Scott Morrison, and David Penneys, *Fusion categories between $\mathcal{C} \boxtimes \mathcal{D}$ and $\mathcal{C} * \mathcal{D}$*, 2013, arXiv:1308.5723.

Masaki Izumi, Scott Morrison, David Penneys, Emily Peters, and Noah Snyder, *Subfactors of index exactly 5*, 2014, arXiv:1406.2389.

Yasuyuki Kawahigashi, Roberto Longo, and Michael Müger, *Multi-interval subfactors and modularity of representations in conformal field theory*, Comm. Math. Phys. **219** (2001), no. 3, 631–669, MR1838752 DOI:10.1007/PL00005565.

Hannah K. Larson, *Pseudo-unitary non-self-dual fusion categories of rank 4*, J. Algebra **415** (2014), 184–213, MR3229513, DOI:10.1016/j.jalgebra.2014.05.032.

Zhengwei Liu, Scott Morrison, and David Penneys, 1*-supertransitive subfactors with index at most* $6\frac{1}{5}$, Comm. Math. Phys. (2014), arXiv:1310.8566, DOI:10.1007/s00220-014-2160-4.

Scott Morrison and David Penneys, *Constructing spoke subfactors using the jellyfish algorithm*, 2013, arXiv:1208.3637, Accepted to Trans. Amer. Math. Soc. Jan. 18, 2013.

______, *2-supertransitive subfactors at index* $3+\sqrt{5}$, 2014, arXiv:1406.3401.

Scott Morrison, David Penneys, Emily Peters, and Noah Snyder, *Subfactors of index less than 5, Part 2: Triple points*, Internat. J. Math. **23** (2012), no. 3, 1250016, 33, MR2902285, arXiv:1007.2240, DOI:10.1142/S0129167X11007586.

Scott Morrison and Noah Snyder, *Subfactors of index less than 5, Part 1: The principal graph odometer*, Comm. Math. Phys. **312** (2012), no. 1, 1–35, MR2914056, arXiv:1007.1730, DOI:10.1007/s00220-012-1426-y.

- Viktor Ostrik, *Fusion categories of rank 2*, Math. Res. Lett. **10** (2003), no. 2-3, 177–183, MR1981895 arXiv:math.QA/0203255.
- Victor Ostrik, *Pivotal fusion categories of rank 3*, 2013, (with an Appendix written jointly with Dmitri Nikshych), arXiv:1309.4822.
- Sorin Popa, *Classification of amenable subfactors of type II*, Acta Math. **172** (1994), no. 2, 163–255, MR1278111, DOI:10.1007/BF02392646.
- David Penneys and Emily Peters, *Calculating two-strand jellyfish relations*, 2013, arXiv:1308.5197.
- David Penneys and James E. Tener, *Subfactors of index less than 5, Part 4: Vines*, Internat. J. Math. **23** (2012), no. 3, 1250017, 18, MR2902286, arXiv:1010.3797, DOI:10.1142/S0129167X11007641.