

1. Compute the following limits. Do NOT use L'hôpital's Rule.

(a)

$$\lim_{r \rightarrow 4} \frac{r^3 - 4r^2}{r^2 - 2r - 8} = \frac{64 - 64}{16 - 8 - 8} = \frac{0}{0}$$

$$= \lim_{r \rightarrow 4} \frac{r^2(r-4)}{(r-4)(r+2)}$$

$$= \lim_{r \rightarrow 4} \frac{r^2}{r+2} = \frac{4^2}{4+2} = \frac{16}{6}$$

(b)

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^6 + 10}}{5 + 4x^3}$$

Since $x < 0$ and
3 is odd
 $x^3 = -\sqrt[3]{x^6}$

$$= \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^6 + 10} \cdot \frac{1}{x^3}}{(5 + 4x^3) \cdot \frac{1}{x^3}} = \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^6 + 10} \cdot \frac{1}{-\sqrt{x^6}}}{\frac{5}{x^3} + 4}$$

$$= \lim_{x \rightarrow -\infty} \frac{-\sqrt{\frac{4x^6 + 10}{x^6}}}{\frac{5}{x^3} + 4} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{4 + \frac{10}{x^6}}}{\frac{5}{x^3} + 4} = \frac{-\sqrt{4 + 0}}{0 + 4} = \frac{-\sqrt{4}}{4} = -\frac{1}{2}$$

(c)

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$$

$\frac{\sqrt{9} - 3}{9 - 9} = \frac{0}{0}$ MORE ALGEBRA!

$$= \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \lim_{x \rightarrow 9} \frac{(\sqrt{x})^2 - 3^2}{(x-9)(\sqrt{x}+3)}$$

$$= \lim_{x \rightarrow 9} \frac{x - 9}{(x-9)(\sqrt{x}+3)} = \lim_{x \rightarrow 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

2. Compute the following limits. You **may** use L'hôpital's Rule.

(a)

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} \quad \frac{0}{0}$$

\curvearrowleft L'hôpital

$$= \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} \quad \left(\frac{1-1}{0} = \frac{0}{0} \right)$$

\curvearrowleft L'hôpital

$$= \lim_{x \rightarrow 0} \frac{-\sin x}{6x} \quad \left(\frac{0}{0} \right)$$

\curvearrowleft L'hôpital

$$= \lim_{x \rightarrow 0} \frac{-\cos x}{6} = \frac{-1}{6}$$

(b)

$$\lim_{x \rightarrow 0^+} \sin(x) \ln(x) \quad \sin(0) \cdot \ln(0^+) = 0 \cdot (-\infty)$$

\curvearrowleft

$$= \lim_{x \rightarrow 0^+} \frac{\ln(x)}{\frac{1}{\sin x}} = \lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x} \quad \begin{array}{l} \text{L'hôpital} \\ \curvearrowleft \end{array}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}$$

\curvearrowleft

$$= \lim_{x \rightarrow 0^+} \frac{-\sin x \cdot \tan x}{x} \quad \begin{array}{l} \text{Simplifying} \\ \curvearrowleft \end{array}$$

$$= \lim_{x \rightarrow 0^+} \frac{-\sin x \cdot \sec^2 x - \cos x \cdot \tan x}{1} = \frac{-0 \cdot 1 - 1 \cdot 0}{1} = 0$$

(c)

$$\lim_{x \rightarrow \infty} (x^2 + 1)^{3/x} \quad (\infty)^0$$

\curvearrowleft

$$= e^{\lim_{x \rightarrow \infty} \ln [(x^2 + 1)^{3/x}]} \quad \begin{array}{l} \text{L} \\ \curvearrowleft \end{array}$$

$L = \lim_{x \rightarrow \infty} \frac{3}{x} \ln(x^2 + 1) = \lim_{x \rightarrow \infty} \frac{3 \ln(x^2 + 1)}{x} \quad \begin{array}{l} (\infty, \text{ so L'hôpital} \\ \text{works}) \end{array}$

$$= \lim_{x \rightarrow \infty} \frac{3 \cdot \frac{1}{x^2 + 1} \cdot 2x}{1} = \lim_{x \rightarrow \infty} \frac{3 \cdot \frac{1}{x^2 + 1} \cdot 2x}{1}$$

\curvearrowleft

$$= \lim_{x \rightarrow \infty} \frac{6x}{x^2 + 1} \quad \begin{array}{l} \text{L'hôpital} \\ \curvearrowleft \end{array} = \lim_{x \rightarrow \infty} \frac{6}{2x} = 0.$$

So ~~for large~~ $\lim_{x \rightarrow \infty} (x^2 + 1)^{3/x} = e^L = e^0 = 1$

3. Use the limit definition of the derivative to compute the derivative of $f(x) = 3 + \frac{2}{x}$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3 + \frac{2}{x+h} - \left(3 + \frac{2}{x}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \lim_{h \rightarrow 0} \frac{\frac{2x}{(x+h)x} - \frac{2(x+h)}{(x+h)x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2x - 2(x+h)}{h(x+h)x} = \lim_{h \rightarrow 0} \frac{2x - 2x - 2h}{h(x+h)x}$$

$$= \lim_{h \rightarrow 0} \frac{-2h}{h(x+h)x} = \lim_{h \rightarrow 0} \frac{-2}{(x+h)x}$$

$$= \frac{-2}{(x+0)x} = \frac{-2}{x^2}$$

4. A curve in the plane is given by the following equation

$$x^2 + 2xy + 3y^2 = 4$$

(a) Find an expression for $\frac{dy}{dx}$ using implicit differentiation.

$$\frac{d}{dx} (x^2 + 2xy + 3y^2) = \frac{d}{dx} 4$$

$$2x + (2x)' \cdot y + 2x(y') + 6y \cdot y' = 0$$

$$2x + 2y + 2x(y') + 6yy' = 0$$

$$2x + 2y + 2x(y') + 6yy' = 0$$

$$(2x + 6y)y' = -2x - 2y$$

$$y' = \frac{-2x - 2y}{2x + 6y}$$

(b) Find the equation of the tangent line to the curve at the point $(2, 0)$.

$$y'|_{(2,0)} = \frac{-2 \cdot 2 - 2 \cdot 0}{2 \cdot 2 + 6 \cdot 0} = \frac{-4}{4} = -1$$

$$y - 0 = -1 \cdot (x - 2)$$

(c) Find the two points on the curve where the tangent line is horizontal.

$$y' = 0 \text{ only when } -2x - 2y = 0 \rightarrow 2y = -2x \rightarrow y = -x$$

~~so~~ ~~so~~ The points also need to be on the curve!

If $y = -x$ on the curve then

$$x^2 + 2x(-x) + 3(-x)^2 = 4$$

$$x^2 - 2x^2 + 3x^2 = 4$$

$$2x^2 = 4$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

so points on the curve with horizontal tangent lines are $(\sqrt{2}, -\sqrt{2})$ & $(-\sqrt{2}, \sqrt{2})$

5. Find the derivative of

$$f(x) = (\ln x)^{7x}, \quad x > 1$$

$$\ln f(x) = \ln((\ln x)^{7x}) = 7x \ln \ln x$$

$$\frac{d}{dx} \ln f(x) = \frac{d}{dx} 7x \ln \ln x$$

$$\frac{f'(x)}{f(x)} = (7x)' \cdot \ln \ln x + 7x \cdot (\ln \ln x)'$$

$$\frac{f'(x)}{f(x)} = 7 \cdot \ln \ln x + 7x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$f'(x) = (\ln x)^{7x} \left[7 \ln \ln x + \frac{7}{\ln x} \right]$$

Find an equation of the tangent line to $f(x)$ at $x = e$.

$$\begin{aligned} f'(e) &= (\ln e)^{7 \cdot e} \left[7 \ln \ln e + \frac{7}{\ln e} \right] \\ &= 1^{7e} \cdot \left[7 \underbrace{\ln 1}_{=0} + \frac{7}{1} \right] = 7 \end{aligned}$$

$$f(e) = (\ln e)^{7e} = 1^{7e} = 1$$

$$Y - 1 = 7(x - e)$$

6. Let $f(x) = 3x^{2/3} - x$. ← Domain is all real numbers

(a) Find all critical points.

$$f'(x) = 3 \cdot \frac{2}{3} \cdot x^{-\frac{1}{3}} - 1 = \frac{2}{x^{\frac{1}{3}}} - 1 = \frac{2 - x^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$f'(x)$ DNE when $x^{\frac{1}{3}} = 0 \rightarrow x = 0$

$f'(x) = 0$ when $2 - x^{\frac{1}{3}} = 0 \rightarrow 2 = x^{\frac{1}{3}} \rightarrow 2^3 = x \rightarrow x = 8$

So $x=0$ & $x=8$ are the critical points.

(b) For each critical point, determine if it is a local max, a local min, or neither.

	0	8	
$2 - x^{\frac{1}{3}}$	+	+	-
$x^{\frac{1}{3}}$	-	+	+
$f'(x)$	-	+	-

By first derivative test,
there is a local min at $x=0$
& a local max at $x=8$.

(c) Does the Mean Value Theorem apply to the function f on the interval $[-1, 8]$? 27

No, the function is not differentiable
at $x=0$.

(d) Find all points c in $(-1, 8)$ where the instantaneous rate of change of f at c is equal to the average rate of change of f on $[-1, 8]$. 27

Average rate of change = ~~$\frac{f(8) - f(-1)}{8 - (-1)} = \frac{3(8)^{\frac{2}{3}} - 27 - (3(-1)^{\frac{2}{3}} + 1)}{9} = \frac{3 \cdot 4 - 27 - (3+1)}{9} = \frac{12 - 27 - 4}{9} = \frac{-19}{9}$~~

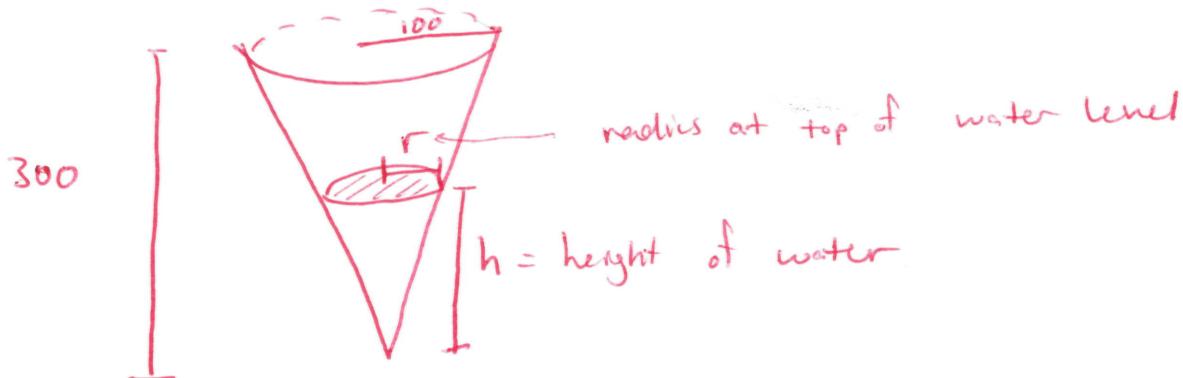
$$\rightarrow = \frac{f(27) - f(-1)}{27 - (-1)} = \frac{3(27)^{\frac{2}{3}} - 27 - (3(-1)^{\frac{2}{3}} + 1)}{28} = \frac{3 \cdot 9 - 27 - (3+1)}{28} = \frac{27 - 27 - 4}{28} = \frac{-4}{28} = -\frac{1}{7}$$

$$f'(c) = \frac{2}{c^{\frac{1}{3}}} - 1 = -\frac{1}{7} \rightarrow \frac{2}{c^{\frac{1}{3}}} = \frac{6}{7} \rightarrow \frac{c^{\frac{1}{3}}}{2} = \frac{7}{6} \rightarrow c^{\frac{1}{3}} = \frac{7}{3}$$

$C = \left(\frac{7}{3}\right)^3$

7. Water is being pumped into an inverted conical tank at a constant rate of 1000π cubic centimeters per minute. The tank has a height of 300cm and a diameter of 200cm. [Note that the volume of a cone with height H and radius R is $\frac{1}{3}\pi R^2 H$]

How fast is the water level rising when the height of the water in the tank is 50cm?



$$V = \text{volume of water} = \frac{\pi}{3} r^2 h.$$

So the question is: What is $\frac{dh}{dt}$ when $\frac{dV}{dt} = 1000\pi$ & $h=50$.

$$\frac{r}{100} = \frac{h}{300} \rightarrow r = \frac{1}{3}h$$

$$V = \frac{\pi}{3} \cdot (\frac{1}{3}h)^2 \cdot h = \frac{\pi}{27} \cdot h^3$$

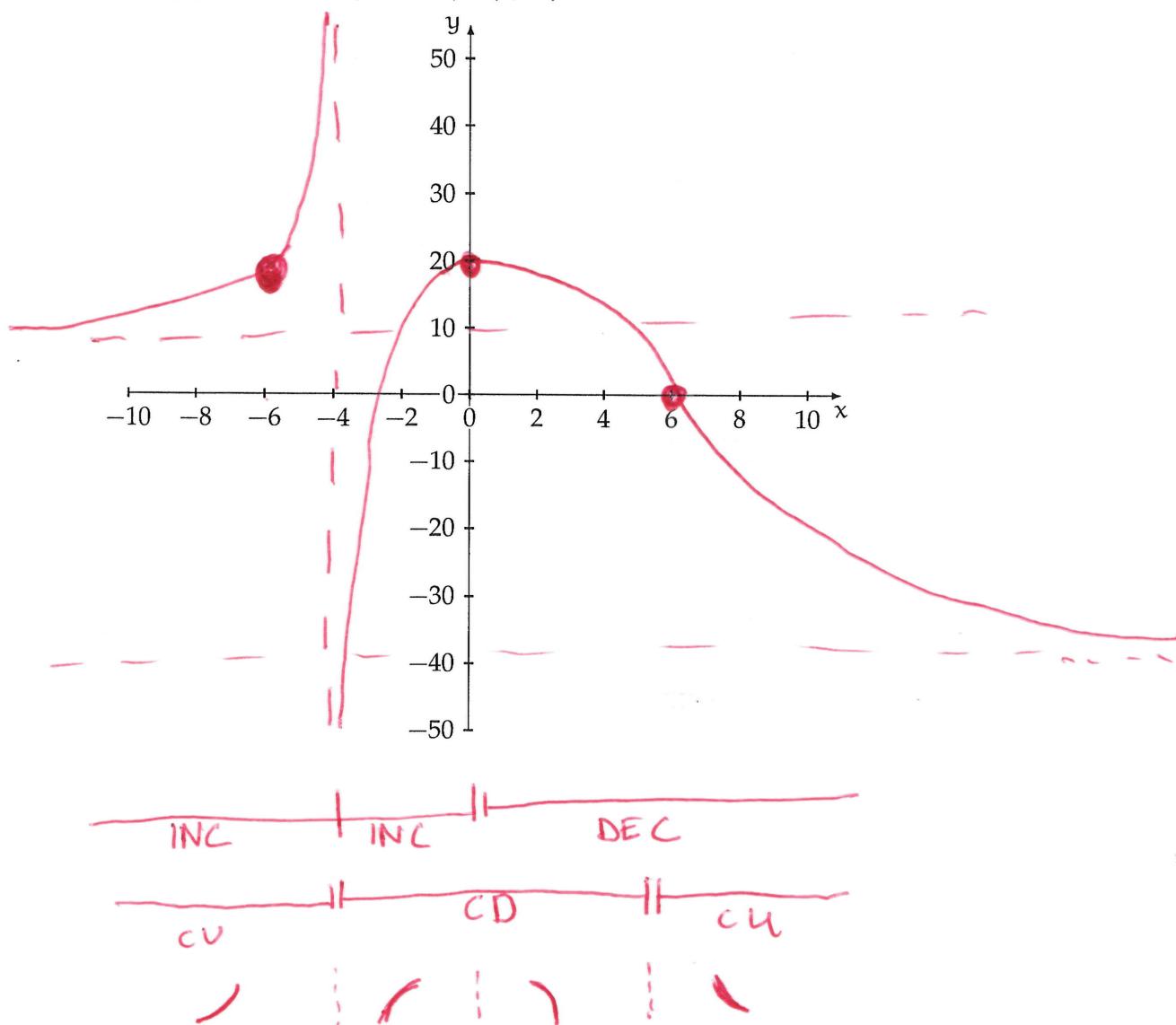
$$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3h^2 \cdot \frac{dh}{dt} = \frac{\pi}{9} \cdot h^2 \frac{dh}{dt}$$

$$1000\pi = \frac{\pi}{9} \cdot 50^2 \cdot \frac{dh}{dt}$$

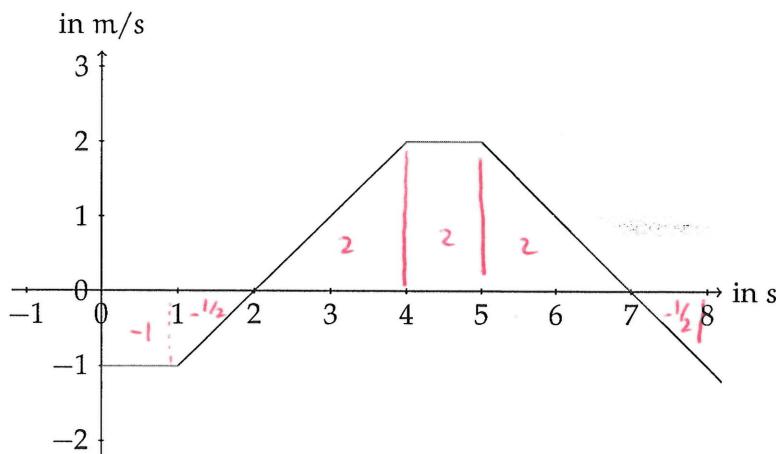
$$\frac{dh}{dt} = \frac{1000\pi}{50^2 \pi} \cdot 9 = \frac{18}{5} \frac{\text{cm}}{\text{min}}$$

8. Graph a function with the following properties:

1. f is continuous on $(-\infty, -4) \cup (-4, \infty)$.
2. $f(-6) = 20, f(0) = 20, f(6) = 0$
3. $\lim_{x \rightarrow -4^-} f(x) = \infty$
4. $\lim_{x \rightarrow -4^+} f(x) = -\infty$
5. $\lim_{x \rightarrow \infty} f(x) = -40$
6. $\lim_{x \rightarrow -\infty} f(x) = 10$
7. $f'(x) < 0$ for x in $(0, \infty)$
8. $f'(x) > 0$ for x in $(-\infty, -4) \cup (-4, 0)$
9. $f''(x) < 0$ for x in $(-4, 6)$
10. $f''(x) > 0$ for x in $(-\infty, -4) \cup (6, \infty)$



9. The graph below represents the velocity $v(t)$ at time t of an object moving in a straight line.



- (a) Use this graph to complete the following table specifying the location $s(t)$ of the object at time t .

t	0	1	2	4	5	7	8
$s(t)$	0	-1	-1.5	0.5	2.5	4.5	4

- (b) Determine the displacement of the object during the time interval $[0, 8]$.

$$\text{Displacement} = \int_0^8 v(u) du = 4$$

- (c) Determine the distance that the object travels during the time interval $[0, 8]$.

$$\begin{aligned} \text{Distance} &= \int_0^8 |v(u)| du \\ &= 1 + \frac{1}{2} + 2 + 2 + 2 + \frac{1}{2} = 8 \end{aligned}$$

10. Consider the following sum

$$\frac{1}{\sqrt{3 + \frac{2}{n}}} \cdot \frac{2}{n} + \frac{1}{\sqrt{3 + \frac{4}{n}}} \cdot \frac{2}{n} + \frac{1}{\sqrt{3 + \frac{6}{n}}} \cdot \frac{2}{n} + \dots + \frac{1}{\sqrt{3 + \frac{2n}{n}}} \cdot \frac{2}{n}$$

(a) Find a definite integral $\int_a^b f(x) dx$ for which the above sum is a right Riemann sum.

[Many right answers here]

$$= \sum_{k=1}^n \underbrace{\frac{1}{\sqrt{3 + \frac{2k}{n}}}}_{f(x_k^*)} \cdot \underbrace{\frac{2}{n}}_{\Delta x}$$

$$f(x_k^*) = \frac{1}{\sqrt{3 + \cancel{2k}} \frac{n}{n}}$$

$$x_k^* = a + k \cdot \Delta x = a + k \cdot \frac{2}{n}$$

~~$$We can set have a=3, f(x)=\frac{1}{\sqrt{x}} \text{ and } \frac{b-a}{n} = \Delta x = \frac{2}{n} \Rightarrow b=5$$~~

So

$$\boxed{\int_3^5 \frac{1}{\sqrt{x}} dx}$$

other possible answers

$$\int_a^b \frac{1}{\sqrt{x+c}} dx \text{ where } b-a=2$$

$c=3-a$

My answer is
for $a=3$

(b) Using the answer from part (a), find the limit of the above sum as $n \rightarrow \infty$.

The limit of this sum is the definite integral!

$$\text{Limit} = \int_3^5 \frac{1}{\sqrt{x}} dx = \int_3^5 x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}} \Big|_3^5$$

$$= 2\sqrt{5} - 2\sqrt{3}$$

11. Evaluate the following integrals.

(a)

$$\int \frac{4x+3}{1+x^2} dx$$

$$\begin{aligned} &= \int \frac{4x}{1+x^2} dx + \int \frac{3}{1+x^2} dx \\ &\quad u = 1+x^2 \\ &\quad du = 2x dx \\ &= \int \frac{2du}{u} + \int \frac{3}{1+x^2} dx = 2 \ln|u| + 3 \arctan x + C \\ &= 2 \ln|1+x^2| + 3 \arctan(x) + C \end{aligned}$$

(b)

$$\begin{aligned} &\int \frac{(8+\ln x)^3}{x} dx \\ &\quad u = 8+\ln x \\ &\quad du = \frac{1}{x} dx \\ &\Rightarrow \int u^3 \cdot du = \frac{1}{4} u^4 + C \\ &= \frac{1}{4} (8+\ln x)^4 + C \end{aligned}$$

(c)

$$\int_0^\pi \frac{d}{dt} \frac{\sin(t/2)}{1+t^2} dt$$

$$= \frac{\sin(t/2)}{1+t^2} \Big|_{t=0}^\pi$$

$$= \frac{\sin(\pi/2)}{1+\pi^2} - \frac{\sin(0)}{1+0^2} = \frac{1}{1+\pi^2}$$

12. Let

$$G(x) = \frac{1}{x^3} \int_{x^2}^4 \frac{u^3}{2+u^3} du$$

(a) Find $G'(2)$

$$\begin{aligned} G'(x) &= \left(\frac{1}{x^3}\right)' \cdot \underbrace{\int_{x^2}^4 \frac{u^3}{2+u^3} du}_{\text{Let } u = 2+u^3} + \frac{1}{x^3} \left[\int_{x^2}^4 \frac{u^3}{2+u^3} du \right]' \\ &= -\frac{3}{x^4} \underbrace{\int_{x^2}^4 \frac{u^3}{2+u^3} du}_{\text{Let } u = 2+u^3} + \frac{1}{x^3} \left[-\frac{x^2}{4} \frac{u^3}{2+u^3} \right]' \\ &= -\frac{3}{x^4} \underbrace{\int_{x^2}^4 \frac{u^3}{2+u^3} du}_{\text{Let } u = 2+u^3} + \frac{1}{x^3} (-1) \cdot \frac{(x^2)^3}{2+(x^2)^3} \cdot 2x \\ G'(2) &= -\frac{3}{2^4} \underbrace{\int_{4}^4 \frac{u^3}{2+u^3} du}_{=0} + \frac{1}{2^3} \cdot (-1) \cdot \frac{2^6}{2+2^6} \cdot 2 \cdot 2 \end{aligned}$$

$$G'(2) = -\frac{1}{2^3} \cdot \frac{2^6}{2+2^6} \cdot 2^2 = -\frac{32}{64} = -\frac{16}{32} \quad \leftarrow \begin{array}{l} \text{[No need to} \\ \text{simplify here]} \end{array}$$

(b) Find the equation of the tangent line to the graph of $G(x)$ at $x = 2$.

$$G(2) = \frac{1}{2^3} \underbrace{\int_4^4 \frac{u^3}{2+u^3} du}_{=0} = 0.$$

$$Y - 0 = -\frac{16}{33} (x - 2)$$

13. Suppose that $f(x)$ is continuous and differentiable everywhere. Below is a table of values for $f(x)$ and $f'(x)$.

x	$f(x)$	$f''(x)$
0	π	3
1	$\pi/2$	2
2	$\pi/4$	10
π	1	-3
$\pi/2$	-1	-2
$\pi/4$	6	5

Compute the following values.

- (a) Let $g(x) = (x + f(x)) \sin x$. Find $g'(\pi)$.

$$g'(x) = [x + f(x)]' \sin x + [x + f(x)] \cdot (\sin x)' \rightarrow g'(\pi) = (1 + f'(\pi)) \sin \pi + (\pi + f(\pi)) \cdot \cos \pi$$

$$g'(x) = (1 + f'(x)) \sin x + (x + f(x)) \cdot \cos x \rightarrow g'(\pi) = (1 + (-3)) \cdot 0 + (\pi + 1) \cdot (-1)$$

- (b) Let $h(x) = f(\tan^{-1} x)$. Find $h'(1)$.

$$h'(x) = f'(\tan^{-1} x) \cdot [\tan^{-1} x]' \rightarrow h'(1) = f'(\tan^{-1} 1) \cdot \frac{1}{1+1} = f'\left(\frac{\pi}{4}\right) \cdot \frac{1}{2}$$

$$= f'(\tan^{-1} x) \cdot \frac{1}{1+x^2} \rightarrow h'(1) = 5 \cdot \frac{1}{2}$$

- (c) Let $H(x) = \tan^{-1}(f(x))$. Find $H'(\pi)$.

$$H'(x) = \frac{1}{1+(f(x))^2} \cdot f'(x) \quad H'(\pi) = \frac{1}{1+(f(\pi))^2} \cdot f'(\pi) = \frac{1}{1+1^2} \cdot (-3) = \frac{-3}{2}$$

(d)

$$\int_0^1 \sin(f(x)) f'(x) dx = \int_{\pi}^{\pi/2} \sin(u) du = -\cos(u) \Big|_{u=\pi}^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos\pi = -0 - 1 = -1$$

$u = f(x)$
 $du = f'(x)dx$
 $x=0 \rightarrow u=f(0)=\pi$
 $x=1 \rightarrow u=f(1)=\frac{\pi}{2}$

(e)

$$\int_0^{\pi/2} f(\cos(x)) \sin(x) dx$$

$u = \cos x$
 $du = -\sin x dx$
 $x=0 \rightarrow u=\cos 0=1$
 $x=\frac{\pi}{2} \rightarrow u=\cos \frac{\pi}{2}=0$

$$= \int_1^0 f(u) (-du)$$

$$= \int_0^1 f'(u) du$$

$$= f(1) - f(0)$$

$$= \frac{\pi}{2} - \pi = -\frac{\pi}{2}$$

14. Consider the following piecewise function (c is some constant)

$$f(x) = \begin{cases} \frac{-8}{x+5} & \text{for } x \geq 3 \\ 2-x & \text{for } -5 \leq x < 3 \\ x^2 + c & \text{for } x < -5 \end{cases}$$

- (a) What is the domain of $f(x)$?

$\frac{-8}{x+5}$ is not defined at $x = -5$, but $f(x) = \frac{-8}{x+5}$ only for $x \geq 3$ so it doesn't matter.
 $2-x$ & $x^2 + c$ are defined for all x .

Domain of $f(x) =$ all real numbers
 $(= \mathbb{R})$

- (b) Determine whether $f(x)$ is continuous at $x = 3$.

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} 2-x = 2-3 = -1 \quad \text{Equal so} \\ \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \frac{-8}{x+5} = \frac{-8}{8} = -1 \quad \lim_{x \rightarrow 3} f(x) = -1,$$

$f(3) = \frac{-8}{8} = -1$ as well, so f is continuous at $x = 3$.

- (c) For which value of c does the limit $\lim_{x \rightarrow -5} f(x)$ exist?

$$\lim_{x \rightarrow -5^-} f(x) = \lim_{x \rightarrow -5^-} x^2 + c = 25 + c$$

$$\lim_{x \rightarrow -5^+} f(x) = \lim_{x \rightarrow -5^+} 2-x = 2-(-5) = 7.$$

So $\lim_{x \rightarrow -5} f(x)$ exists only if $25+c=7$

$$(C = -18)$$

15. Use the linear approximation of $f(x) = \sqrt[3]{x}$ at $a = 27$ to approximate $\sqrt[3]{28}$.

$$L(x) = f(27) + f'(27)(x - 27)$$

$$f(27) = \sqrt[3]{27} = 3$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \quad f'(27) = \frac{1}{3}\frac{1}{(27)^{\frac{2}{3}}} = \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$$

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

$$\sqrt[3]{28} \approx f(28) = 3 + \frac{1}{27}(28 - 27) = 3 + \frac{1}{3}$$

16. Let $f(x) = 2x + e^x$. Note that $f(x)$ is an one-to-one function and that $f(1) = 2 + e$. Find

$$\frac{d}{dx} f^{-1}(2 + e).$$

$$\rightarrow \frac{1}{f'(f^{-1}(2+e))} = \frac{1}{f'(1)}$$

$$f'(x) = 2 + e^x$$

$$\rightarrow \frac{d}{dx} f'(2+e) = \frac{1}{2+e^1} = \frac{1}{2+e}$$

17. Let

$$f(x) = \frac{x}{1+x^2}$$

continuous everywhere

(a) Determine when $f(x)$ is increasing and when it is decreasing.

$$f'(x) = \frac{1 \cdot (1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

bottom is never zero.

$$1-x^2=0 \Leftrightarrow (1-x)(1+x)=0$$

$$x=1, -1$$

$f' \leftarrow \begin{matrix} -1 & + & 1 \\ - & + & - \end{matrix} \Rightarrow \boxed{\begin{matrix} f(x) \text{ is} \\ \text{increasing on } (-1, 1) \\ \text{and decreasing} \\ \text{on } (-\infty, -1), (1, \infty) \end{matrix}}$

(b) Where is $f(x)$ concave up and where is it concave down?

$$f''(x) = \frac{-2x(1+x^2)^2 - (1-x^2) \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^3} = \frac{-2x[1+x^2+2-2x^2]}{(1+x^2)^3}$$

$$= \frac{-2x(3-x^2)}{(1+x^2)^3} = \frac{-2x(3-x^2)}{(1+x^2)^3}$$

top = 0

$$-2x(3-x^2) = 0 \quad \downarrow$$

$$\begin{matrix} x=0 \\ x=\pm\sqrt{3} \end{matrix}$$

(c) Find all local maxima and local minima of $f(x)$.By 1st derivative test,local max at $x=1$ local min at $x=-1$ (d) Find all inflection points of $f(x)$.

Inflection points at

 $x=-\sqrt{3}, 0, \sqrt{3}$ $f'' \leftarrow \begin{matrix} - & + & - & + \end{matrix}$ f concave down on $(-\infty, -\sqrt{3}), (0, \sqrt{3})$ f concave up on $(-\sqrt{3}, 0), (\sqrt{3}, \infty)$ (e) Find all horizontal asymptotes of $f(x)$

$$\lim_{x \rightarrow \infty} \frac{x}{1+x^2} = \lim_{x \rightarrow \infty} \frac{1}{2x} = 0$$

L'Hopital

$$\lim_{x \rightarrow -\infty} \frac{x}{1+x^2} = \lim_{x \rightarrow -\infty} \frac{1}{2x} = 0$$

$y=0$ is
only horizontal asymptote