Classifying small index subfactors

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Invariants of subfactors

$$
\begin{tikzcd}[column sep=huge]
\begin{array}{c}\text{subfactor}\\ A \subset B\end{array} \arrow[r, bend left] & \begin{array}{c}\text{planar algebra}\\ (P_+, P_-)\end{array} \arrow[l, bend left] \arrow[r, bend left] & \begin{array}{c}\text{principal graphs}\\ (\Gamma+, \Gamma_-)\end{array} \arrow[l, bend left, dotted, "??"]
\end{tikzcd}
$$

II_1-factors

Definitions

- A von Neumann algebra is a $*$-closed subalgebra $M \subseteq B(H)$ such that $M = M''$.
- A factor is a von Neumann algebra with trivial center $Z(M) = M' \cap M = \mathbb{C}1$.
- A II_1-factor M is an infinite dimensional factor with a tracial state $\mathrm{tr} : M \to \mathbb{C}$.
- A subfactor is a unital inclusion of factors.

Our subfactors will be II_1-subfactors.

Index for subfactors

Definition

Given a II_1-subfactor $A \subset B$, we say it has finite index if B is a finitely generated projective A-module. The index $[B : A]$ is the trace of the corresponding idempotent in $K_0(A)^+$.

Theorem [Jon83]

For a II_1-subfactor $A \subset B$,

$$[B\colon A] \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \middle| n = 3, 4, \ldots \right\} \cup [4, \infty].$$

Where do subfactors come from?

Some examples include:

- Groups – from $G \curvearrowright R$, we get $R^G \subset R$ and $R \subset R \rtimes_\alpha G$.
- finite dimensional unitary Hopf/Kac algebras
- Quantum groups
- Conformal field theory
- endomorphisms of Cuntz C*-algebras

However, there are certain possible infinite families without uniform constructions.

Remark

Just as von Neumann algebras come in pairs (M, M'), subfactors come in pairs $(A \subset B, B' \subset A')$.

The standard invariant

Definition

The standard invariant of $A \subset B$ is the the following unitary 2-category:

- Objects: A and B
- 1-Morphisms: bimodules generated by ${}_A B_B$ and ${}_B B_A$ (take tensor products and decompose into irreducible summands)
- 2-Morphisms: bimodule intertwiners
- unitary structure: Adjoint on A, B is identity. Adjoint on bimodules is the contragredient, giving an involution on $A - A$ and $B - B$ bimodules, but swaps $A - B$ and $B - A$ bimodules. Adjoint on intertwiners is usual adjoint.

Choosing our favorite 1-morphism ${}_A B_B$ gives a planar algebra.

Principal graphs

Definition

Given the 1-morphism ${}_A B_B$, we define the principal graph of $A \subset B$ as follows.

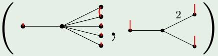
- Even vertices: isomorphism classes of simple $A-A$ bimodules.
- Odd vertices: isomorphism classes of simple $A-B$ bimodules.
- Edges: $\dim(\mathrm{Hom}_{A-B}(P \otimes_A B, Q))$ unoriented edges from ${}_A P_A$ to ${}_A Q_B$.

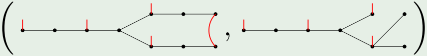
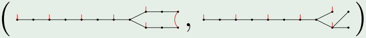
- Get the dual principal graph by looking at ${}_B B_A$ together with $B-B$ and $B-A$ bimodules.
- Can define the fusion graph with respect to any simple bimodule.

Examples

Examples of principal graphs

- index < 4: A_n, D_{2n}, E_6, E_8. No D_{odd} or E_7.
- index $= 4$: $A^{(1)}_{2n-1}, D^{(1)}_{n+2}, E^{(1)}_6, E^{(1)}_7, E^{(1)}_8, A_\infty, A^{(1)}_\infty, D_\infty$
- Graphs for $R \subset R \rtimes G$ obtained from G and Rep(G).



- Haagerup 333 
- extended Haagerup 733 

- First graph is principal, second is dual principal.
- Leftmost vertex is the trivial bimodule ${}_AA_A$, ${}_BB_B$ resp.
- Red tags for duality of $A-A$ and $B-B$ vertices.
- Duality of $A-B$ to $B-A$ is by depth and height

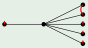
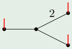
Finite depth

Definition

If the principal graph is finite, then the subfactor and standard invariant/planar algebra are called finite depth.

Example: $R \subset R \rtimes G$ for finite G

For $G = S_3$:

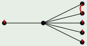
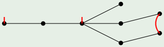
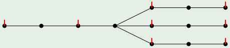
- Principal graph: 
- Dual principal graph: 

Supertransitivity

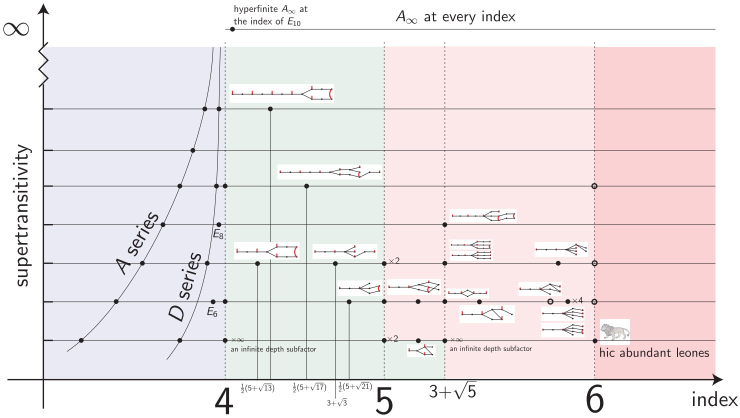
Definition

We say a principal graph is n-supertransitive if it begins with an initial segment consisting of the Coxeter-Dynkin diagram A_{n+1}, i.e., an initial segment with n edges.

Examples

-  is 1-supertransitive
-  is 2-supertransitive
-  is 3-supertransitive

Known subfactors



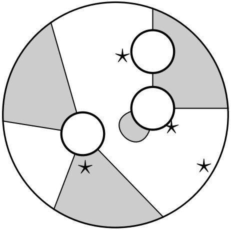
- Recent classification to index 5 (contributed to parts 2 and 4) [MS12, MPPS12, IJMS12, PT12]
- Map of known subfactors from Jones-Morrison-Snyder survey [JMS13], to appear Bulletin AMS.

Planar algebras [Jon99]

Definition

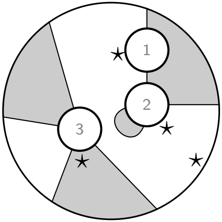
A shaded planar tangle has

- a finite number of inner boundary disks
- an outer boundary disk
- non-intersecting strings
- a marked interval $\star$ on each boundary disk
- a checkerboard shading

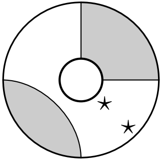


Composition of tangles

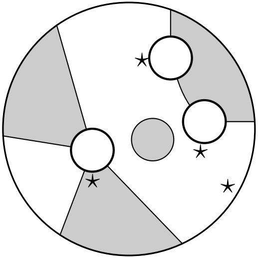
We can compose planar tangles by insertion of one into another if the number of strings matches up:



$\circ_2$



$=$



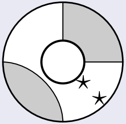
Definition

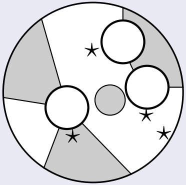
The *shaded planar operad* consists of all shaded planar tangles (up to isotopy) with the operation of composition.

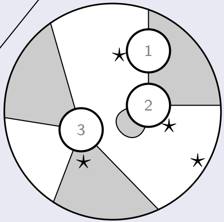
Definition

A *planar algebra* is a family of vector spaces $P_{k,\pm}$, $k = 0, 1, 2, \ldots$ and an action of the shaded planar operad.

$$\begin{tikzcd}
P_{2,-} \times P_{1,+} \times P_{1,+} \arrow[rr] \arrow[dr] & & P_{3,+} \\
& P_{2,-} \times P_{2,+} \times P_{1,+} \arrow[ur] &
\end{tikzcd}$$





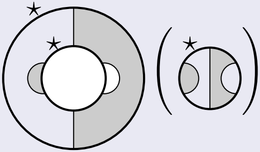
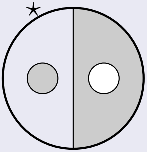
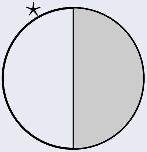


Example: Temperley-Lieb

$TL_{n,\pm}(\delta)$ is the complex span of non-crossing pairings of $2n$ points arranged around a circle, with formal addition and scalar multiplication.

$TL_{3,+}(\delta) = \mathrm{Span}_{\mathbb{C}} \Big\{$  $\Big\}$.

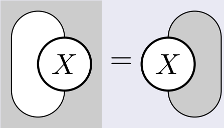
Planar tangles act on TL by inserting diagrams into empty disks, smoothing strings, and trading closed loops for factors of δ.

 $=$  $= \delta^2$ 

Subfactor planar algebras

Definition

A planar algebra $P_\bullet$ is a subfactor planar algebra if it is:

- Finite dimensional: $\dim(P_{k,\pm}) < \infty$ for all k
- Evaluable: $P_{0,\pm} \cong \mathbb{C}$ by sending the empty diagram to $1_{\mathbb{C}}$
- Sphericality: 
- Positivity: each $P_{k,\pm}$ has an adjoint $*$ such that the sesquilinear form $\langle x, y\rangle := \mathrm{Tr}(y^* x)$ is positive definite

From these properties, it follows that closed circles count for a multiplicative constant $\delta \in \{2\cos(\pi/n) | n \geq 3\} \cup [2, \infty)$.

Planar algebras from tensor categories

Definition

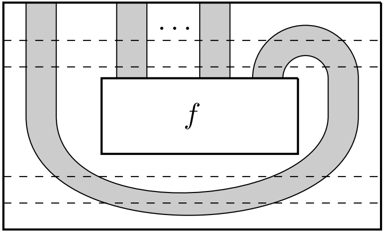
Given a unitary fusion category and a choice of simple object X, we get a planar algebra by setting

$$P_{n,+} = \mathrm{Hom}(1, (X \otimes \overline{X})^{\otimes n}) \text{ and } P_{n,-} = \mathrm{Hom}(1, (\overline{X} \otimes X)^{\otimes n})$$

- The strand is the identity 1-morphsim: $\mathrm{id}_X = |$ and $\mathrm{id}_{\overline{X}} = |$
- Caps are evaluation $\mathrm{ev}_X = \frown$ and $\mathrm{ev}_{\overline{X}} = \frown$
- Cups are coevaluation $\mathrm{coev}_X = \smile$ and $\mathrm{coev}_{\overline{X}} = \smile$
- Vertical join is composition $gf = \begin{array}{c}\boxed{g}\\ \boxed{f}\end{array}$
- Horizontal join is tensor product $f \otimes g = \boxed{f}\,\boxed{g}$

An example tangle

If $f \in P_{n,+} = \mathrm{Hom}(1, (X \otimes \overline{X})^{\otimes n})$, the tangle below is a composite map, read from bottom to top:



$$\mathrm{id}_{(X\otimes\overline{X})^{\otimes n}} \otimes \mathrm{ev}_{\overline{X}}$$

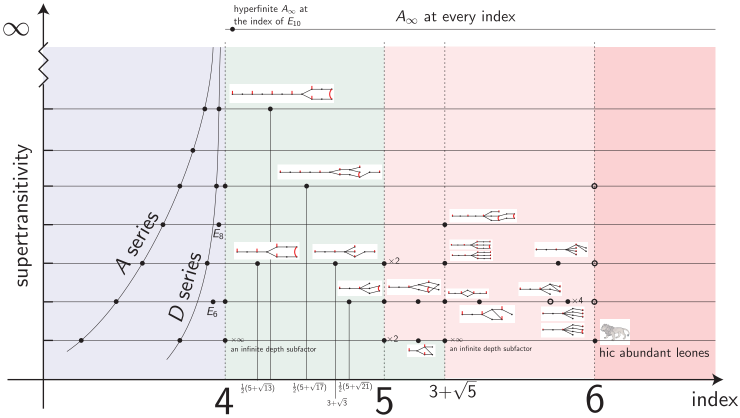
$$\mathrm{id}_{(X\otimes\overline{X})^{\otimes n}\otimes X} \otimes \mathrm{ev}_X \otimes \mathrm{id}_{\overline{X}}$$

$$\mathrm{id}_{X\otimes\overline{X}} \otimes f \otimes \mathrm{id}_{X\otimes\overline{X}}$$

$$\mathrm{id}_X \otimes \mathrm{coev}_{\overline{X}} \otimes \mathrm{id}_{\overline{X}}$$

$$\mathrm{coev}_X$$

Small index subfactor classification program



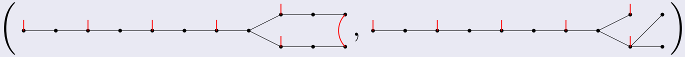
Focuses of the classification program:

- Enumerate graph pairs and apply obstructions.
- Construct examples when graphs survive.
- Place exotic examples into families.

The extended Haagerup subfactor

Bigelow-Morrison-Peters-Snyder, [BMPS12]

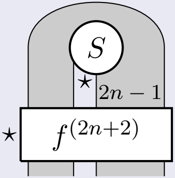
The extended Haagerup subfactor is the unique subfactor with principal graphs

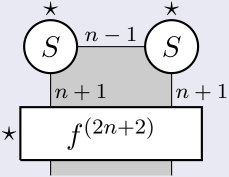


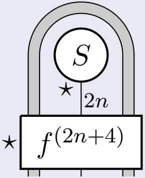
- Last remaining possible graph in Haagerup's classification to $3+\sqrt{3}$ [Haa94] by work of Asaeda-Yasuda [AY09].
- Largest known supertransitivity outside the A and D series.
- Its planar algebra was constructed using Bigelow's jellyfish algorithm.

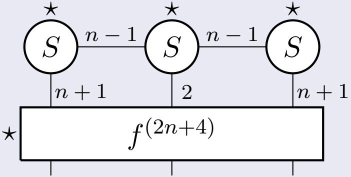
Bigelow-Morrison-Peters-Snyder, [BMPS12]

The Haagerup and extended Haagerup subfactor planar algebras have a generator $S \in P_{n,+}$ where $n = 4, 8$ respectively satisfying:

- 

$= i\dfrac{\sqrt{[n][n+2]}}{[n+1]}$


- 

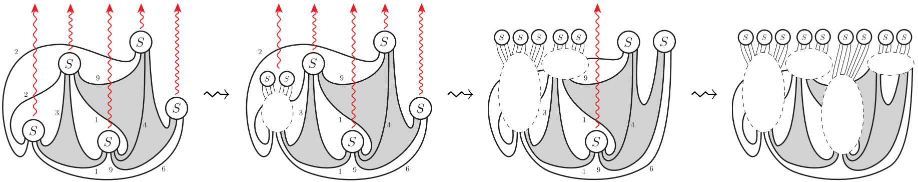
$= \dfrac{[2][2n+4]}{[n+1][n+2]}$


- (Absorption) capping S gives zero and $S^2 = f^{(n)} \in TL_{n,+}$.

The jellyfish algorithm

We can evaluate all closed diagrams as follows:

1. First, pull all generators to the outside using the jellyfish relations



2. Second, reduce the number of generators using the capping and absorption (multiplication) relations.

Consistency and positivity

Theorem [Jones-Penneys [JP11], Morrison-Walker]

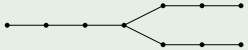
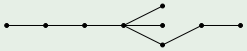
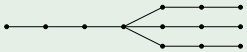
Every subfactor planar algebra embeds in the graph planar algebra of its principal graph.

This serves two purposes:

1. To show the planar algebra is non-zero, give a representation.
2. Graph planar algebras are always finite dimensional, spherical, and positive. Only need to check evaluable.

Spoke graphs

Examples of spoke principal graphs

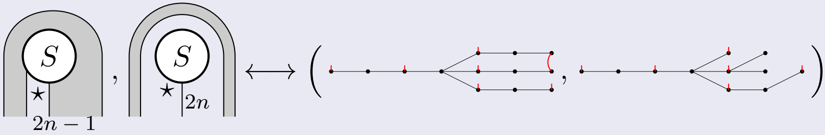
- $A_n, D_{2n}, E_6, E_8,$
- $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$
- $A_\infty, A_\infty^{(1)}, D_\infty$
- Principal graphs for $R \subset R \rtimes G$, G finite (, )
- 2221 
- Haagerup 333 
- 3311 
- 3333 
- 4442 
- extended Haagerup 733 

Spokes and jellyfish

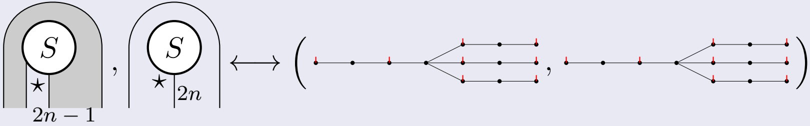
Assume all generators of $P_\bullet$ are at the same depth n.

Theorem [Bigelow-Penneys [BP14]]

- $P_\bullet$ has 2-strand jellyfish relations $\Leftrightarrow$ one graph is a spoke.



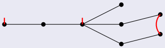
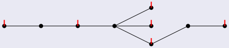
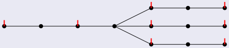
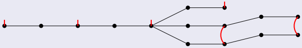
- $P_\bullet$ has 1-strand jellyfish relations $\Leftrightarrow$ both graphs are spokes.



Constructing spoke subfactors with jellyfish

Theorem [Morrison-Penneys [MP13]]

We automate finding 1-strand relations for these subfactors:

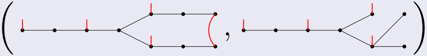
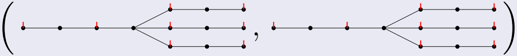
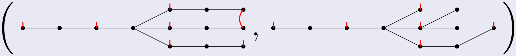
- Izumi-Xu 2221  [Han11]
- [GdlHJ89] 3311 
- Izumi $3^{\mathbb{Z}/2\times\mathbb{Z}/2}$  (index $3+\sqrt{5}$)
- 4442  (index $3+\sqrt{5}$)

For the above, both principal graphs are the same spoke graph.

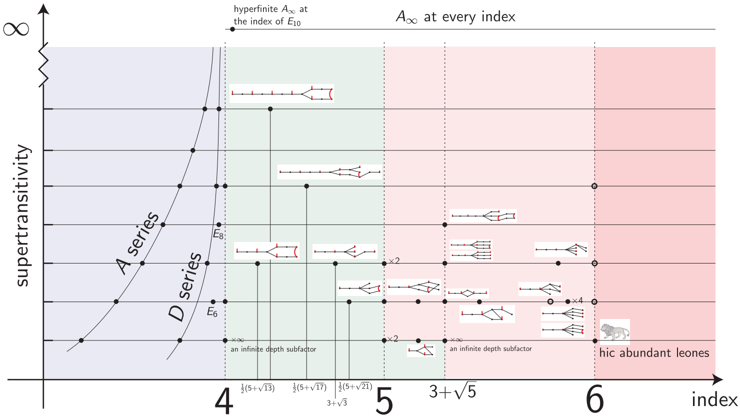
Constructing spoke subfactors with jellyfish, part 2

Theorem [Penneys-Peters [PP13]]

We give explicit 2-strand relations for the following subfactors:

- Haagerup 333 $(\mathbb{Z}/3)$ 
- 3333 $(\mathbb{Z}/2 \times \mathbb{Z}/2)$ 
- 3333 $(\mathbb{Z}/4)$ 

Small index subfactor classification program



Focuses of the classification program:

- Enumerate graph pairs and apply obstructions.
- Construct examples when graphs survive.
- Place exotic examples into families.

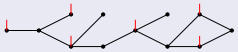
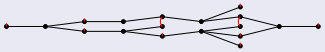
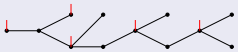
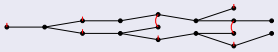
Why do we care about index $3+\sqrt{5}$?

- Standard invariants at index 4 are completely classified.
  - $\mathbb{Z}/2 * \mathbb{Z}/2 = D_\infty$ is amenable
- Standard invariants at index 6 are wild.
  - There is (at least) one standard invariant for every normal subgroup of the modular group $\mathbb{Z}/2 * \mathbb{Z}/3 = PSL(2, \mathbb{Z})$
  - There are unclassifiably many distinct hyperfinite subfactors with the same standard invariant [BNP07, BV13]
- $4 = 2 \times 2$ and $6 = 2 \times 3$ are composite indices, as is $3+\sqrt{5} = 2\tau^2$ where $\tau = \frac{1+\sqrt{5}}{2}$.

1-supertransitive subfactors at index $3+\sqrt{5}$

Theorem [Liu13], partial proof by Izumi-Morrison-Penneys [IMP13]

There are exactly seven 1-supertransitive subfactor planar algebras with index $3+\sqrt{5}$:

- ( , ) self-dual
- ( , ) and its dual
- ( , ) and its dual
- ( $\cdots$,  $\cdots$) and its dual

These are all the standard invariants of composed inclusions of A_3 and A_4 subfactors.

Open question

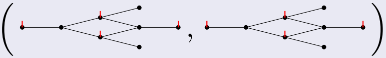
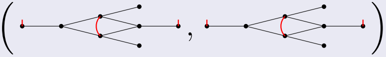
Are there infinitely many distinct hyperfinite subfactors with the same standard invariant at index $3+\sqrt{5}$?

- $A_3 * A_4$ and $A_2 * T_2$ are not amenable [HI98].

1-supertransitive with index at most $6\frac{1}{5}$

Theorem [Liu-Morrison-Penneys [LMP13]]

An exactly 1-supertransitive subfactor planar algebra with index at most $6\frac{1}{5}$ either comes from a composed inclusion (and has index $3+\sqrt{5}$ or 6), or is one of 3 self-dual planar algebras at index $3+2\sqrt{2}$:

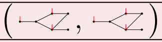
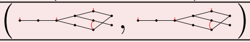
- 
-  two complex conjugate

- Can push classification results above index 6!
- Could hope that the only wildness at index 6 is "group-like"

Index $(5, 3+\sqrt{5})$

Conjecture [Morrison-Peters] [MP12]

There are exactly two non Temperley-Lieb subfactor planar algebras in the index range $(5, 3+\sqrt{5})$:

name	Principal graphs	Index	Constructed
$SU(2)_5$		5.04892	[Wen90], [MP12]
$SU(3)_4$		5.04892	[Wen88], [MP12]

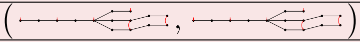
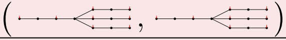
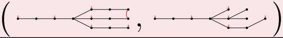
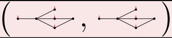
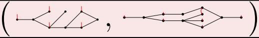
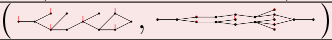
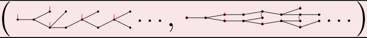
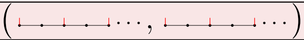
Theorem [Morrison-Peters] [MP12]

There is exactly one 1-supertransitive subfactor in the index range $(5, 3+\sqrt{5})$

Subfactor planar algebras at index $3+\sqrt{5}$

Conjecture [Morrison-Penneys]

At $3+\sqrt{5}$, we have only the following subfactor planar algebras:

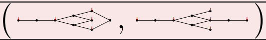
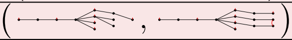
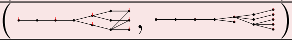
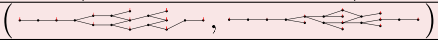
name	Principal graphs	#	Constructed
4442		2	[MP13], Izumi
$3^{\mathbb{Z}/2\times\mathbb{Z}/2}$		2	Izumi, [MP13]
$3^{\mathbb{Z}/4}$		2	Izumi, [PP13]
$2D2$		2	Izumi, [MPP]
$A_3\otimes A_4$		1	$\otimes$
fish 2		2	BH
fish 3		2	[IMP13]
$A_3 * A_4$		2	[BJ97]
A_∞		1	[Pop93]

- The 1-supertransitive case is known by [Liu13, IMP13]

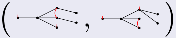
Subfactor planar algebras at index 6

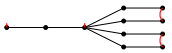
Wildly optimistic conjecture [Penneys-Peters-Snyder]

At index 6, a ≥ 2-supertransitive subfactor planar algebra is one of

name	Principal graphs
$A_5 b$	
$S_5 b$	
$A_5 \subset A_6$	
$S_5 \subset S_6$	

Theorem [LMP13]

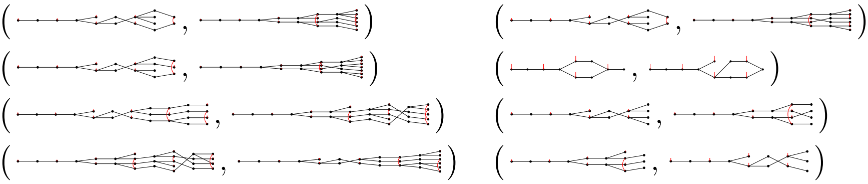
If a subfactor has principal graphs an extension of , then it is a Bisch-Haagerup subfactor of the form $R^{\mathbb{Z}/2} \subset R \rtimes \mathbb{Z}/3$.

- We do not yet understand composed inclusions of A_3 and A_5.
- No subfactor with principal graph  [EG12].

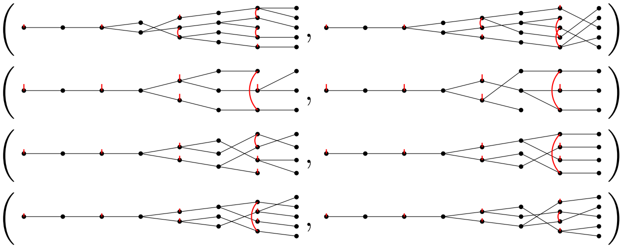
How can we prove these conjectures?

Biggest hurdle: need to eliminate certain weeds.

$*10$ weeds:



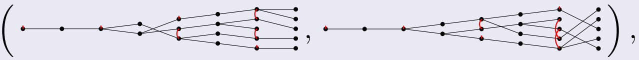
$*11$ weeds:



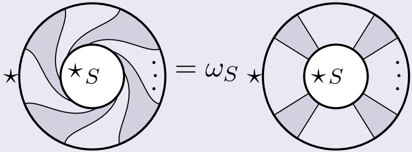
Triple points

Fact

If the graph starts with a triple point at depth $n-1$, e.g.



then the planar algebra has an uncappable rotational eigenvector at depth n with eigenvalue ω_S where $\omega_S^n = 1$.

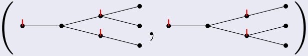


- If there is no merging two past the branch, we get a strong constraint in terms of the structure of the graph and ω_S.

New obstruction

Theorem [Pen13]

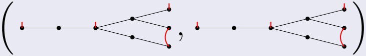
(1) If (Γ_+, Γ_-) is a translated extension of



, then

$$(\check{r}-1)\frac{r}{\check{r}} + \frac{(\sigma_S + \sigma_S^{-1})}{[n]}\frac{\sqrt{r}}{\sqrt{\check{r}}} = \frac{r[n]-[n+2]}{[n]}.$$

(2) If (Γ_+, Γ_-) is a translated extension of



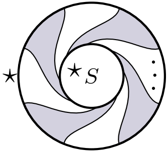
, then

$$(r-1) + \frac{(\sigma_S + \sigma_S^{-1})}{[n]} = \frac{[n+2]-r[n]}{r[n]}.$$

σ_S is the chirality (σ_S^2 is rotational eigenvalue)

$r, \check{r}$ are the branch factors (ratio of dimensions past branch)

Remaks on the new obstruction

- The key is the rotation 

- The obstruction is far more general. Recovers results of Jones and Snyder.
- Key relation in the proof due to Liu, which is a variant of a relation due to Wenzl.
- This obstruction eliminates the $*11$ weeds
- Can obtain rotational eigenvalues for most small index subfactors.
- Gives easy non-existence result for D_{odd} and E_7.

Thank you for listening!

Recent articles:

- with Bigelow - Spokes and jellyfish - **Math. Ann.** - MR3157990
- with Morrison - Constructing spokes with 1-strand jellyfish - to appear **Trans. AMS** - arXiv:1208.3637
- with Peters - Constructing spokes with 2-strand jellyfish - Submitted - arXiv:1308.5197
- with Izumi and Morrison - 1-supertransitive at $3+\sqrt{5}$ - Submitted - arXiv:1308.5723
- with Liu and Morrison - 1-supertransitive below $6\frac{1}{5}$ - Submitted - arXiv:1310.8566
- new obstruction - Submitted - arXiv:1307.5890

- Marta Asaeda and Seidai Yasuda, *On Haagerup's list of potential principal graphs of subfactors*, Comm. Math. Phys. **286** (2009), no. 3, 1141–1157, MR2472028 DOI:10.1007/s00220-008-0588-0 arXiv:0711.4144.
- Dietmar Bisch and Vaughan F. R. Jones, *Algebras associated to intermediate subfactors*, Invent. Math. **128** (1997), no. 1, 89–157, MR1437496.
- Stephen Bigelow, Scott Morrison, Emily Peters, and Noah Snyder, *Constructing the extended Haagerup planar algebra*, Acta Math. **209** (2012), no. 1, 29–82, MR2979509, arXiv:0909.4099, DOI:10.1007/s11511-012-0081-7. MR 2979509
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Masaki Izumi, Vaughan F. R. Jones, Scott Morrison, and Noah Snyder, *Subfactors of index less than 5, Part 3: Quadruple points*, Comm. Math. Phys. **316** (2012), no. 2, 531–554, MR2993924, arXiv:1109.3190, DOI:10.1007/s00220-012-1472-5. MR 2993924

Masaki Izumi, Scott Morrison, and David Penneys, *Fusion categories between $C \boxtimes D$ and $c * d$.*, 2013, arXiv:1308.5723.

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- Zhengwei Liu, *Composed inclusions of a_3 and a_4 subfactors*, 2013, arXiv:1308.5691.
- Zhengwei Liu, Scott Morrison, and David Penneys, *1-supertransitive subfactors with index at most $6\frac{1}{5}$*, 2013, arXiv:1310.8566.
- Scott Morrison and Emily Peters, *The little desert? Some subfactors with index in the interval $(5, 3+\sqrt{5})$*, 2012, arXiv:1205.2742.
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- Scott Morrison, David Penneys, and Emily Peters, *Equivariantizations and 3333 spoke subfactors at index $3+\sqrt{5}$*, In preparation.

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