Classifying small index subfactors

UCLA Workshop on von Neumann algebras and ergodic theory

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Where do subfactors come from?

Some examples include:

- Groups – from $G \curvearrowright R$, we get $R^G \subset R$ and $R \subset R \rtimes_\alpha G$.
- finite dimensional unitary Hopf/Kac algebras
- Quantum groups – Rep$(\mathcal{U}_q(\mathfrak{g}))$
- Conformal field theory
- endomorphisms of Cuntz C*-algebras
- composites of known subfactors

However, there are certain possible infinite families without uniform constructions.

Remark

Just as von Neumann algebras come in pairs (M, M'), subfactors come in pairs $(A \subset B, B' \subset A')$.

Index for subfactors

Theorem (Jones [Jon83])

For a II_1-subfactor $A \subset B$,

$$[B\colon A] \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \middle| n = 3, 4, \ldots \right\} \cup [4, \infty].$$

Moreover, there exists a subfactor at each index.

Definition

The Jones tower of $A = A_0 \subset A_1 = B$ (finite index) is given by

$$A_0 \subset A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \overset{e_3}{\subset} \cdots$$

where e_i is the projection in $B(L^2(A_i))$ with range $L^2(A_{i-1})$.

The standard invariant: two towers of centralizer algebras

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \cup & & \cup & & \\
P_{3,+} & = & A_0' \cap A_3 & \supset & A_1' \cap A_3 & = & P_{2,-} \\
 & & \cup & & \cup & & \\
P_{2,+} & = & A_0' \cap A_2 & \supset & A_1' \cap A_2 & = & P_{1,-} \\
 & & \cup & & \cup & & \\
P_{1,+} & = & A_0' \cap A_1 & \supset & A_1' \cap A_1 & = & P_{0,-} \\
 & & \cup & & & & \\
P_{0,+} & = & A_0' \cap A_0 & & & &
\end{array}$$

These centralizer algebras are finite dimensional [Jon83], and they form a planar algebra [Jon99].

Popa's reconstruction theorem

Popa axiomatized the standard invariant of a subfactor, and showed how to reconstruct a subfactor from an abstract standard invariant.

Theorem (Popa [Pop94])

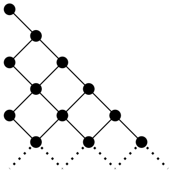
Every (strongly) amenable standard invariant is realized by a unique subfactor of R up to conjugacy.

Principal graphs

The complex $*$-algebras $P_{n,\pm}$ are all finite dimensional. The tower

$$P_{0,+} \subset P_{1,+} \subset P_{2,+} \subset \cdots$$

is described by its Bratteli diagram (and the trace).



Principal graphs

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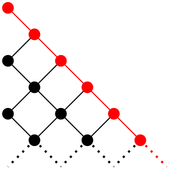
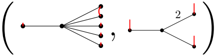
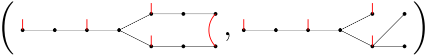
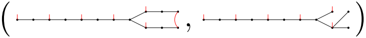


- The non-reflected part is the principal graph Γ_+.
- Get the dual principal graph Γ_- by looking at the Bratteli diagram for the tower $(P_{n,-})$.

Examples of principal graphs

- index < 4: A_n, D_{2n}, E_6, E_8. No D_{odd} or E_7.
- index $= 4$: $A^{(1)}_{2n-1}, D^{(1)}_{n+2}, E^{(1)}_6, E^{(1)}_7, E^{(1)}_8, A_\infty, A^{(1)}_\infty, D_\infty$
- Graphs for $R \subset R \rtimes G$ obtained from G and $\mathrm{Rep}(G)$.



- Haagerup 333 
- extended Haagerup 733 

- First graph is principal, second is dual principal.
- Leftmost vertex corresponds to $P_{0,\pm} \cong \mathbb{C}$ (factoriality).
- Red tags for duality of even vertices ($x \mapsto Jx^*J$).
- Duality of odd vertices by depth and height

Rep($A \subset B$)

Definition

The representation 2-category of $A \subset B$ is given by

- 0-morphisms: $\{A, B\}$
- 1-morphisms: bimodule summands of $L^2(A_k)$ for some $k \geq 0$
- 2-morphisms: intertwiners (elements of $A_0' \cap A_k$ and $A_1' \cap A_{k+1}$ for $k \geq 0$)

This 2-category is semi-simple, unitary, rigid (duals are well behaved), pivotal, sometimes spherical (iff $A \subset B$ extremal).

Theorem (Popa [Pop94])

There is a Tannaka-Krein like duality between (strongly) amenable subfactors and their representation 2-categories.

Principal graphs revisited

Let $X =_A L^2(B)_B$.

Definition

The principal graph Γ_+ has one vertex for each isomorphism class of simple ${}_AP_A$ and ${}_AQ_B$. There are

$$\dim(\mathrm{Hom}_{A-B}(P \otimes X, Q))$$

edges from P to Q.
The dual principal graph Γ_- is defined similarly using $B - B$ and $B - A$ bimodules.

- $\Gamma_\pm$ is pointed, where the base point is ${}_AL^2(A)_A$, ${}_BL^2(B)_B$ respectively.
- Duality is given by contragredient, which is always at the same depth, although duals at odd depths of $\Gamma_\pm$ are on $\Gamma_\mp$.

Fact

The dual graph of $A_0 \subset A_1$ is the principal graph of $A_1 \subset A_2$.

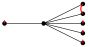
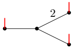
Finite depth

Definition

If the principal graph is finite, then the subfactor and standard invariant are called <u>finite depth</u>.

Example: $R \subset R \rtimes G$ for finite G

For $G = S_3$:

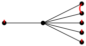
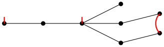
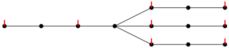
- Principal graph: 
- Dual principal graph: 

Supertransitivity

Definition

We say a principal graph is n-supertransitive if it begins with an initial segment consisting of the Coxeter-Dynkin diagram A_{n+1}, i.e., an initial segment with n edges.

Examples

-  is 1-supertransitive
-  is 2-supertransitive
-  is 3-supertransitive

Small index subfactor classification program

Steps of subfactor classifications:

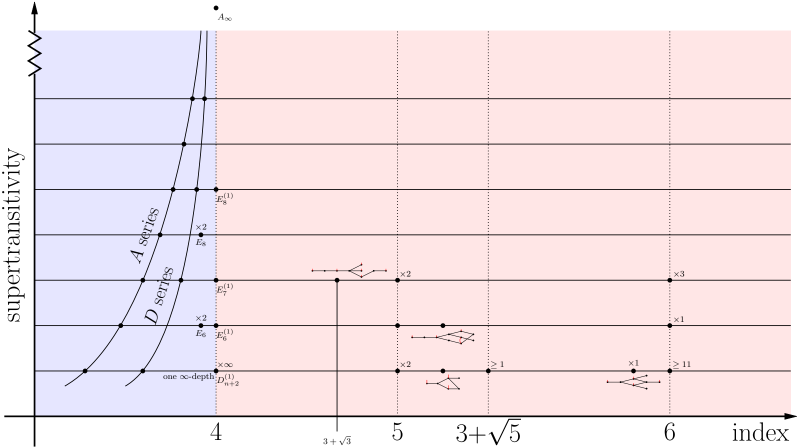
1. Enumerate graph pairs which survive obstructions.
2. Construct examples when graphs survive.

Fact (Popa [Pop94])

For a subfactor $A \subset B$, $[B : A] \geq \|\Gamma_+\|^2 = \|\Gamma_-\|^2$.

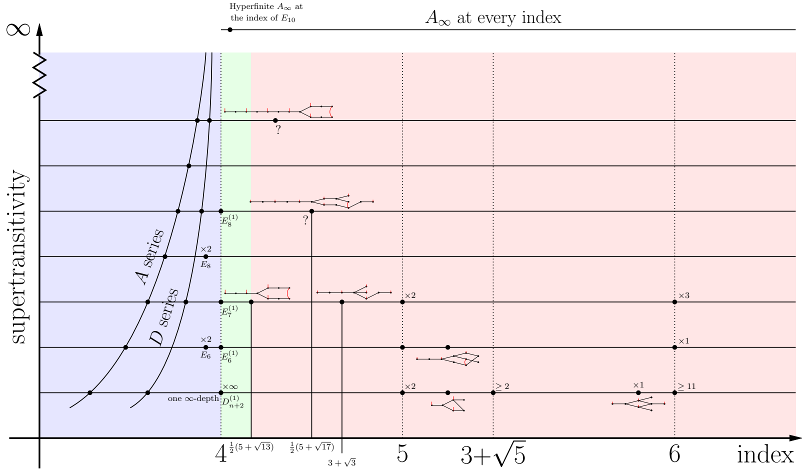
If we enumerate all graph pairs with norm at most r, we have found all principal graphs with index at most r^2.

Known small index subfactors, 1991



- ADE for index ≤ 4 (GHJ)
- No D_{odd}, E_7 (Ocneanu)
- Subgroup subfactors
- Composition (e.g., $\otimes$)
- Quantum groups (Wenzl)
- GHJ subfactors

Known small index subfactors, 1994

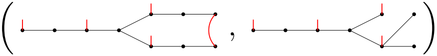
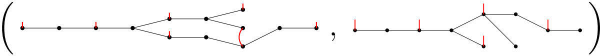
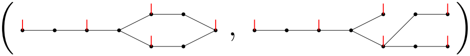


- Haagerup's partial classification to $3+\sqrt{3}$
- Popa's A_∞ at all indices
- Bisch-Haagerup example at $3+\sqrt{5}$

Haagerup's enumeration

Theorem (Haagerup [Haa94])

Any non A_∞-standard invariant in the index range $(4, 3+\sqrt{2})$ must have principal graphs a translation of one of

- 
- 
- 

Translation means raising the supertransitivity of both graphs by the same even amount.

Definition (Morrison-Snyder [MS12])

A vine is a graph pair which represents an infinite family of graph pairs obtained by translation.

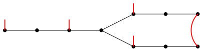
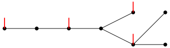
Main tools for Haagerup's enumeration

Play associativity off of Ocneanu's triple point obstruction.

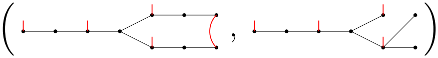
- Associativity: graphs must be similar
- Ocneanu's triple point obstruction: graphs must be different!

The consequence is a strong constraint.

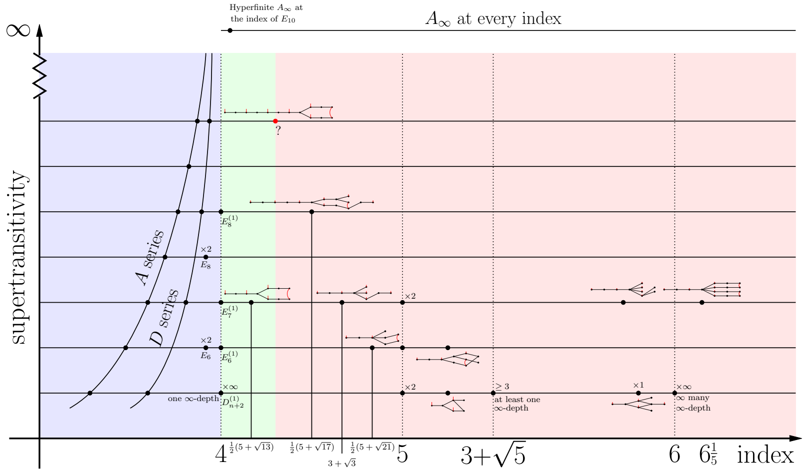
Example

 and  cannot be paired with themselves.

They must be paired with each other:

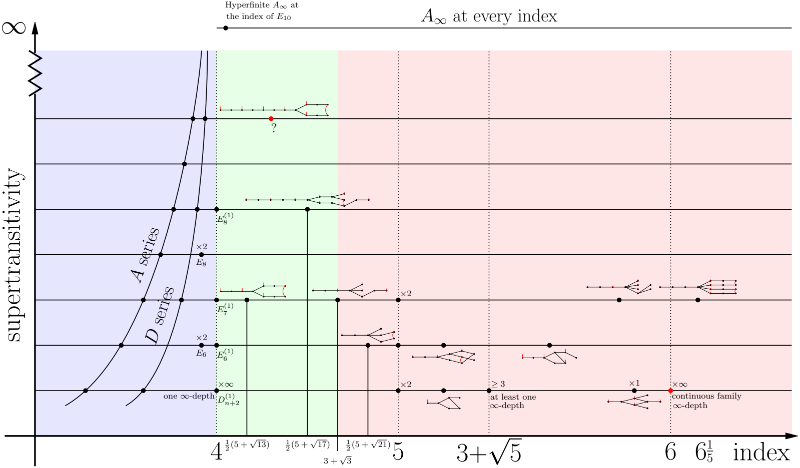


Known small index subfactors, 2001



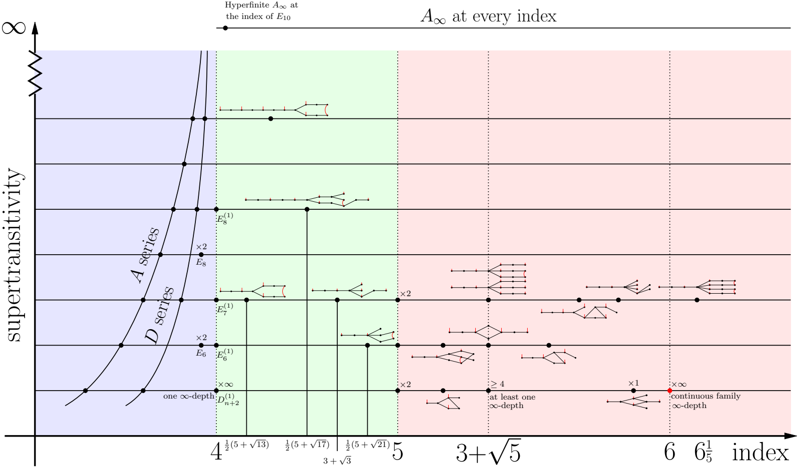
- Bisch-Haagerup
- Asaeda-Haagerup
- Izumi's Cuntz algebra examples
- Xu's examples from conformal inclusions
- Bisch-Jones Fuss-Catalan
- Bisch kills Hexagon vine

Known small index subfactors, 2007



- Asaeda-Yasuda eliminate Haagerup vine
- Haagerup + 1 (Grossman-Izumi)
- Bisch-Nicoara-Popa's continuous family with same standard invariant at index 6

Known small index subfactors, 2011



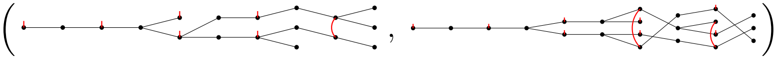
- Extended Haagerup
- Classification to index 5 (Izumi, Jones, Morrison, P, Peters, Snyder, Tener)
- Asaeda-Haagerup + 1 (Asaeda-Grossman)
- More Cuntz algebra examples from Izumi

Weeds and vines

The classification to index 5 introduced the terminology of weeds and vines.

Definition

A <u>weed</u> is a graph pair which represents an infinite family of graph pairs obtained by translation and extension.
An <u>extension</u> of a graph pair adds new vertices and edges at strictly greater depths than the maximum depth of any vertex in the original pair.

$\mathcal{F} =$ 

Using weeds allows us to bundle hard cases together, ensuring the enumerator terminates.

Eliminating vines with number theory

We can uniformly treat vines using number theory, based on the following theorem inspired by Asaeda-Yasuda [AY09]:

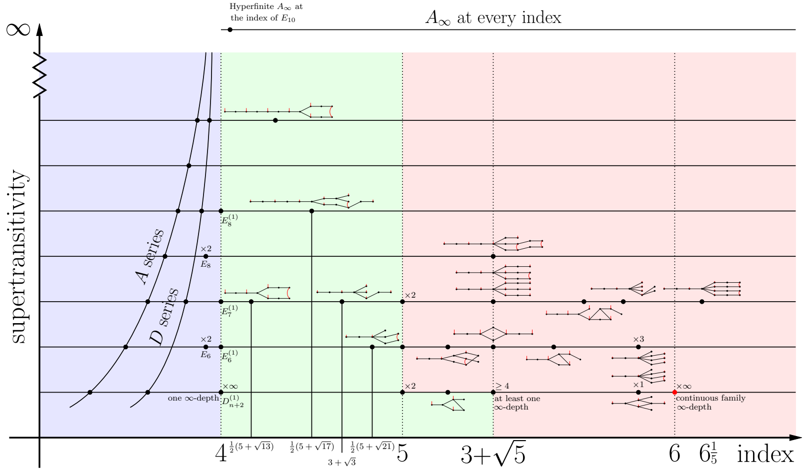
Theorem (Calegari-Morrison-Snyder [CMS11])

For a fixed vine $\mathcal{V}$, there is an effective (computable) constant $\mathcal{R}(\mathcal{V})$ such that any n-translate with $n > \mathcal{R}(\mathcal{V})$ has norm squared which is not a cyclotomic integer.

Theorem [CG94, ENO05]

The index of a finite depth subfactor (which is equal to the norm squared of the principal graph) must be a cyclotomic integer.

Known small index subfactors, 2012



- Morrison-Peters 1-supertransitive classification to $3+\sqrt{5}$
- 4442 (Morrison-P)
- Evans-Gannon near groups

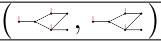
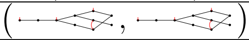
Why do we care about index $3+\sqrt{5}$?

- Standard invariants at index 4 are completely classified.
  - $\mathbb{Z}/2 * \mathbb{Z}/2 = D_\infty$ is amenable
- Standard invariants at index 6 are wild.
  - There is (at least) one standard invariant for every normal subgroup of the modular group $\mathbb{Z}/2 * \mathbb{Z}/3 = PSL(2,\mathbb{Z})$
  - There are unclassifiably many distinct hyperfinite subfactors with standard invariant $A_3 * D_4$ (Brothier-Vaes [BV13])
- $4 = 2 \times 2$ and $6 = 2 \times 3$ are composite indices, as is $3+\sqrt{5} = 2\tau^2$ where $\tau = \frac{1+\sqrt{5}}{2}$.

Index $(5, 3+\sqrt{5})$

Conjecture (Morrison-Peters [MP12b])

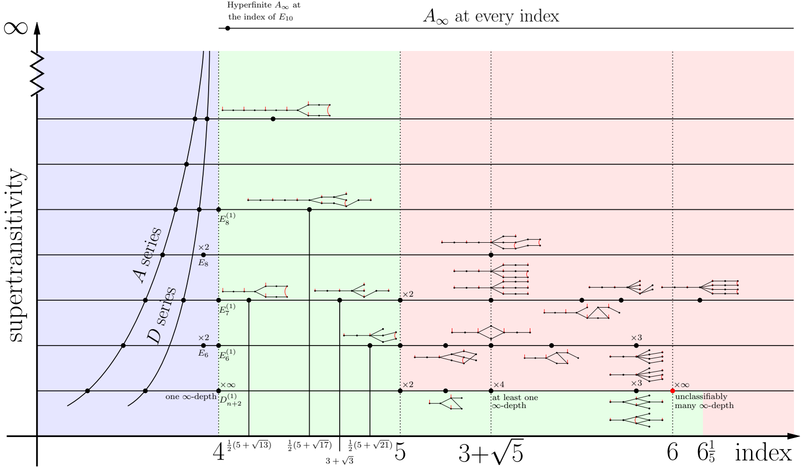
There are exactly two non-A_∞ standard invariants in the index range $(5, 3+\sqrt{5})$:

name	Principal graphs	Index	Existence, Uniqueness
$SU(2)_5$		5.04892	[Wen90], [MP12b]
$SU(3)_4$		5.04892	[Wen88], [MP12b]

Theorem [Morrison-Peters [MP12b]]

There is exactly one 1-supertransitive subfactor in the index range $(5, 3+\sqrt{5})$

Known small index subfactors, 2013

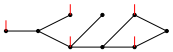
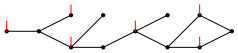
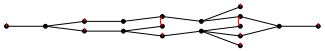
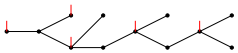
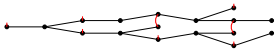


- Brothier-Vaes unclassifiably many subfactors with standard invariant $A_3 * D_4$ at index 6
- Liu classified composite standard invariants from A_3 and A_4
- 1-supertransitive classification to index $6\frac{1}{5}$ (Liu-Morrison-P)

1-supertransitive subfactors at index $3+\sqrt{5}$

Theorem (Liu [Liu13a], partial proof by [IMP13])

There are exactly seven 1-supertransitive standard invariants with index $3+\sqrt{5}$:

- ( , ) self-dual
- ( , ) and its dual
- ( , ) and its dual
- ( $\cdots$,  $\cdots$) and its dual ($A_3 * A_4$)

These are all the standard invariants of composed inclusions of A_3 and A_4 subfactors.

Open question

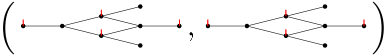
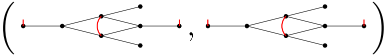
How many hyperfinite subfactors have Bisch-Jones' Fuss-Catalan $A_3 * A_4$ standard invariant at index $3+\sqrt{5}$?

- $A_3 * A_4$ and $A_2 * T_2$ are not amenable [Pop94, HI98].

1-supertransitive with index at most $6\frac{1}{5}$

Theorem (Liu-Morrison-P [LMP13])

An exactly 1-supertransitive standard invariant with index at most $6\frac{1}{5}$ either comes from a composed inclusion (and has index $3+\sqrt{5}$ or 6), or is one of 3 self-dual standard invariants at index $3+2\sqrt{2}$:

- 
-  two complex conjugate

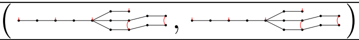
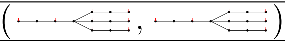
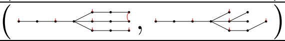
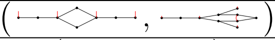
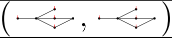
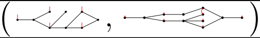
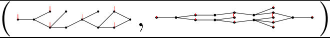
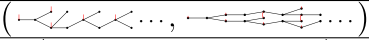
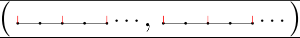
This result uses Liu's virtual normalizers for 1-supertransitive subfactors [Liu13b] (generalization of [PP86]), which force existence of intermediate subfactors.

- Can push classification results above index 6!
- Could hope that the only wildness at index 6 is "group-like"

Standard invariants at index $3 + \sqrt{5}$

Conjecture (Morrison-P [MP14])

At $3 + \sqrt{5}$, we have only the following standard invariants:

name	Principal graphs	#	Existence/Uniqueness
4442		1	[MP12a, MP14], Izumi
$3^{\mathbb{Z}/2\times\mathbb{Z}/2}$		1	Izumi, [MP12a]
$3^{\mathbb{Z}/4}$		2	Izumi, [PP13]
$2D2$		2	Izumi, [MP14]
$A_3 \otimes A_4$		1	$\otimes$, [Liu13a, IMP13]
fish 2		2	BH, [Liu13a, IMP13]
fish 3		2	[IMP13, Liu13a]
$A_3 * A_4$		2	[BJ97],
A_∞		1	[Pop93]

- 1-supertransitive case known by [Liu13a, IMP13, LMP13]

Methods to push classification results further

- 1-supertransitive classification to $6\frac{1}{5}$ [LMP13]
- Popa's principal graph stability [Pop95, BP14] $\longrightarrow$ cylinders
- New number theory approach to cylinders (Calegari-Guo)
- New high-tech graph pair enumerator, based on Brendan McKay's isomorph free enumeration by canonical construction paths. Two independent implementations, same results. (Afzaly-Morrison-P)
- Tail enumerator for periodic graphs (Afzaly-Morrison-P)
- The non-initial triple point obstruction [Haa94]
- New general initial triple point obstruction [Pen13]

Popa's principal graph stability

Definition

We say $\Gamma_\pm$ is stable at depth n if every vertex at depth n connects to at most one vertex at depth $n+1$, no two vertices at depth n connect to the same vertex at depth $n+1$, and all edges between depths n and $n+1$ are simple.

Theorem (Popa [Pop95], Bigelow-P [BP14])

Suppose $A \subset B$ (finite index) has principal graphs (Γ_+, Γ_-). Suppose that the truncation $\Gamma_\pm(n+1) \neq A_{n+2}$ and $\delta > 2$.

(1) If $\Gamma_\pm$ are stable at depth n, then $\Gamma_\pm$ are stable at depth k for all $k \geq n$, and $\Gamma_\pm$ are finite.

(2) If Γ_+ is stable at depths n and $n+1$, then $\Gamma_\pm$ are stable at depth $n+1$.

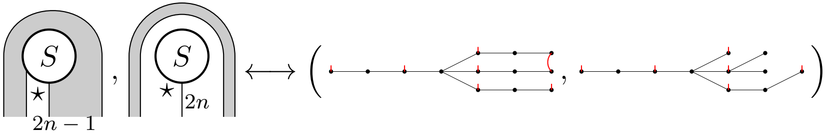
Part (2) uses the 1-click rotation in the planar algebra.

Bigelow's jellyfish algorithm

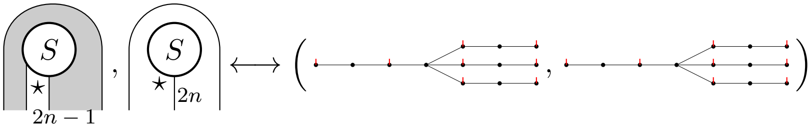
First used by Bigelow-Morrison-Peters-Snyder to construct extended Haagerup [BMPS12].

Theorem (Bigelow-P [BP14])

- $P_\bullet$ has 2-strand jellyfish relations $\Leftrightarrow$ one graph is a spoke.



- $P_\bullet$ has 1-strand jellyfish relations $\Leftrightarrow$ both graphs are spokes.



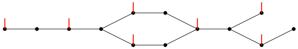
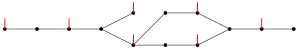
Theorem (Morrison-P [MP14])

A variation of the jellyfish algorithm is universal for finite depth subfactor planar algebras.

Cylinders

Definition

A <u>cylinder</u> is a graph pair which represents an infinite family of graph pairs obtained by translation and finite stable extension.

$\mathcal{C} = \Big($  ,  $\Big)$

Theorem (Guo)

Let $\mathcal{S}_M$ be the class of finite graphs satisfying:

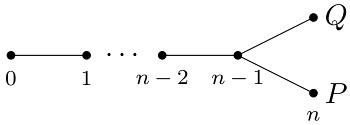
1. all vertices have valence at most M, and
2. at most M vertices have valence > 2.

Then ignoring A_n, D_n, $A_n^{(1)}$, and $D_n^{(1)}$, only finitely many graphs in $\mathcal{S}_M$ have norm squared which is a cyclotomic integer.

- This result is in principle effective, but not yet practical.
- Calegari-Guo eliminate our troublesome cylinder $\mathcal{C}$ by hand.

New triple point obstruction

Suppose $A \subset B$ (finite index) has principal graphs (Γ_+, Γ_-) starting with a triple point:



Theorem [Pen13]

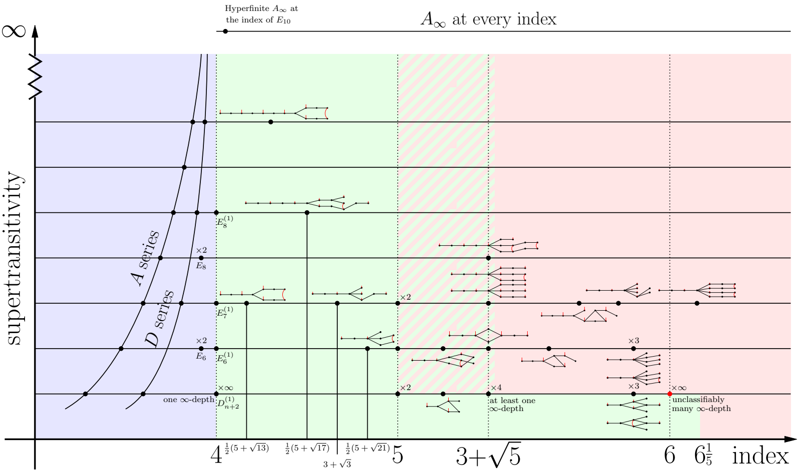
Suppose that for each R at depth $n+1$ connected to P, there is a unique vertex $E(\overline{R})$ at depth n connected to the dual vertex $\overline{R}$ of R. Then there is an explicit formula for $\sigma_A + \sigma_A^{-1}$ in terms of the traces of the projections of $\Gamma_\pm$ with depth at most $n+1$.

Here, σ_A is the <u>chirality</u>, related to 1-click rotation.

Moral:

This formula gives the chirality, which is a priori hidden in the planar algebra structure, in terms of visible combinatorial data of the principal graph.

Known small index subfactors, today



Theorem (Afzaly-Morrison-P)

The conjectures of Morrison-Peters (up to index $5\frac{1}{4} > 3+\sqrt{5}$) and Morrison-P hold with at most finitely many exceptions.

Thank you for listening!

Slides available at
`http://www.math.ucla.edu/~dpenneys/PenneysUCLA2014.pdf`

My recent articles referenced herein:

- with Liu and Morrison - 1-supertransitive below $6\frac{1}{5}$ - to appear **Comm. Math. Phys.** - `arXiv:1310.8566`
- with Bigelow - Spokes and jellyfish - **Math. Ann.** - `MR3157990`
- with Morrison - Constructing spokes with 1-strand jellyfish - to appear **Trans. AMS** - `arXiv:1208.3637`
- with Peters - Constructing spokes with 2-strand jellyfish - Submitted - `arXiv:1308.5197`
- with Izumi and Morrison - 1-supertransitive at $3+\sqrt{5}$ - Submitted - `arXiv:1308.5723`
- new obstruction - Submitted - `arXiv:1307.5890`
- with Izumi, Morrison, Peters, and Snyder - Subfactors of index exactly 5 - Submitted - `arXiv:1406.2389`
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