Introduction to the classification program for subfactors

Topological Quantum Groups, C*-Tensor Categories, and Subfactors

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Introduction

This mini-course will focus on the theory of II_1-subfactors and unitary tensor categories. The goal of the first talk will be to prepare everyone for many talks throughout the workshop on subfactors and unitary tensor categories. The goal of the second talk will be to go through many equivalent notions of the standard invariant of a subfactor, by example. Finally, the third talk will focus on classification and construction techniques for the small index classification program.

1 II_1-subfactors and unitary tensor categories

In this section, we define the notions of II_1 subfactor and unitary tensor category. We then discuss how a finite index (extremal) II_1 subfactor $A \subset B$ is the same data as a triple $(\mathcal{C}, Q, F)$ where $\mathcal{C}$ is a unitary tensor category, Q is a suitably nice algebra object in $\mathcal{C}$ (namely a simple normalized Q-system), and $F : \mathcal{C} \to \mathsf{Bim}(A)$ is a unitary tensor functor.

In this lecture, H will denote a separable Hilbert space.

1.1 von Neumann algebras and factors

Definition 1.1. A *von Neumann algebra* is a unital $*$-closed subalgebra $A \subseteq B(H)$ which is closed in the topology of *pointwise convergence*, i.e., $a_i \to a$ if and only if $a\xi \to a\xi$ for all $\xi \in H$. By von Neumann's Bicommutant Theorem, this property is equivalent to $A = A''$, where for a subset $S \subset B(H)$,

$$S' = \{x \in B(H) | xs = sx \text{ for all } s \in S\}.$$

Exercise 1.2. Show that if $S \subset T$, then $T' \subset S'$. Then show that $S' = S'''$ for any subset $S \subset B(H)$.

Thus von Neumann algebras come in pairs, A and A'. The *center* of a von Neumann algebra is $Z(A) = A' \cap A$, which is the center of both A and A'.

Definition 1.3. A von Neumann algebra A is called a *factor* if $Z(A) = \mathbb{C}1$.

There are 3 types of factors:

(1) A is type I_n if $A \cong B(H)$ where $\dim(H) = n$ with $n \in \mathbb{N} \cup \{\infty\}$.

(2) A is type II if A has a normal, semifinite tracial weight $\mathrm{tr} : A^+ \to [0, \infty]$. A is called type II_1 if this trace can be normalized so $\mathrm{tr}(1_A) = 1$. Otherwise A is called type II_∞.

(3) If A is not type I or type II, then A is type III.

We will focus mainly on type II_1.

Fact 1.4. The (normalized) trace in a II_1 factor is unique. Moreover, it is *normal*, i.e., is is SOT-continuous on the norm-closed unit ball.

Example 1.5. Suppose Γ is a countable icc group (all conjugacy classes infinite except for the class of $e \in \Gamma$). Consider the left regular action of Γ on $\ell^2\Gamma$ by $(\lambda_g f)(h) = f(g^{-1}h)$. Then $L\Gamma = \lambda\Gamma''$ is a II_1-factor with $\mathrm{tr}_{L\Gamma}(x) := \langle x\delta_e, \delta_e\rangle$, where δ_e is the indicator function at $e \in \Gamma$. For finite sums, $\mathrm{tr}_A\left(\sum c_g \lambda_g\right) = c_e$.

Definition 1.6. A II_1-factor is called *hyperfinite* if it generated by an increasing union of finite dimensional algebras.

Fact 1.7. There is a unique hyperfinite II_1-factor R up to isomorphism by [MvN43]. In fact, $R \cong L(S_\infty)$, where S_∞ is the group of finite permutations of $\mathbb{N}$, and also $R \cong \bigotimes_{i=1}^{\infty} M_2(\mathbb{C})$.

Fact 1.8. $\{\mathrm{tr}(p) | p \text{ is a projection in } A\} = [0, 1]$.

Exercise 1.9. Construct a projection of arbitrary trace in LS_∞ and in $L\mathbb{F}_2$.

1.2 The standard representation and modules

Given a II_1-factor A and its (unique) trace tr_A, we define its standard representation on L^2A via the GNS construction.

We define a sesquilinear form on A by $\langle a, b\rangle := \mathrm{tr}_A(b^*a)$. Since tr_A is faithful, there are no zero-length vectors. Define $L^2A = L^2(A, \mathrm{tr}_A)$ to be the completion of A in the 2-norm given by $\|a\|_2 = \mathrm{tr}_A(a^*a)^{1/2}$. We denote the image of $1 \in A$ in L^2A by Ω, which allows us to differentiate between $a \in A$ and $a\Omega \in L^2A$.

Now A acts on $A\Omega$ by left multiplication: $a \cdot b\Omega := ab\Omega$. Since tr_A is a normal state on A, this action is by bounded operators, and thus the action extends to a normal action of A on L^2A. Moreover, the operator norm on L^2A agrees with the operator norm on A coming from any other representation.

Definition 1.10. The standard representation of A is the left regular representation on L^2A.

Since tr_A is a trace, the map $a\Omega \mapsto a^*\Omega$ is isometric, and thus extends to an anti-linear unitary J on L^2A called the modular conjugation.

We now consider the action of Ja^*J on L^2A:

$$Ja^*Jb\Omega = Ja^*b^*\Omega = Ja^*b^*\Omega = ba\Omega,$$

i.e., Ja^*J is right multiplication by a. Since right multiplication commutes with left multiplication, we have $JAJ \subset A' \cap B(L^2A)$.

Fact 1.11. $JAJ = A' \cap B(L^2A)$.

Definition 1.12. Any A-module H_A is isomorphic to $\bigoplus^n L^2A \oplus pL^2A$ for some orthogonal projection $p \in A$. The (right) *von Neumann dimension* of H is defined as $\dim({}_AH) := n + \mathrm{tr}(p)$. There is a similar notion of left von Neumann dimension for left modules.

Fact 1.13. $\dim(H_A)$ is finite if and only if $A' \cap B(H)$ is a II_1 factor.

Given an isomorphism $H_A \cong \bigoplus^n L^2A \oplus pL^2A$, let $\beta_i = (0, \dots, 0, \Omega, 0, \dots, 0)$ where Ω is in the i-th slot for $1 \le i \le n$, and let $\beta_{n+1} = (0, \dots, 0, p\Omega)$ (if $n \ne \infty$ and $p \ne 0$). Then each β_i defines a bounded map $L_{\beta_i} : L^2A_A \to H_A$ by $x\Omega \mapsto \beta_i x$, and $\sum_i L_{\beta_i} L_{\beta_i}^* = 1_H$. We call $\{\beta_i\}$ an *orthonormal H_A-basis*.

1.3 II_1-subfactors, index, and the basic construction

We now consider a II_1-*subfactor*, i.e., a unital inclusion of II_1-factors $A \subseteq B$. Since A, B have unique traces $\mathrm{tr}_A, \mathrm{tr}_B$ respectively, we must have $\mathrm{tr}_B|_A = \mathrm{tr}_A$.

Definition 1.14 ([Jon83])**.** The *index* of the subfactor $A \subset B$ is $[B : A] := \dim({}_AL^2B)$. By [[PP86], the index $[B : A]$ is finite if and only if B is a finitely generated projective left A-module if and only if $A' \cap B(L^2B)$ is a II_1 factor.

Examples 1.15.

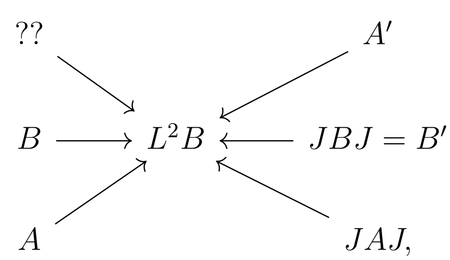
(1) (amplification) $A \subset M_n(A)$ has index n^2.

(2) (locally trivial) For any finitely presented group $\Gamma := \langle \theta_1, \dots, \theta_{n-1} \rangle \subset \mathrm{Aut}(A)$, we get an index n^2 subfactor
$$\{\mathrm{diag}(a, \theta_1(a), \dots, \theta_{n-1}(a)) | a \in A\} \subset M_n(A).$$

(3) Given an outer action of a finite group G on A, then $A^G \subset A$ and $A \subset A \rtimes G$ have index $|G|$.

(4) If in the previous case, $H \le G$ is a subgroup, then both $R^G \subset R^H$ and $A \rtimes H \subset A \rtimes G$ have index $[G : H]$.

Let us now look more closely at the standard representation of B on L^2B in the presence of a finite index subfactor $A \subseteq B$. We immediately see the action of five II_1-factors:



and we see there should be one more II_1-factor in this story.

Definition 1.16. The *basic construction* of $A \subseteq B$ is the II_1-factor $JA'J$ acting on L^2B.

Example 1.17. Suppose $\alpha : G \to \mathrm{Aut}(A)$ is an outer action where G is a finite group. The basic construction of $A^G \subseteq A$ is isomorphic to $A \rtimes G$. The basic construction of $A \subseteq A \rtimes G$ is isomorphic to $A \otimes B(\ell^2(G))$.

Just as von Neumann algebras come in pairs A, A', we see that subfactors also come in pairs: $A \subseteq B$ and $B' \subseteq A'$, which is conjugate (anti-isomorphic) to $B = JB'J \subseteq JA'J$.

Alternatively, since $\mathrm{tr}_B|_A = \mathrm{tr}_A$, the natural inclusion $A \subset B$ gives rise to a canonical inclusion $\iota_A : L^2A \subset L^2B$, which is $A - A$ bimodular. The adjoint $\iota_A^* : L^2B \to L^2A$ is the $A - A$ bimodular projection onto L^2A. We define the *Jones projection* $e_A := \iota_A \iota_A^* \in B(L^2B)$.

Fact 1.18 ([Jon83])**.** $JA'J = \langle B, e_A \rangle$, the von Neumann algebra generated by B and e_A.

For each $b \in B$, observe that $E_A(b) := \iota_A^* b \iota_A \in B(L^2A)$ is A-linear, and thus defines an operator in $JAJ' = A$. Moreover, $e_A b\Omega = E_A(b)\Omega$. The map $E_A : B \to A$ is clearly normal, $A - A$ bimodular, and unital completely positive, i.e., it is a normal *conditional expectation*. One can also show E_A is faithful.

Fact 1.19 ([Jon83])**.** $[\langle B, e_A \rangle : B] = [B : A]$, and $\mathrm{tr}_{\langle B, e_A \rangle}(e_A x) = [B : A]^{-1} \mathrm{tr}_B(x)$ for all $x \in B$.

Hence we may iterate the basic construction of $A_0 = A \subseteq B = A_1$ to get the *Jones tower* of II_1-factors $(A_n)_{n \geq 0}$ where $[A_{n+1} : A_n] = [B : A]$, and $A_{n+1} = \langle A_n, e_n \rangle$, where $e_n : L^2A_n \to L^2A_n$ is the orthogonal projection with range L^2A_{n-1}.

The first sign that something genuinely interesting is going on is the following proposition.

Proposition 1.20 ([Jon83])**.** *The projections $(e_i)_{i \geq 0}$ satisfy the Jones-Temperley-Lieb relations [TL71] for $d = [B : A]^{-1/2}$:*

(1) $e_i^2 = e_i^* = e_i$,

(2) $e_i e_j = e_j e_i$ when $|i - j| > 1$, and

(3) $e_i e_{i \pm 1} e_i = d^{-2} e_i$.

Thus a subfactor gives a semisimple quotient of the *Temperley-Lieb-Jones algebra* $TLJ_n(d)$ for $d = [B : A]^{1/2}$ for every $n \geq 0$, which we will discuss in the second lecture, leading to Jones' famous *index rigidity theorem*.

1.4 Unitary tensor categories

We can trade a finite index (extremal) II_1 subfactor for three pieces of data:

(1) The unitary tensor category $\mathcal{C}$,

(2) A suitably nice algebra object $Q \in \mathcal{C}$ (namely a simple normalized Q-system), and

(3) a unitary tensor functor $F : \mathcal{C} \to \mathsf{Bim}(A)$.

Given a subfactor $A \subset B$, $\mathcal{C}$ is the tensor category of $A - A$ bimodules generated by L^2B, $Q = L^2B$ equipped with the unit $\iota_A : L^2A \hookrightarrow L^2B$ and multiplication $\mu : L^2B \boxtimes_A L^2B \to L^2B$ given by $\mu(x\Omega \boxtimes y\Omega) := xy\Omega$, and the unitary tensor functor F arises as $\mathcal{C}$ lives inside $\mathsf{Bim}(A)$.

The pair $(\mathcal{C}, Q)$ is an 'algebraic' invariant, while the tensor functor F is an 'analytic' object which can be viewed as a generalized fiber functor. The *standard invariant* is the algebraic data $(\mathcal{C}, Q)$ which forgets F, and the only data needed to reconstruct the original subfactor is F. This viewpoint divides subfactor classification into two parts: the algebraic categorical classification of standard invariants, and the functional analytic problem of actions of unitary tensor categories on factors.

Given (the dual of) a compact quantum group $(\mathcal{C}, F)$ where $\mathcal{C}$ is a unitary tensor category and $F : \mathcal{C} \to \mathsf{Hilb}$ is a fiber functor, we can reconstruct the polynomial Hopf algebra as a coend $\bigoplus_{c\in\operatorname{Irr}(\mathcal{C})} F(c) \otimes F(c)^*$. Similarly, we can reconstruct the over-factor B from $(\mathcal{C}, Q, F)$ by a co-end construction

$$B = \bigoplus_{c\in\operatorname{Irr}(\mathcal{C})} \operatorname{Hom}_{\mathbb{C}-A}(L^2A \to F(c)) \otimes \operatorname{Hom}_{\mathcal{C}}(c \to Q) = \operatorname{Hom}_{\mathbb{C}-A}(L^2A \to F(Q)).$$

This point of view is called *Q-system realization*.

Examples 1.21.

(1) For amplifications $A \subset M_n(A)$, $\mathcal{C} = \mathsf{Hilb}$, $Q = M_n(\mathbb{C})$, and $F : \mathbb{C} \mapsto L^2A$.

(2) For locally trivial subfactors $\{\operatorname{diag}(a, \theta_1(a), \dots, \theta_n(a) | a \in A\} \subset M_n(A)$ and $\Gamma = \langle 1, \theta_1, \dots, \theta_n\rangle$, $\mathcal{C} = \mathsf{Hilb}(\Gamma)$, $Q = \mathbb{C}[(\sum \theta_i)(\sum \theta_j)^{-1}]$, and $F : \theta \mapsto L^2A_\theta$ where $a \cdot x\Omega \cdot b := ax\theta(b)\Omega$.

(3) For $A^G \subset A$, $\mathcal{C} = \mathsf{Rep}(G)$, $Q = C(G)$ with $(g \cdot \xi)(h) = \xi(g^{-1}h)$.
For $A \subset A \rtimes G$, $\mathcal{C} = \mathsf{Hilb}(G)$, $Q = \mathbb{C}[G]$, and $F : g \mapsto L^2A_g$.

(4) For $R^G \subset R^H$, $\mathcal{C} \subset \mathsf{Rep}(G)$,[1] $Q = \operatorname{Ind}_H^G(\operatorname{Res}_H^G(1_G))$, and $F : g \mapsto L^2A_g$.

Definition 1.22. A *unitary tensor category* is a semisimple (equivalently idempotent complete) rigid C* tensor category $\mathcal{C}$ with simple unit object.

In more detail,

- (semisimple) $\mathcal{C}$ is a *linear category*, where every hom space $\mathcal{C}(a \to b)$ is a finite dimensional vector space. For simplicity, we assume $\mathcal{C}$ admits direct sums of objects, and all idempotents split, i.e., if $p \in \mathcal{C}(a \to a)$ is an idempotent, there is a $b \in \mathcal{C}$ and maps $r : b \to a$ and $s : a \to b$ such that $s \circ r = \mathrm{id}_a$ and $r \circ s = p$.

  Every object of c can be written as a finite direct sum $c = \bigoplus c_i$ of *simple objects* which satisfy

$$\operatorname{Hom}_{\mathcal{C}}(c_i \to c_j) \cong \begin{cases} \mathbb{C} & \text{if } c_i \cong c_j. \\ 0 & \text{else.} \end{cases}$$

[1]Here, $\mathcal{C}$ is actually the unitary tensor subcategory of $\mathsf{Rep}(G)$ generated by the connected component of the trivial G-representation in the induction-restriction graph of $H \leq G$.

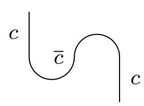
- (C*) The category $\mathcal{C}$ has a *dagger structure*, i.e., for all $a, b \in \mathcal{C}$, there is an anti-linear map $\dagger : \mathcal{C}(a \to b) \to \mathcal{C}(b \to a)$ such that $f^{\dagger\dagger} = f$ and $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$ whenever f, g are composable. With this dagger structure, $\mathcal{C}$ is C*, i.e., every finite dimensional endomorphism algebra of $\mathcal{C}$ is a C*-algebra.

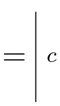
- (tensor) There is a $\dagger$-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product*, and a distinguished object $1_\mathcal{C} \in \mathcal{C}$ called the *unit object*, which we require to be simple. We have *associator* and *unitor* unitary natural isomorphisms

$$\alpha_{a,b,c} : (a \otimes b) \otimes c \longrightarrow a \otimes (b \otimes c)$$
$$\lambda_a : 1_\mathcal{C} \otimes a \longrightarrow a$$
$$\rho_a : a \otimes 1_\mathcal{C} \longrightarrow a$$

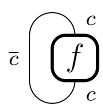
  There are pentagon and triangle coherence axioms, which basically allow the tensor product and unit to function as a higher multiplication with an identity.

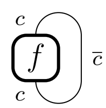
- (rigid) Every object $c \in \mathcal{C}$ admits a *dual object* $\overline{c} \in \mathcal{C}$ together with maps $\mathrm{ev}_c : \overline{c} \otimes c \to 1_\mathcal{C}$ and $\mathrm{coev}_c : 1 \to c \otimes \overline{c}$ satisfying the snake equations, which are best depicted graphically:



$$= (\mathrm{id}_c \otimes \mathrm{ev}_c) \circ \alpha_{c,\overline{c},c} \circ (\mathrm{coev}_c \otimes \mathrm{id}_c) = \mathrm{id}_c =$$


  and a similar reflected equation swapping $c, \overline{c}$. Moreover, these maps can be chosen such that for every $c \in \mathcal{C}$,



$$= \mathrm{ev}_c \circ (\mathrm{id}_{\overline{c}} \otimes f) \circ \mathrm{ev}_c^\dagger = \mathrm{coev}_c^\dagger \circ (f \otimes \mathrm{id}_{\overline{c}}) \circ \mathrm{coev}_c =$$


$$\forall f \in \mathrm{End}_\mathcal{C}(c).$$

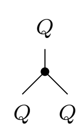
  In this case, the above function on $\mathrm{End}_\mathcal{C}(c)$ is a positive definite trace $\mathrm{tr}_\mathcal{C}$. For each $c \in \mathcal{C}$, we define its *quantum dimension* as $d_c := \mathrm{tr}_\mathcal{C}(\mathrm{id}_c)$.

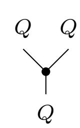
We will show in the next lecture that the $A - A$ bimodules generated by L^2B is a unitary tensor category.

Definition 1.23. An *algebra object* $Q \in \mathcal{C}$ is an object equipped with a multiplication $\mu : Q \otimes Q \to Q$ and a unit $\iota : 1 \to Q$. We represent the unit by a univalent vertex and the multiplication by a trivalent vertex, and their adjoints by the vertical reflections:

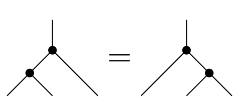
$\iota =$ 

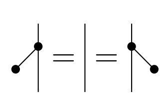
$\iota^\dagger =$ 

$\mu =$ 

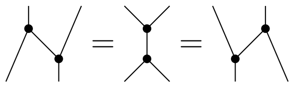
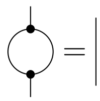
$\mu^\dagger =$ 

The unit and multiplication satisfy associativity and unitality axioms:





We call Q:

- *simple* if $\mathrm{End}_{Q-Q}(Q) = \mathbb{C}\,\mathrm{id}_Q$,
- C^*-*Frobenius* if 
- *separable* if  and
- a *normalized Q-system* if Q is a separable C^*-Frobenius algebra such that $\iota^* \circ \iota = d_Q$.

2 Various notions of the standard invariant

2.1 Historical development of the standard invariant

The standard invariant of a finite index (extremal) II_1 subfactor has seen many equivalent definitions and axiomatizations. We attempt to provide a synoptic list here relying roughly on publication date for chronology, together with a chart to explain equivalences between the notions.

- [Jon87] Jones defines standard invariant as the lattice of *higher relative commutants* associated to the subfactor.

$$
\begin{array}{ccccccccc}
A_0' \cap A_0 & \subset & A_0' \cap A_1 & \subset & A_0' \cap A_2 & \subset & A_0' \cap A_3 & \subset & \cdots \\
 & & \cup & & \cup & & \cup & & \\
 & & A_1' \cap A_1 & \subset & A_1' \cap A_2 & \subset & A_1' \cap A_3 & \subset & \cdots \\
 & & & & \cup & & \cup & & \\
 & & & & A_2' \cap A_2 & \subset & A_2' \cap A_3 & \subset & \cdots \\
 & & & & & & \cup & & \\
 & & & & & & A_3' \cap A_3 & \subset & \cdots
\end{array}
$$

- [Ocn88] Ocneanu defines the notions of *paragroup* and *flat biunitary connection*, formalized by Kawahigashi [Kaw95].
- [Pop90] Popa shows that a *finite depth* subfactor ($\dim(A_0' \cap A_n)$ bounded as $n \to \infty$) is completely classified by its canonical commuting square.

$$
\begin{array}{ccc}
A_0' \cap A_n & \subset & A_0' \cap A_{n+1} \\
\cup & & \cup \\
A_1' \cap A_n & \subset & A_1' \cap A_{n+1}
\end{array}
\qquad\qquad n \text{ past depth}
$$

  Popa actually finds a *generating Jones tunnel* rather than using the Jones tower.

- [Pop95] Popa axiomatizes the standard invariant as a standard λ-lattice, which also includes the data of a trace, conditional expectations, Jones projections implementing the conditional expectations, and a commutation rule.

$$\begin{array}{ccccccccc} A_{00} & \subset & A_{01} & \subset & A_{02} & \subset & A_{03} & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & A_{11} & \subset & A_{12} & \subset & A_{13} & \subset & \cdots \\ & & & & \cup & & \cup & & \\ & & & & A_{22} & \subset & A_{23} & \subset & \cdots \\ & & & & & & \cup & & \\ & & & & & & A_{33} & \subset & \cdots \\ & & & & & & & & \ddots \end{array} \qquad A_{ij} \leftrightarrow A_i' \cap A_j$$

Theorem 2.1 ([Pop95]). *Every standard λ-lattice arises as the standard invariant of an extremal finite index II_1-subfactor $A \subset B$. If the λ-lattice is* finite depth *($\dim(Z(A_{0k}))$ is bounded as $k \to \infty$), then A, B can be taken to be hyperfinite.*

- [Jon21] (∼1999) Jones shows that the standard invariant forms a planar algebra.
- [Müg03] Müger explains connections between subfactor standard invariants, pivotal 2-categories with a generator, and tensor categories with Frobenius algebra objects/Longo's Q-systems [Lon89].

2.2 The A_4 commuting square

A commuting square is a square of finite dimensional C*-algebras

$$\begin{array}{ccc} C & \subset & D \\ \cup & & \cup \\ A & \subset & B \end{array}$$

together with a trace on D such that the trace-preserving conditional expectations E_C and E_B commute. Moreover, the square is *non-degenerate* if $BC = D (= CB)$.

The A_4 commuting square is given by

$$(A, b) \longmapsto A \oplus \begin{pmatrix} A & \\ & b \end{pmatrix}$$

$$\begin{array}{ccccccc} u \begin{pmatrix} b & \\ & a \end{pmatrix} u^* \oplus b & M_2(\mathbb{C}) \oplus \mathbb{C} & \subset & M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) & uAu^* \oplus w \begin{pmatrix} A & \\ & b \end{pmatrix} w^* \\ \uparrow & \cup & & \cup & \uparrow \\ (a, b) & \mathbb{C} \oplus \mathbb{C} & \subset & M_2(\mathbb{C}) \oplus \mathbb{C} & (A, b) \end{array}$$

$$(a, b) \longmapsto \begin{pmatrix} a & \\ & b \end{pmatrix} \oplus b$$

where

$$u = \begin{pmatrix} -\phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & \phi^{-1} \end{pmatrix} \qquad w = \begin{pmatrix} \phi^{-2} & -\phi^{-1/2} & -\phi^{-3/2} \\ -\phi^{-3/2} & \phi^{-1} & \phi^{-1} \\ \phi^{-1/2} & 0 & \phi^{-1} \end{pmatrix} \qquad \phi = \frac{1+\sqrt{5}}{2}.$$

Iterating the basic construction simultaneously for the top and bottom rows gives a subfactor $A \subset B$ with ${}_A\mathcal{C}_A$ the Fibonacci unitary fusion category.

2.3 The Temperley-Lieb standard invariant and graphical calculus

Recall that the n-th Temperley-Lieb-Jones algebra $TLJ_n(d)$ is the unital complex $*$-algebra generated by $e_1, \dots, e_{n-1}$ satisfying the following relations:

(1) $e_i^2 = e_i^* = e_i$,

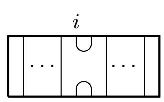
(2) $e_i e_j = e_j e_i$ when $|i-j| > 1$, and

(3) $e_i e_{i\pm1} e_i = d^{-2} e_i$.

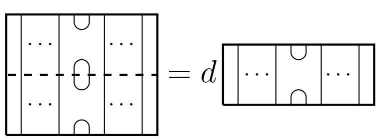
Exercise 2.2. Prove that $\dim(TLJ_n(d)) \leq \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.

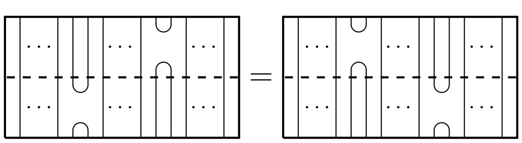
The n-th *diagrammatic* Temperley-Lieb-Jones algebra has as its basis all non-crossing pairings of n points on the lower boundary of a rectangle and n points on the upper boundary of a rectangle, up to isotopy. We multiply by stacking and trading closed bubbles for a multiplicative factor of d. Observe that this algebra has dimension $\frac{1}{n+1}\binom{2n}{n}$.

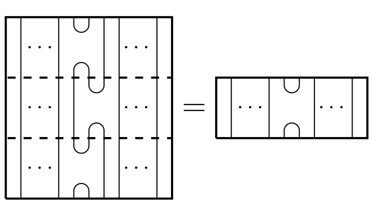
Exercise 2.3. For $i = 1, \dots, n-1$, show that the elements

$E_i =$ 

satisfy the following relations:

(1) $E_i^2 =$  $= dE_i = dE_i^*$,

(2) $E_iE_j =$  $= E_jE_i$ if $|i-j| > 1$, and

(3) $E_iE_{i\pm1}E_i =$  $= E_i$.

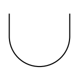
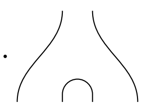
We thus have a unital $*$-homomorphism from $TLJ_n(d)$ to the diagrammatic algebra by $e_i \mapsto d^{-1}E_i$.

Exercise 2.4. Prove this map is surjective, and use a dimension count to deduce it is a $*$-isomorphism.

The standard invariant of a finite index II_1 subfactor always contains the Jones projections. Popa noted that the (semisimple quotients of the) algebras $TLJ_n(d)$ form a standard λ-lattice, and so there is an extremal subfactor with this standard invariant. It is still an open question whether these factors can be taken to be hyperfinite when $d > 2$.

We can see that Temperley-Lieb-Jones gives a unitary tensor category $\mathcal{TLJ}(d)$ and a Q-system Q using graphical calculus. First, we consider the objects $[n]$ for $n \in \mathbb{N}_{\geq 0}$. The maps $[m] \to [n]$ are formal linear combinations of non-crossing pair partitions with m points on the bottom and n points on top. (Observe that $\mathrm{Hom}([m] \to [n]) = 0$ unless $m \equiv n \mod 2$.) Composition is stacking using the rule that a closed loop counts for a multiplicative factor of d. Tensor product is horizontal juxtaposition, and $\dagger$ is the anti-linear extension of vertical reflection of diagrams. One then quotients out length zero morphisms under $\|\cdot\|_2$, where $\langle x, y\rangle = \mathrm{tr}_m(y^*x)$ for $x, y : [m] \to [n]$. This gives us a rigid C* tensor category with simple unit, which turns out to be additive, but not yet idempotent complete. One takes the formal unitary idempotent completion to get a unitary tensor category, which we call $\mathcal{TLJ}(d)$.

The Q-system is $[2]$ with unit and multiplication

$\iota = d^{1/2} \cdot$  $\qquad \mu = \frac{1}{d^{1/2}} \cdot$  (1)

2.4 Higher relative commutants and bimodules

The standard invariant of a finite index (extremal) II_1-subfactor $A \subseteq B$ can also be defined, roughly speaking, as the collection of $A-A$, $A-B$, $B-A$, and $B-B$ bimodules generated by L^2B under:

- Connes' fusion relative tensor product $\boxtimes_A$ and $\boxtimes_B$,
- taking sub-bimodules, i.e., $H \subset \boxtimes_A^n L^2(B)$ for some n as a bimodule,
- direct sums, i.e., if H, K are bimodules with same algebras acting on the left and right, so is $H \oplus K$,
- conjugates, e.g., if H is an $A-B$ bimodule, then $\overline{H}$ is a $B-A$ bimodule with $b \cdot \overline{\xi} \cdot a := \overline{a^*\xi b^*}$.

together with bounded bimodule intertwiners. This structure assembles into a *unitary 2-category* where the objects are A, B, the 1-morphisms are bimodules, and the 2-morphisms are intertwiners. Importantly, we must remember the *generator* ${}_AL^2B_B$.

If we consider the von Neumann algebra $A \oplus B$, then we can view the standard invariant as a *unitary multitensor category* (semisimple rigid C* tensor category without simple unit) $\mathcal{C}(A \subset B) \subset \mathsf{Bim}(A \oplus B)$ with unit object $L^2A \oplus L^2B$, where we have a 2×2 decomposition

$$\mathcal{C}(A \subset B) = \begin{pmatrix} {}_A\mathcal{C}_A & {}_A\mathcal{C}_B \\ {}_B\mathcal{C}_A & {}_B\mathcal{C}_B \end{pmatrix}.$$

Here, ${}_C\mathcal{C}_D$ refers to the $C - D$ bimodules for $C, D \in \{A, B\}$.

Fact 2.5 ([PP86, JS97])**.** $L^2A_n \cong \boxtimes_A^n L^2B$, the Connes' fusion, and $A_n \cong \bigotimes_A^n B$, the algebraic tensor product, where (A_n) is the Jones tower of $A \subset B$. Moreover, $A_i' = J_n A_{2n-i} J_n$ acting on L^2A_n:

$$
\begin{array}{ccccc}
A_{2n} & & & & JA_{2n}J \\
& \searrow & & \swarrow & \\
A_n & \longrightarrow & L^2A_n & \longleftarrow & JA_nJ = B' \\
& \nearrow & & \nwarrow & \\
A_0 & & & & JA_0J,
\end{array}
$$

This means we have isomorphisms between the higher relative commutants and endomorphisms of $L^2A_n \cong \boxtimes_A^n L^2B$:

$$
\begin{aligned}
A_0' \cap A_{2n} &\cong \mathrm{End}_{A-A}\left(\boxtimes_A^n L^2B\right) \\
A_0' \cap A_{2n-1} &\cong \mathrm{End}_{A-B}\left(\boxtimes_A^n L^2B\right) \\
A_1' \cap A_{2n} &\cong \mathrm{End}_{B-A}\left(\boxtimes_A^n L^2B\right) \\
A_1' \cap A_{2n-1} &\cong \mathrm{End}_{B-B}\left(\boxtimes_A^n L^2B\right)
\end{aligned}
$$

Thus the simple summands of $\boxtimes_A^n L^2B$ correspond to minimal projections in the higher relative commutants:

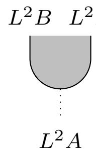
$$
\begin{aligned}
p \in A_0' \cap A_{2n} &\longleftrightarrow pL^2A_n \qquad A-A \text{ bimodule} \\
p \in A_0' \cap A_{2n-1} &\longleftrightarrow pL^2A_n \qquad A-B \text{ bimodule} \\
p \in A_1' \cap A_{2n-1} &\longleftrightarrow pL^2A_n \qquad B-B \text{ bimodule} \\
p \in A_1' \cap A_{2n} &\longleftrightarrow pL^2A_n \qquad B-A \text{ bimodule}
\end{aligned}
$$

Fact 2.6 (Semi-simplicity [Jon83])**.** For every $0 \le i \le j$, $A_i' \cap A_j$ is finite dimensional, and consequently, $L^2A_n \cong \boxtimes_A^n L^2B$ splits as a finite direct sum of $C-D$ bimodules for $C, D \in \{A, B\}$.

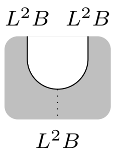
Example 2.7. Consider the subfactor $A = B^G \subset B$. Then ${}_AL^2B_B$ is irreducible and $\mathrm{End}_{A-A}(L^2B) \cong \mathbb{C}[G]$, the group algebra. Thus ${}_AL^2B_A \cong \bigoplus n_\pi H_\pi$ where $H_\pi \in \mathsf{Bim}(A)$ is the image of an irrep π of G, which appears with multiplicity $n_\pi = \dim(\pi)$. We saw that the basic construction of $B^G \subset B$ is $B \rtimes G$. We have $L^2B \boxtimes_A L^2B \cong L^2(B \rtimes G) \cong \bigoplus_g L^2B_g$ as $B-B$ bimodules. We thus see that the simples for $\mathcal{C}(A \subset B)$ can be described by:

$$
\begin{pmatrix} {}_A\mathcal{C}_A & {}_A\mathcal{C}_B \\ {}_B\mathcal{C}_A & {}_B\mathcal{C}_B \end{pmatrix} = \begin{pmatrix} \mathsf{Rep}(G) & \mathsf{Hilb} \\ \mathsf{Hilb} & \mathsf{Hilb}(G) \end{pmatrix}.
$$

We now discuss dualizability. Consider our generator ${}_A L^2 B_B$. We get bounded maps



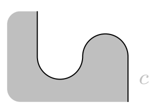
$: {}_A L^2 A_A \longrightarrow {}_A L^2 B_A = {}_A L^2 B \boxtimes_B L^2 B_A$ given by $d^{1/2}\iota_A$  $= A$



$: {}_B L^2 B_B \longrightarrow L^2 B \boxtimes_A L^2 B$ given by $x\Omega \mapsto \dfrac{1}{d^{1/2}} \sum x b_i \Omega \boxtimes b_i^* \Omega$  $= B$

Here, $\iota_A : L^2 A \to L^2 B$ is the isometric inclusion, and $\{b_i \Omega\}$ is a finite orthonormal $L^2 B_A$-basis ($\{b_i\}$ is a Pimnser-Popa basis for B over A). Since ι_A is an isometry, and our normalization makes the shaded loop equal to d. The unshaded cup may be viewed as a scalar multiple of the inclusion $L^2 B = L^2 A_1 \to L^2 A_2$. Its adjoint is the multiplication map $L^2 B \boxtimes_A L^2 B \to L^2 B$, suitably normalized. Since $\sum b_i b_i^* = d^2$, the unshaded loop is also a factor of d.

We now verify one zig-zag axiom:

$\mathrm{End}_{A-A}(L^2 B) \ni$ 

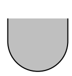
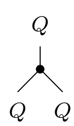
$: x\Omega \mapsto \sum b_i \Omega \boxtimes_A b_i^* \Omega \boxtimes_B x\Omega \mapsto \sum b_i E_A(b_i^* x)\Omega = x\Omega.$

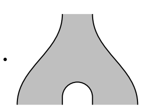
Exercise 2.8. Verify the other zig-zag axioms.

Remark 2.9. When $A \subset B$ is *irreducible*, i.e., $A' \cap B = \mathbb{C}$, then this cup and cap are *standard solutions* to the conjugate equations, meaning the corresponding left and right quantum traces are equal. This will also be the case when $A \subset B$ is extremal.

Remark 2.10. Starting with $\mathcal{C}(A \subset B)$, we get $(\mathcal{C}, Q)$ by taking $\mathcal{C} = {}_A \mathcal{C}_A$ and $Q = {}_A L^2 B_A$ with unit and multiplication similar to (1):

$\iota =$ 

$= d^{1/2} \cdot$  $\mu =$ 

$= \dfrac{1}{d^{1/2}} \cdot$ 

In the graphical calculus, closed bubbles both count for a factor of $d := [B : A]^{1/2}$.

Conversely, starting with $(\mathcal{C}, Q)$, we get a 2×2 unitary mutlitensor category by taking

$$\begin{pmatrix} \mathcal{C} & \mathsf{Mod} - Q \\ Q - \mathsf{Mod} & \mathsf{Bim}(Q) \end{pmatrix}$$

with generator Q as a $1 - Q$ bimodule.

2.5 The principal graphs

The standard invariant of a finite index (extremal) subfactor $A \subset B$ is too much information for effectively classifying hyperfinite subfactors of small index. From the standard invariant, we extract two bipartite graphs called the *principal graphs*, with a little extra structure.

Definition 2.11. The principal graph Γ_+ is the bipartite graph defined as follows. The even vertices are the isomorphism classes of simple $A-A$ bimodules in ${}_A\mathcal{C}_A$. A bimodule X is simple if $\mathrm{End}(X) \cong \mathbb{C}1$. The odd vertices are the isomorphism classes of simple $A-B$ bimodules in ${}_A\mathcal{C}_B$. There are $\dim(\mathrm{Hom}_{A-B}(X \otimes_A L^2B, Y))$ edges from ${}_AX_A$ to ${}_AY_B$.

The dual principal graph Γ_- is defined similarly using $B-B$ and $B-A$ bimodules. The dual principal graph is also the principal graph for the inclusion $B' \subseteq A'$ or its conjugate $B \subseteq \langle B, e_A \rangle$.

We also record the data of the duality on $A-A$ and $B-B$ vertices by marking tags on the even vertices, and we order the odd vertices to encode the duality between the $A-B$ and $B-A$ vertices by lexicographic order.

Each of $\Gamma_\pm$ has a distinguished vertex corresponding to the trivial bimodules L^2A and L^2B. The depth of a vertex is its distance to the trivial, which corresponds to the minimal n such that $X \subseteq \overline{L^2A_n}$. Since A_n is a $*$-algebra, the dual of an even vertex of $\Gamma_\pm$ is a vertex of $\Gamma_\pm$ at the same depth. The dual of an odd vertex of $\Gamma_\pm$ is a vertex of $\Gamma_\mp$ at the same depth.

A subfactor has *finite depth* if and only if the principal graph is finite.

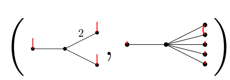
Fact 2.12. The principal graph is finite if and only if the dual principal graph is finite. In this case, the difference in their depth is at most one.

Example 2.13. Consider the outer action $\alpha : G \to \mathrm{Aut}(B)$. The principal graphs of $A = B^G \subseteq B$ can be described as follows.

The $A-A$ vertices correspond to irreducible representations (irreps) of G, and duality is taking the contragredient. There is a single $A-B$ vertex, and the number of edges to an irrep is the dimension of the irrep.

The $B-B$ vertices correspond to the elements of G, and duality is taking the inverse. There is only one $B-A$ vertex, with a single edge to each $g \in G$.

For $G = S_3$, the principal graphs are given by


.

The principal graphs $\Gamma_\pm$ also have some additional properties and structure that make them particularly useful.

Fact 2.14. $\dim(A_0' \cap A_n)$ is equal to the number of loops of length $2n$ on $\Gamma_\pm$ starting at $\star$.

We saw that irreducible summands of L^2A_n correspond to minimal projections in the higher relative commutants. The traces of these projections give a *dimension function* dim on the vertices of $\Gamma_\pm$ which satisfies

$$d \cdot \dim(Y) = \sum_{Z \in V(\Gamma)} N_{Y,L^2B}^{Z} \dim(Z)$$

for all simple $A - A$ bimodules Y and simple $A - B$ bimodules Z, where N^{Z}_{Y,L^2B} is the number of copies of Z inside $Y \boxtimes_A L^2B$. The above condition says that the dimension function gives an eigenvector for the adjacency matrix of $\Gamma_\pm$.

When $\Gamma_\pm$ is finite, then this dimension is the unique *Frobenius-Perron eigenvector* with positive entries, normalized so that the dimension of L^2A is 1 [BH12]. In this case, we write FPdim instead of dim.

Fact 2.15. The index $[B : A] \geq \|\Gamma_\pm\|^2$, where the norm of $\Gamma_\pm$ is the operator norm of the adjacency matrix acting on ℓ^2 of the vertices [Pop94]. If $\Gamma_\pm$ is finite, then $[B : A] = \|\Gamma_\pm\|^2$, and the norm of $\Gamma_\pm$ is the largest eigenvalue of the adjacency matrix [Jon87].

Remark 2.16. In the infinite depth case, it is important to distinguish between the Frobenius-Perron dimension function FPdim corresponding to the norm of the adjacency matrix of $\Gamma_\pm$ and the quantum dimension function dim which may not agree. In fact, they agree if and only if the subfactor is amenable [Pop94]. Finite depth subfactors are amenable by uniqueness of the Frobenius-Perron dimension function [Jon87, EK98].

The principal graphs are much less information than the standard invariant, which is exactly why they are so useful. One of the main motivating questions in subfactor theory is the following.

Question 2.17. *Given a pair of bipartite graphs (Γ_+, Γ_-) with dual data satisfying some additional properties, are they the principal graphs of a finite index subfactor? If so, for how many subfactors?*

Fact 2.18. If (Γ_+, Γ_-) are finite, then by Ocneanu Rigidity [ENO05], the answer is always *at most finitely many*

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