The classification of standard invariants to index at most 5.25

David Penneys

Topological quantum groups, C^*-tensor categories, and subfactors

May 28, 2022

The Ohio State University

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(Fantastic beasts and where to find them)

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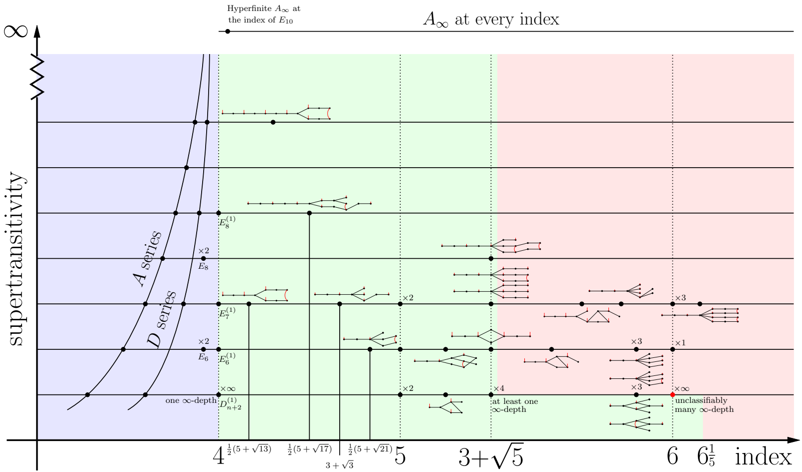
Quotes from the AMS memorial article for Vaughan:

- “Every trip to a wonderland requires a wizard and our journey started with Vaughan”
  -Vanderbilt students (Corey Jones, Bin Gui, Yunxiang Ren, Sayan Das, and Zhengwei Liu)
- “My older sister Bethany once told our father he was our Gandalf, or our Aslan. It moved him deeply. Like Tolkien’s wizard or Lewis’ lion, ... there was a sense of magic whenever he was around, and a mystique to his work”
  -Ian Jones

Irreducible standard invariants with index at most $5\frac{1}{4}$



Theorem [AMP15, Liu15], building on work of many others

We know all standard invariants up to index $5\frac{1}{4} > 3 + \sqrt{5}$, the first interesting composite index.

Outline

1. Known families of standard invariants
2. How do we classify standard invariants?
3. Constructing standard invariants

Known families of standard invariants

All known subfactor standard invariants fit into 4 families.
We list those with index at most $5\frac{1}{4}$ here.

1. **Groups:** integer index examples

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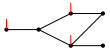
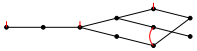
2. **Quantum groups:** ADE index< 4, Fuss-Catalan and quotients, 3311, $PSU(2)_5 \rightsquigarrow$ , $SU(3)_4 \rightsquigarrow$ 

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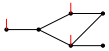
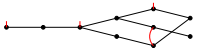
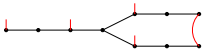
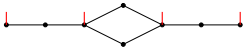
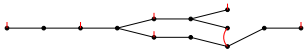
2. **Quantum groups:** ADE index< 4, Fuss-Catalan and quotients, 3311, $PSU(2)_5 \rightsquigarrow$ , $SU(3)_4 \rightsquigarrow$ 
3. **Haagerup-Izumi quadratic categories:** 'spokes': 2221, $3^{\mathbb{Z}/3} =$ , $3^{\mathbb{Z}/2\times\mathbb{Z}/2}, 3^{\mathbb{Z}/4}, 4442; 2D2 =$ ; $\mathcal{AH} =$ 

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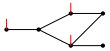
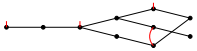
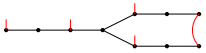
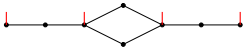
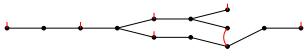
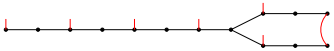
2. **Quantum groups:** ADE index< 4, Fuss-Catalan and quotients, 3311, $PSU(2)_5 \rightsquigarrow$ , $SU(3)_4 \rightsquigarrow$ 
3. **Haagerup-Izumi quadratic categories:** 'spokes': 2221, $3^{\mathbb{Z}/3} =$ , $3^{\mathbb{Z}/2\times\mathbb{Z}/2}, 3^{\mathbb{Z}/4}, 4442; 2D2 =$ ; $\mathcal{AH} =$ 
4. **Extended Haagerup/Haagerup-Peters:** 

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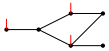
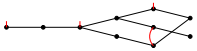
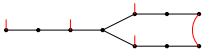
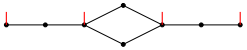
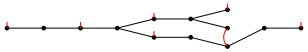
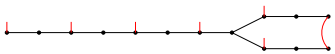
2. **Quantum groups:** ADE index< 4, Fuss-Catalan and quotients, 3311, $PSU(2)_5 \rightsquigarrow$ , $SU(3)_4 \rightsquigarrow$ 
3. **Haagerup-Izumi quadratic categories:** 'spokes': 2221, $3^{\mathbb{Z}/3} =$ , $3^{\mathbb{Z}/2\times\mathbb{Z}/2}, 3^{\mathbb{Z}/4}, 4442; 2D2 =$ ; $\mathcal{AH} =$ 
4. **Extended Haagerup/Haagerup-Peters:** 

Here, 'fit into' means it can be constructed from known examples by known construction techniques, including:

- dual, tensor product, free product, intermediate subfactors, reduced subfactors, Morita equivalence, equivariantization, de-equivariantization, maximal atlas search ...

There is currently no uniform construction for the Haagerup-Izumi quadratic categories, and many candidates remain unconstructed.

Standard invariants in math and physics

Subfactor standard invariants are **group-like objects** which encode **quantum symmetry**. They have applications to many areas of mathematics and physics. Some include:

- invariants of knots and links via Jones' polynomial
- topological quantum field theory
- conformal field theory
- topological phases of matter

How do we classify standard invariants?

$$A \subset B \qquad \rightsquigarrow \qquad \mathcal{C}(A \subset B) \qquad \rightsquigarrow \qquad (\Gamma_+, \Gamma_-)$$

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Definition

The **principal graph** Γ_+ of $\mathcal{C}(A \subset B)$ has vertices the irreducible $A - A$ and $A - B$ bimodules, and $\dim(\mathrm{Hom}(H \boxtimes_A L^2 B, K))$ edges from ${}_A H_A$ to ${}_A K_B$.

The **dual graph** Γ_- of $\mathcal{C}(A \subset B)$ is defined similarly using $B - B$ and $B - A$ bimodules.

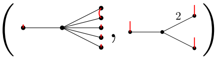
- $\Gamma_\pm$ is **pointed** with basepoint the trivial bimodule ${}_A L^2(A)_A$, ${}_B L^2(B)_B$ respectively.
- The **depth** of a bimodule is its distance from the basepoint.
- **Duality** is given by $\overline{\cdot}$, which is always at the same depth, although duals at odd depths of $\Gamma_\pm$ are on $\Gamma_\mp$.

Fact

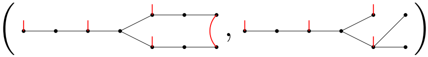
The dual graph of $A_0 \subset A_1$ is the principal graph of $A_1 \subset A_2$.

Examples of principal graphs

- index < 4: A_n, D_{2n}, E_6, E_8. No D_{odd} or E_7.
- Graphs for $R \subset R \rtimes G$ obtained from G and Rep(G).



$G = S_3$

- Haagerup $3^{\mathbb{Z}/3}$ 

- First graph is principal, second is dual principal.
- Leftmost vertex is the trivial bimodule.
- Red tags for duality (conjugates of bimodules).
- Duality of odd vertices by depth and height

Steps of small index standard invariant classification

1. **Enumerate** possible graph pairs. 

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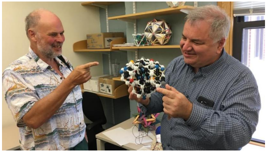
Theorem [Pop94]

For a subfactor $A \subset B$, $[B : A] \geq \|\Gamma_+\|^2 = \|\Gamma_-\|^2$.

- If we enumerate all graph pairs with norm at most r, we have found all principal graphs of subfactors with index at most r^2.

Theorem (Ocneanu Rigidity [ENO05])

There are only finitely many standard invariants with the same finite principal graphs.



Classification to index 4

1. Bipartite graphs with norm at most 2 have an ADE classification [GdlHJ89].
2. There are obstructions to the existence of a standard invariant with principal graphs D_{odd} or E_7 [Ocn88, Izu91].
3. All other examples are realizable [Kaw95].

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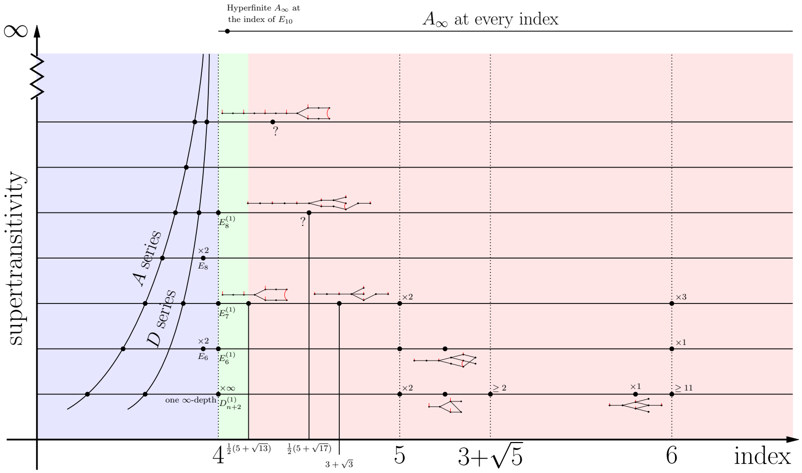
name	#	∃, !
A_n	1	[Jon83, Ocn88]
D_{2n}	1	[Ocn88, Kaw95]
E_6, E_8	2	[Ocn88, BN91, Izu94, Kaw95]
$A^{(1)}_{2n-1}$	n	[Pop94]
$D^{(1)}_{n+2}$	n	[Pop89, IK93]
$E^{(1)}_6, E^{(1)}_7, E^{(1)}_8$	1	[GdlHJ89, Kaw95]
$A_\infty, A^{(1)}_\infty, D_\infty$	1	[Pop89, Pop94]

Known small index standard invariants, 1994



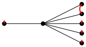
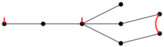
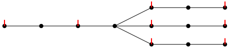
- Haagerup [Haa94, AH99]
- A_∞ everywhere [Pop93]
- Quantum groups [Wen88, Wen90]
- GHJ 3311 [GdlHJ89]

Supertransitivity

Definition

A graph pair (Γ_+, Γ_-) is n-**supertransitive** if it begins with an initial segment consisting of the Coxeter-Dynkin diagram A_{n+1}, i.e., an initial segment with n edges.

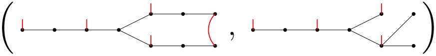
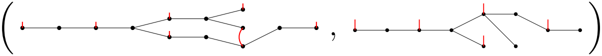
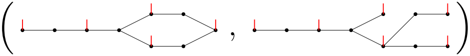
Examples

-  is 1-supertransitive
-  is 2-supertransitive
-  is 3-supertransitive

Haagerup's enumeration

Theorem [Haa94]

Any non A_∞-standard invariant in the index range $(4, 3+\sqrt{2})$ must have principal graphs a **translation** of one of

- 
- 
- 

Translation means raising the supertransitivity of both graphs by the same even amount.

Definition [MS12]

A **vine** is a graph pair which represents an infinite family of graph pairs obtained by translation.

Main tool for Haagerup's enumeration

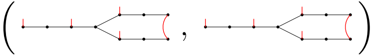
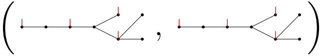
Play **associativity** off of Ocneanu's **triple point obstruction**.

- Associativity: graphs must be similar
- Ocneanu's triple point obstruction: graphs must be different!

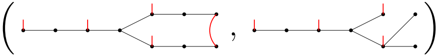
The consequence is a strong constraint.

Example

The following pairs are not allowed:

 and 

They must be paired with each other:



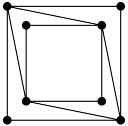
Associativity

Start with an $A - A$ bimodule, tensor on left and right with L^2B:

$$\begin{tikzcd} {}_A X_A \arrow[r, "- \boxtimes_A L^2B"] \arrow[d, "L^2B \boxtimes_A -"] & {}_A X \boxtimes_A L^2B_B \arrow[d, "L^2B \boxtimes_A -"] \\ {}_B L^2B \boxtimes_A X_A \arrow[r, "- \boxtimes_A L^2B"] & {}_B L^2B \boxtimes_A X \boxtimes_A L^2B \end{tikzcd}$$

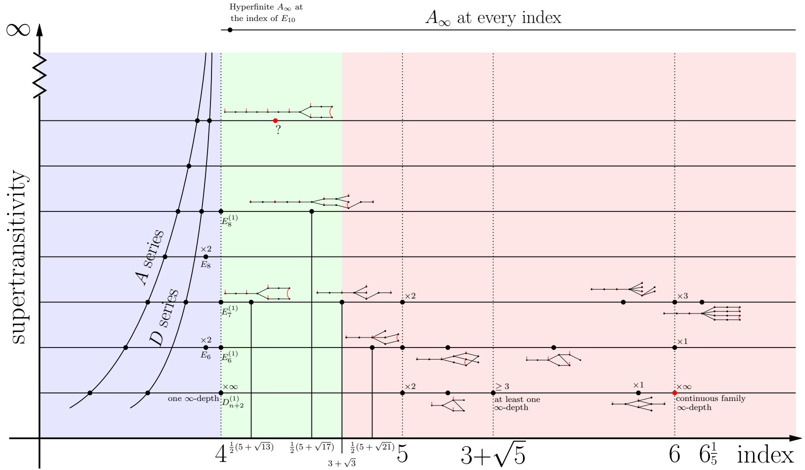
Same simple $B - B$ summands must appear going either way.
Similar result going from $A - B$ to $B - A$ bimodules.

$$\begin{tikzcd} A - A \arrow[r, "\Gamma_+"] \arrow[d, "\Gamma_+"] & A - B \arrow[d, "\Gamma_-"] \\ B - A \arrow[r, "\Gamma_-"] & B - B \end{tikzcd}$$

e.g.,  $\Gamma_\pm = A_4$

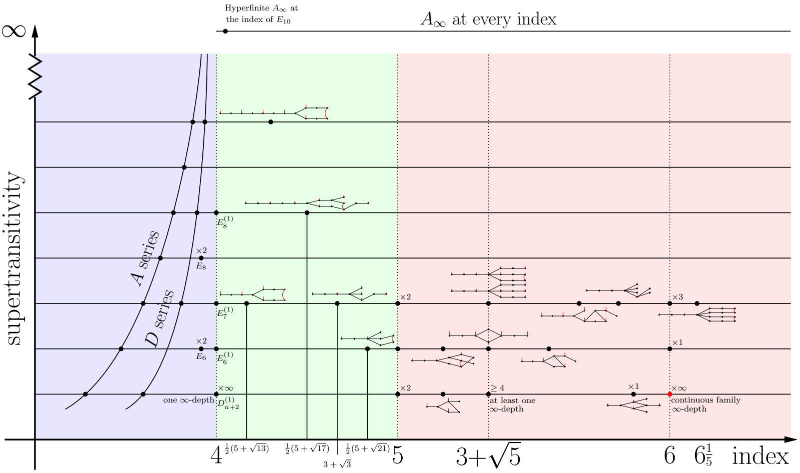
- **Associativity**: same number of paths going either way between vertices on opposite corners

Known small index subfactors, 2007



- Asaeda-Haagerup [AH99]
- No Hexagon vine [Bis98]
- Trimed Haagerup vine [AY09]
- Izumi-Xu 2221 [Izu01]
- 3^{odd} examples [Izu01]
- Wildness at index 6 [BNP07]

Known small index subfactors, 2011



- Extended Haagerup [BMPS12]
- Classification to index 5
- 3^{even} examples [Izu18]
- Haagerup+1 [GS12]

Small index 1ST standard invariants

For many small index 1-supertransitive standard invariants, one can prove there is a corresponding **intermediate subfactor**. This forces the index to be **composite**.

- $2 \times 2 = 4$
- $2 \times \phi^2 = 3 + \sqrt{5} \simeq 5.236$

Theorem [MS12]

There are no 1ST standard invariants with index in $(4, 5)$.

Sketch of proof.

Look at depth 2 objects.

- If all have dimension 1, index is an integer.
- If some have index 1 and some are larger, have an intermediate by [PP86, Bis94].
- An object with dimension in $(1, 2)$ implies the index is too big.
- 2 objects with dimension 2 implies the index is too big. $\square$

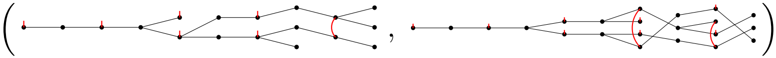
Weeds and vines

The classification to index 5 introduced **weeds** and **vines**.

Definition

A **weed** is a graph pair which represents an infinite family of graph pairs obtained by translation and extension.
An **extension** of a graph pair adds new vertices and edges at strictly greater depths than the maximum depth of any vertex in the original pair.

$\mathcal{F} =$



Using weeds allows us to bundle hard cases together, ensuring the enumerator terminates.

Eliminating vines with number theory

We can uniformly treat vines using number theory, based on the following theorem inspired by Asaeda-Yasuda [AY09]:

Theorem [CMS11]

For a fixed vine $\mathcal{V}$, there is an effective (computable) constant $\mathcal{R}(\mathcal{V})$ such that any n-translate with $n > \mathcal{R}(\mathcal{V})$ has norm squared which is not a cyclotomic integer.

Theorem [CG94, ENO05]

Dimensions of objects in fusion categories are cyclotomic integers.

- The index of a finite depth subfactor (which equals the norm squared of the principal graph) must be a cyclotomic integer.

Why do we care about index $3+\sqrt{5}$?

- Standard invariants at index 4 are completely classified.
  - $\mathbb{Z}/2 * \mathbb{Z}/2 = D_\infty$ is amenable
- Standard invariants at index 6 are wild.
  - There is (at least) one standard invariant for every normal subgroup of the modular group $\mathbb{Z}/2 * \mathbb{Z}/3 = PSL(2,\mathbb{Z})$
  - There are unclassifiably many distinct hyperfinite subfactors with standard invariant $A_3 * D_4$ [BV15].
    (Unpublished joint work shows this holds for $A_3 * A_5$ too.)
- $4 = 2\times 2$ and $6 = 2\times 3$ are composite indices, as is $3+\sqrt{5} = 2\phi^2$ where $\phi = \frac{1+\sqrt{5}}{2}$.

Open Question: How many hyperfinite II_1 subfactors have standard invariant Bisch-Jones' Fuss-Catalan $A_3 * A_4$ standard invariant at index $3+\sqrt{5}$?

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index	Haagerups
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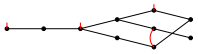
$3+\sqrt{4} = 5$	5     
$3+\sqrt{5} \approx 5.236$	70

Index $(5, 3+\sqrt{5})$

There are exactly two non-A_∞ irreducible standard invariants in the index range $(5, 3+\sqrt{5})$:

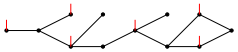
name	Principal graphs	Index	$\exists, !$
$\mathrm{Alt}(PSU(2)_5)$	( , )	5.04892	[Wen90], [MP14]
$\mathrm{Alt}(SU(3)_4)$	( , )	5.04892	[Wen88], [MP14]

Theorem [MP14]

There is exactly one 1-supertransitive subfactor in the index range $(5, 3+\sqrt{5})$

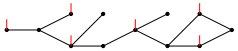
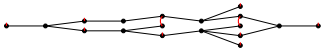
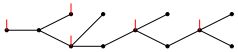
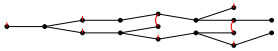
1-supertransitive subfactors at index $3+\sqrt{5}$

Bisch-Haagerup found an infinite family of 1ST graph pairs at index $3+\sqrt{5}$ which admit unique biunitary connections

 ,  ,  , $\cdots$

Theorem [Liu15]

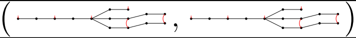
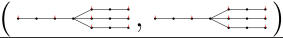
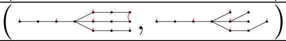
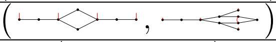
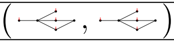
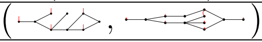
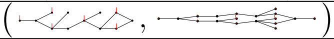
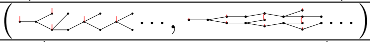
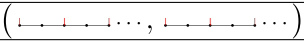
There are exactly seven 1-supertransitive standard invariants with index $3+\sqrt{5}$:

- ( , ) self-dual
- ( , ) and its dual
- ( , ) and its dual
- ( $\cdots$,  $\cdots$) and its dual $(A_3 * A_4)$

These are all the standard invariants of composed inclusions of A_3 and A_4 subfactors.

Standard invariants at index $3+\sqrt{5}$

At $3+\sqrt{5}$, we have only the following standard invariants:

name	Principal graphs	#	$\exists, !$
4442		1	[MP15b, MP15a, Izu18]
$3^{\mathbb{Z}/2\times\mathbb{Z}/2}$		1	[Izu18], [MP15b]
$3^{\mathbb{Z}/4}$		2	[Izu18], [PP15]
$2D2$		2	[Izu18], [MP15a]
$A_3 \otimes A_4$		1	$\otimes$, [Liu15, IMP16]
fish 2		2	[BH], [Liu15, IMP16]
fish 3		2	[IMP16, Liu15]
$A_3 * A_4$		2	[BJ97],
A_∞		1	[Pop93]

Methods to push classification results further

Enumeration:

- 1-supertransitive classification to $6\frac{1}{5}$ [LMP15]
- High-tech graph pair enumerator, based on Brendan McKay's isomorph free enumeration by canonical construction paths [McK98]. Two independent implementations, same results. (Afzaly and Morrison-P)

- Popa's principal graph stability [Pop95, BP14]

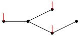
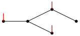
Obstructions:

- Number theory for stable weeds [CG18]
- Morrison's hexagon obstruction [Mor14]
- Powerful triple point obstruction [Pen15]

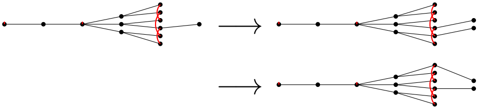
Why better combinatorics are needed

Three ways we produce redundant isomorphism classes of graphs:

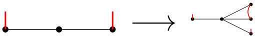
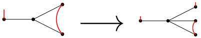
(1) Equivalent generating steps from same object give isomorphic results.

 and 

(2) Two inequivalent generating steps applied to the same object can yield isomorphic objects.



(3) Starting with two non-isomorphic objects and applying a generating step can result in isomorphic objects.

 and 

Problems fixed by McKay's isomorph-free enumeration [McK98]!

Popa's principal graph stability

Definition

We say $\Gamma_\pm$ is **stable at depth** n if every vertex at depth n connects to at most one vertex at depth $n+1$, no two vertices at depth n connect to the same vertex at depth $n+1$, and all edges between depths n and $n+1$ are simple.

Theorem [Pop95, BP14]

Suppose $A \subset B$ (finite index) has principal graphs (Γ_+, Γ_-). Suppose that the truncation $\Gamma_\pm(n+1) \neq A_{n+2}$ and $\delta > 2$.

(1) If $\Gamma_\pm$ are stable at depth n, then $\Gamma_\pm$ are stable at depth k for all $k \geq n$, and $\Gamma_\pm$ are finite.

(2) If Γ_+ is stable at depths n and $n+1$, then $\Gamma_\pm$ are stable at depth $n+1$.

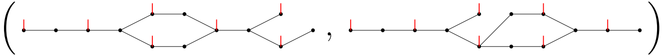
Part (2) uses the 1-click rotation in the planar algebra and the jellyfish algorithm [BMPS12].

Stable weeds

Definition

A **stable weed** represents an infinite family of graph pairs obtained by translation and finite stable extension.

$\mathcal{C} =$



Theorem [CG18]

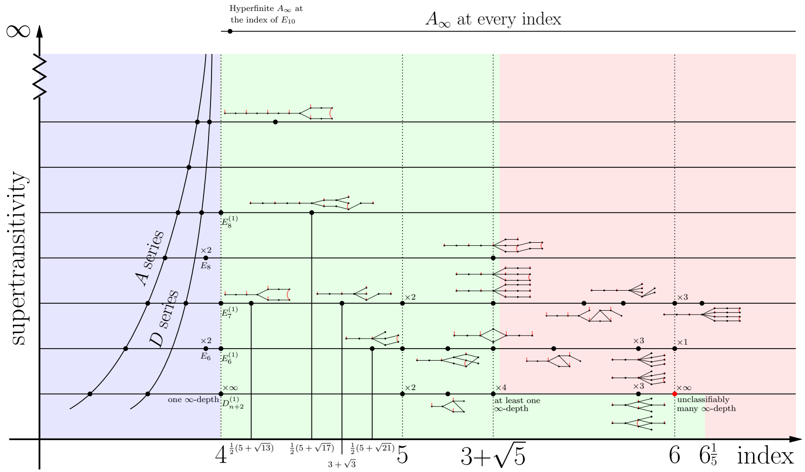
Let $\mathcal{S}_M$ be the class of finite graphs satisfying:

1. all vertices have valence at most M, and
2. at most M vertices have valence > 2.

Then ignoring A_n, D_n, $A_n^{(1)}$, and $D_n^{(1)}$, only finitely many graphs in $\mathcal{S}_M$ have norm squared which is a cyclotomic integer.

- Result is effective for a given fixed stable weed.

Irreducible standard invariants with index at most $5\frac{1}{4}$



Theorem [AMP15, Liu15], building on work of many others

We know all standard invariants up to index $5\frac{1}{4} > 3+\sqrt{5}$, the first interesting composite index.

Recent progress

In joint work with Grossman, Morrison, Peters, and Snyder [GMP+18], we find all standard invariants related to the exotic Extended Haagerup.

- Many new standard invariants and fusion categories

Question (Vaughan Jones, ∼2000)

In which graph planar algebras can you embed a subfactor planar algebra?

Module Embedding Theorem [GMP+18]

A subfactor planar algebra embeds in a graph planar algebra if and only if the graph is a fusion graph for a module category.

The latest classification result

While classifying fusion categories of rank 4 remains out of reach, we have the following recent result.

Theorem [EMIP21]

We have a complete classification of all **quadratic** $\mathbb{Z}/2$ categories, whose simple objects consist of the group $\mathbb{Z}/2$ together with one other orbit under the group action.

$$\text{simples } = \mathbb{Z}/2 \cup \mathbb{Z}/2\{\rho\}$$

- Building on [Lar14], this completes the classification of rank 4 unitary fusion categories with a dual pair of simples.

Thank you for listening!

Full slides available at
https://people.math.osu.edu/penneys.2/talks/PenneysWaterloo2022Talk3.pdf

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