EXTENDING PROPERTIES TO RELATIVELY HYPERBOLIC GROUPS

DANIEL A. RAMRAS AND BOBBY W. RAMSEY

ABSTRACT. Consider a finitely generated group G that is relatively hyperbolic with respect to a family of subgroups $H_1, \ldots, H_n$. We present an axiomatic approach to the problem of extending metric properties from the subgroups H_i to the full group G. We use this to show that both (weak) *finite decomposition complexity* and *straight finite decomposition complexity* are extendable properties. We also discuss the equivalence of two notions of straight finite decomposition complexity.

1. INTRODUCTION

The concept of relative hyperbolicity was proposed by Gromov in [10], as a generalization of hyperbolicity. Farb, Bowditch, Osin, and Mineyev–Yemen, [1, 7, 18, 16], have developed this in various directions, which are equivalent for finitely generated groups. We follow the approach to relatively hyperbolicity given by Osin [18].

Say G is a finitely generated group that is relatively hyperbolic with respect to a family of subgroups $\{H_i\}_{i=1}^n$, as defined in Section 2. Various authors have considered the problem of extending metric properties of the subgroups H_i to the full group G. In particular, *finite asymptotic dimension*, *coarse embeddability* (also known as *uniform embeddability*), and *exactness* are all known to be extendable [17, 3, 19]. The main goal of this article is to show that *finite decomposition complexity* [12, 13] and *straight finite decomposition complexity* [5] are extendable properties. For finite decomposition complexity, this was previously observed by Sisto [22].

In this article, we present an axiomatic approach to the problem of extendability. This approach is similar in spirit to [11], where properties $\mathcal{P}$ of *metric families* (that is, sets of metric spaces) are studied from the point of view of permanence. We say that a metric space X has the property $\mathcal{P}$ if the metric family $\{X\}$ has $\mathcal{P}$. We identify several conditions that such a property $\mathcal{P}$ may satisfy, which together imply the extendability of $\mathcal{P}$ for relatively hyperbolic groups. These conditions are Coarse Inheritance, the Finite Union Theorem, the Union Theorem, and the Transitive Fibering Theorem, which are defined in Section 3. The Transitive Fibering Theorem is a weak version of Fibering Permanence from [11], and is needed for our study of straight finite decomposition complexity (variants of the other conditions also appear in [11]). We also assume that $\mathcal{P}$ is satisfied by all metric spaces with finite

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asymptotic dimension. Our main tool for extending such properties is the work of Osin [17] regarding the *relative Cayley graph* of a relatively hyperbolic group.

Finite decomposition complexity (FDC) and its weak version (wFDC) were introduced in [12] as natural generalizations of finite asymptotic dimension (FAD), and were used to study rigidity properties of manifolds. The more general notion of straight finite decomposition complexity (sFDC) was recently introduced in [5]. (We review the definitions in Section 4.) The class of groups with FDC is already quite large, and contains all countable linear groups [12, Theorem 3.1]. By [5, Theorem 3.4], all metric spaces with sFDC satisfy Yu's Property A, so finitely generated groups with sFDC satisfy the coarse Baum–Connes conjecture [23]. We have the following chain of implications relating these concepts:

$$\text{FAD} \Longrightarrow \text{FDC} \Longrightarrow \text{wFDC} \Longrightarrow \text{sFDC} \Longrightarrow \text{Property A.}$$

The fact that weak FDC implies straight FDC is proven in Proposition 4.7. It is also possible to formulate a weak version of sFDC, although it was shown by Dydak and Virk that this weak version is in fact equivalent to sFDC [6].

Our results on extendability interact nicely with recent work in algebraic K–theory. It was shown in [20] that the Integral K–theoretic Novikov Conjecture (injectivity of the K–theoretic assembly map) holds for all group rings $R[G]$, where R is a unital ring and G has finite decomposition complexity and a finite classifying space $K(G,1)$. If G is torsion-free and relatively hyperbolic with respect to subgroups $\{H_i\}_{i=1}^n$ satisfying the conditions of this theorem, then G also satisfies the conditions: by Corollary 3.11, G has finite decomposition complexity, and by [8, Theorem A.1], there exists a finite $K(G,1)$[1]. In related work, Goldfarb [9] showed that finitely generated groups with sFDC satisfy *weak regular coherence*, which guarantees the existence of projective resolutions of finite length for certain $R[\Gamma]$–modules over sufficiently well-behaved coefficient rings R. This work is part of a program for proving surjectivity of assembly maps [2].

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2. Relatively Hyperbolic Groups

Suppose G is a finitely generated group with a finite symmetric generating set S, and let $\{H_i\}_{i=1}^k$ be a family of finitely generated subgroups. Then G is a quotient of the free product $F = F(S) * H_1 * H_2 * \cdots * H_k$, where $F(S)$ is the free group on S. Say that G is *finitely presented relative to* $\{H_i\}_{i=1}^k$ if the kernel of the projection $F \to G$ is the normal closure of a finite subset $\mathcal{R}$ in F. (Note that if G is finitely presented, then it is also finitely presented relative to $\{H_i\}_{i=1}^k$).

Set $\mathcal{H} = \sqcup_{i=1}^k (H_i \setminus \{1\})$. If a word w in the alphabet $S \cup \mathcal{H}$ represents the identity element of G, it can be expressed in the form $w = \prod_{j=1}^m a_i^{-1} r_i^{\pm 1} a_i$ where $r_i \in \mathcal{R}$ and $a_i \in F$ for $i = 1, \ldots, m$. The smallest possible number m in such a representation of w is the *relative area* of w, denoted by $Area_{rel}(w)$.

[1]Kasprowski [15] has shown that the Integral K–theoretic Novikov Conjecture holds for all groups G with finite decomposition complexity and a *finite-dimensional* classifying space. Hence it would be interesting to know whether the property of having a finite-dimensional classifying space is extendable (at least for torsion-free groups).

Definition 2.1. *G is* hyperbolic relative to the collection of subgroups $\{H_i\}_{i=1}^k$ *if it is finitely presented relative to $\{H_i\}_{i=1}^k$ and there is a constant K such that every word w in $S \cup \mathcal{H}$ that represents the identity in G satisfies $Area_{rel}(w) \leq K\|w\|$, where $\|w\|$ represents the length of the word in $S \cup \mathcal{H}$.*

A key construction in relatively hyperbolic groups is the relative Cayley graph, $\Gamma(G, S \cup \mathcal{H})$; that is, the Cayley graph of G with respect right multiplication by elements in the generating set $S \cup \mathcal{H}$. This graph is not locally finite. However Osin has proven the following.

Theorem 2.2 ([17, Theorem 17])**.** *The relative Cayley graph $\Gamma(G, S \cup \mathcal{H})$ has finite asymptotic dimension.*

The existence of constants L and ε involved in the following two lemmas (from [17]) will be necessary in what follows, though the results themselves will not be mentioned again. The terminology and notation is taken from [17].

Lemma 2.3. *Suppose that a group G is generated by a finite set S and is hyperbolic relative to $\{H_i\}_{i=1}^k$. Then there is a constant $L > 0$ such that for every cycle q in $\Gamma(G, S \cup \mathcal{H})$, every $i \in \{1, \ldots, k\}$, and every set of isolated H_i-components $p_1, \ldots, p_m$ of q, we have*

$$\sum_{j=1}^{m} d_S\left((p_j)_-, (p_j)_+\right) \leq L\|q\|.$$

Lemma 2.4. *Suppose that a group G is generated by a finite set S and is hyperbolic relative to $\{H_i\}_{i=1}^k$. Then for any $s \geq 0$, there is a constant $\varepsilon = \varepsilon(s) \geq 0$ such that the following condition holds. Let p_1 and p_2 be two geodesics in $\Gamma(G, S \cup \mathcal{H})$ such that $d_S\left((p_1)_-, (p_1)_+\right) \leq s$ and $d_S\left((p_2)_-, (p_2)_+\right) \leq s$. Let c be a component of p_1 such that $d_S\left(c_-, c_+\right) \geq \varepsilon$. Then there is a component of p_2 connected to c.*

3. Extendable properties

Many properties can be extended from the peripheral subgroups $H_1, \ldots, H_n$ to the group G. Coarse embeddability [3], exactness [19], finite asymptotic dimension [17], and combability [14] are just a few examples of such properties. An analysis of [3] and [17] shows much similarity in method.

Given a countable group Γ, we will view Γ as a metric space with respect to a proper left-invariant metric. Any two such metrics are coarsely equivalent, and the properties under consideration here are all coarsely invariant, so the choice of metric will not matter.

Suppose that $\mathcal{P}$ is some property of metric families. We isolate a few features that may hold for $\mathcal{P}$, which will be of interest. Recall that a map between metric spaces, $f\colon X \to Y$, is *uniformly expansive* if there exists a nondecreasing function $\rho\colon [0, \infty) \to [0, \infty)$ such that for all $x, x' \in X$, $d_Y(f(x), f(x')) \leq \rho(d_X(x, x'))$. Such a map is *homogeneous* if for all $y_1, y_2 \in \mathrm{im}(f) \subset Y$ there exist isometries $\phi\colon X \to X$ and $\bar{\phi}\colon Y \to Y$ such that

- $f \circ \phi = \bar{\phi} \circ f$, and
- $\bar{\phi}(y_1) = y_2$.

Lemma 3.1. *Let G be a finitely generated group, with finite symmetric generating set S, and let $\mathcal{H}$ be a finite family of subgroups. Then the map $p\colon G \to \Gamma(G, S \cup \mathcal{H})$, which sends a group element to the vertex it represents, is homogeneous.*

Proof. Let $g, g' \in G$. Denote by v_g and $v_{g'}$ the vertices in $\Gamma(G, S \cup \mathcal{H})$ identified with g and g', respectively. As p is equivariant with respect to left multiplication in G, we define $\phi\colon G \to G$ and $\bar{\phi}\colon \Gamma(G, S \cup \mathcal{H}) \to \Gamma(G, S \cup \mathcal{H})$ through left multiplication by the element $g' g^{-1}$. Thus $\bar{\phi}(g) = g'$, and $p \circ \phi = \bar{\phi} \circ p$. □

There are several versions of the Fibering Theorem. We will establish the following version for straight finite decomposition complexity in Section 5. Recall that we say a metric space X has $\mathcal{P}$ if the family $\{X\}$ has $\mathcal{P}$.

Definition 3.2 (Homogeneous Fibering Theorem)**.** *Say that* $\mathcal{P}$ satisfies the Homogeneous Fibering Theorem *if the following holds.*

> *Let $f\colon E \to B$ be a uniformly expansive, homogeneous map. Assume B has property $\mathcal{P}$ and for each bounded subset $D \subset B$, the inverse image $f^{-1}(D)$ has property $\mathcal{P}$. Then E has property $\mathcal{P}$.*

A significantly weaker version of the above will suffice for studying extendability. We say that a map $f\colon X \to Y$ of metric spaces is *contractive*, or a *contraction*, if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. Such maps are uniformly expansive.

Definition 3.3 (Transitive Fibering Theorem)**.** *Say that* P satisfies the Transitive Fibering Theorem *if the following holds.*

> *Let Γ be a countable group acting isometrically on E and B, and assume B has finite asymptotic dimension and that Γ acts transitively on B. Let $f : E \to B$ be a contractive, Γ–equivariant map. If for each bounded subset $D \subset B$, $f^{-1}(D)$ has property $\mathcal{P}$, then E has property $\mathcal{P}$.*

We note that the maps p considered in the Transitive Fibering Theorem are automatically homogeneous, since Γ is acting by isometries.

Definition 3.4 (Finite Union Theorem)**.** *Say that* $\mathcal{P}$ satisfies the Finite Union Theorem *if the following holds.*

> *Let X be a metric space written as a finite union of metric subspaces $X = \cup_{i=1}^{n} X_i$. If each X_i has $\mathcal{P}$ then so does X.*

The next property addresses more general unions. Recall that two subsets A, B of a metric space X are said to be r–disjoint if $d(A, B) > r$.

Definition 3.5 (Union Theorem)**.** *Say that* $\mathcal{P}$ satisfies the Union Theorem *if the following holds.*

> *Let X be a metric space written as a union of metric subspaces $X = \cup_{i \in \mathcal{I}} X_i$. Suppose that $\{X_i\}_{i \in \mathcal{I}}$ has $\mathcal{P}$ and that for every $r > 0$ there exists a metric subspace $Y(r) \subset X$ with $\mathcal{P}$ such that the sets $Z_i(r) = X_i \setminus Y(r)$ are pairwise r–disjoint. Then X has $\mathcal{P}$.*

Definition 3.6 (Coarse Inheritance)**.** *Say that* $\mathcal{P}$ satisfies Coarse Inheritance *if the following holds.*

> *Let X and Y be metric spaces. If there is a coarse embedding from X to Y and Y has $\mathcal{P}$, then so does X.*

Note that if $\mathcal{P}$ satisfies Coarse Inheritance, then it is a coarsely invariant property.

Definition 3.7. *Say that* $\mathcal{P}$ *is* axiomatically extendable *if it satisfies the Transitive Fibering Theorem, the Finite Union Theorem, the Union Theorem, and Coarse Inheritance, and every metric space with finite asymptotic dimension has* $\mathcal{P}$.

Proposition 3.8. *Coarse embeddability, exactness, and finite decomposition complexity (see Definition 4.4) are axiomatically extendable properties.*

Coarse embeddability and exactness for metric families are defined in [3, Definitions 2.2 and 2.8], where they are referred to as 'equi-embeddability' and 'equi-exactness'.

Proof. For coarse embeddability, the Coarse Inheritance property is clear. The Finite Union Theorem and the Union Theorem are Corollaries 4.5 and 4.6 of [3]. The Transitive Fibering Theorem is a special case of Corollary 4.7 of [3]. Finally, spaces of finite asymptotic dimension are coarsely embeddable [21].

We now turn to exactness. Again, the Coarse Inheritance property follows easily from the definition. Metric spaces of finite asymptotic dimension are exact, by Proposition 4.3 of [3]. The Finite Union Theorem, Union Theorem, and Transitive Fibering Theorem come from Corollaries 4.5, 4.6, and 3.4 of [3].

For finite decomposition complexity, Coarse Inheritance, the Finite Union Theorem, the Union Theorem, and a stronger version of the Fibering Theorem appear in Section 3.1 of [13]. That spaces of finite asymptotic dimension have finite decomposition complexity is proven in [13, Section 4], using [4]. □

Theorem 3.9. *Suppose that* $\mathcal{P}$ *is an axiomatically extendable property. If* G *is relatively hyperbolic with respect to* $H_1, \ldots, H_n$ *and each* H_i *has* $\mathcal{P}$*, then* G *has* $\mathcal{P}$.

We begin by proving an auxiliary lemma. Let

$$B(n) = \{g \in G \,:\, d_{S\cup\mathcal{H}}(e, g) \leq n\}\,.$$

That is, $B(n)$ is the closed ball around e of radius n in $\Gamma(G, S \cup \mathcal{H})$. We consider $B(n)$ as a metric subspace of G, with the word metric associated to S.

Lemma 3.10. *Suppose that each* H_i *has* $\mathcal{P}$*. For any integer* $n > 0$*,* $B(n)$ *has* $\mathcal{P}$.

Note that the word metric on G restricts to give a proper left-invariant metric on each H_i, and we will choose this as our metric on H_i.

Proof. The argument is based on the proof of [17, Lemma 3.2]. Proceed by induction on n. For $n = 1$, $B(1) = S \cup \left(\cup_{i=1}^k H_i\right)$ has $\mathcal{P}$ by the Finite Union Theorem. Let $n > 1$ and assume $B(m)$ has $\mathcal{P}$ for all positive integers $m < n$. We have

$$B(n) = \left(\bigcup_{i=1}^{k} B(n-1)H_i\right) \cup \left(\bigcup_{s\in S} B(n-1)s\right).$$

As each $B(n-1)s$ is coarsely equivalent to $B(n-1)$ and S is finite, $\bigcup_{s\in S} B(n-1)s$ has $\mathcal{P}$ by the Finite Union Theorem and the induction hypothesis. It remains to check that $\bigcup_{i=1}^k B(n-1)H_i$ has $\mathcal{P}$.

Fix $i \in \{1, \ldots, k\}$ and let $R(n-1)$ be a subset of $B(n-1)$ such that

$$B(n-1)H_i = \bigsqcup_{r\in R(n-1)} rH_i.$$

Fix an $s > 0$ and set

$$T_s = \{g \in G \,:\, d_S(e, g) \leq \max\{\varepsilon, 2L(s+1)\}\}\,,$$

where L and $\varepsilon = \varepsilon(s)$ are the constants from Lemmas 2.3 and 2.4 respectively. Let $Y_s = B(n-1)T_s$. As T_s is finite, Y_s has $\mathcal{P}$. Osin shows in [17, Lemma 3.2] that the sets $\{rH_i \setminus Y_s \,:\, r \in R(n-1)\}$ are s–disjoint, so $B(n-1)H_i$ has $\mathcal{P}$ by the Union Theorem. The Finite Union Theorem then shows $\bigcup_{i=1}^{k} B(n-1)H_i$ has $\mathcal{P}$. □

Proof of Theorem 3.9. Consider the map $p\colon G \to \Gamma(G, S \cup \mathcal{H})$. This is a contraction, thus it is uniformly expansive. By Theorem 2.2, $\Gamma(G, S \cup \mathcal{H})$ has finite asymptotic dimension, so $\Gamma(G, S \cup \mathcal{H})$ has the property $\mathcal{P}$ as well.

For each bounded subset Z of $\Gamma(G, S \cup \mathcal{H})$ there is an n such that $p^{-1}(Z)$ lies in $B(n)$. By Lemma 3.10, $B(n)$ has $\mathcal{P}$, and $p^{-1}(Z)$ has $\mathcal{P}$ as well by Coarse Inheritance. Consider the map $p\colon G \to \mathrm{Image}(p)$, which is equivariant with respect to the transitive left-translation actions of G (in fact, p is simply the identity map on underlying set G). Since $\Gamma(G, S \cup \mathcal{H})$ has finite asymptotic dimension, so does $\mathrm{Im}(p) \subset \Gamma(G, S \cup \mathcal{H})$. By the Transitive Fibering Theorem, G has the property $\mathcal{P}$. □

Corollary 3.11. *Suppose G is relatively hyperbolic with respect to $H_1, \ldots, H_n$. If each H_i has finite decomposition complexity, so does G.*

The same argument shows that this result holds with FDC replaced by either of the weak versions (k–FDC or wFDC) discussed in the next section, since the extendability arguments for FDC in [13] all apply to these weak versions as well.

4. Straight finite decomposition complexity

We recall the definition of finite decomposition complexity from [13].

Definition 4.1. *An (k, r)–*decomposition *of a metric space X over a metric family $\mathcal{Y}$ is a decomposition*

$$X = X_0 \cup X_1 \cup \cdots \cup X_{k-1}, \quad X_i = \bigsqcup_{r-disjoint} X_{ij},$$

*where each $X_{ij} \in \mathcal{Y}$. A metric family $\mathcal{X}$ is (k, r)–*decomposable *over $\mathcal{Y}$ if every member of $\mathcal{X}$ admits a (k, r)–decomposition over $\mathcal{Y}$.*

We will write

$$\mathcal{X} \xrightarrow{(k,r)} \mathcal{Y}$$

to indicate that $\mathcal{X}$ admits a (k, r)–decomposition over $\mathcal{Y}$. When $k = 2$, we recover the notion of r–decomposition from [13]. In this case, we will write

$$\mathcal{X} \xrightarrow{r} \mathcal{Y} \tag{1}$$

to mean that $\mathcal{X}$ admits an r–decomposition over $\mathcal{Y}$.

Remark 4.2. *If X admits a (k, r)–decomposition over a metric family $\mathcal{Y}$, then it also admits a (k', r)–decomposition over $\mathcal{Y}$ for each $k' \geqslant k$, since we may repeat the spaces X_i appearing in the decomposition (or add copies of the empty set).*

Definition 4.3. *Let $\mathfrak{U}$ be a collection of metric families. A metric family $\mathcal{X}$ is* k–decomposable *over $\mathfrak{U}$ if, for every $r > 0$, there is a metric family $\mathcal{Y}_r \in \mathfrak{U}$ and a (k, r)–decomposition of $\mathcal{X}$ over $\mathcal{Y}_r$. The collection $\mathfrak{U}$ is* stable under k–fold decomposition *if every metric family which k–decomposes over $\mathfrak{U}$ actually belongs to $\mathfrak{U}$.*

A metric family is weakly decomposable *over* $\mathfrak{U}$ *if it is* k*–decomposable over* $\mathfrak{U}$ *for some* $k \in \mathbb{N}$.

Recall that a metric family $\mathcal{Z}$ is *uniformly bounded* if

$$\sup\{\operatorname{diam}(Z) : Z \in \mathcal{Z}\} < \infty.$$

Definition 4.4. *The collection* $\mathfrak{D}^k$ *of metric families with* k*–fold finite decomposition complexity* (k*–FDC*) *is the smallest collection of metric families that contains the uniformly bounded metric families and is stable under* k*–fold decomposition. When* $k = 2$*, we recover the notion of FDC from* [12, 13].

The collection w$\mathfrak{D}$ *of metric families with weak finite decomposition complexity* (*wFDC*) *is the smallest collection of metric families that contains the uniformly bounded metric families and is stable under weak decomposition.*

Remark 4.5. *As explained in* [13] *and in* [20, Section 6] *for the case of FDC, the collections* $\mathfrak{D}^k$ *are unions of collections of families* $\mathfrak{D}^k_\alpha$ *indexed by countable ordinals* α*. One starts with* $\mathfrak{D}^k_0 = \mathcal{B}$*, the collection of uniformly bounded metric families, and then inductively defines* $\mathfrak{D}^k_{\alpha+1}$ *to be the set of metric families that* k*–decompose over* $\mathfrak{D}^k_\alpha$ *(for limit ordinals* β*, one may simply set* $\mathfrak{D}^k_\beta = \bigcup_{\alpha<\beta} \mathfrak{D}_\alpha$*). One then checks that the union of the union of the collections* $\mathfrak{D}^k_\alpha$*, taken over all countable ordinals* α*, is stable under* k*–fold decomposition. The same remark applies to* w$\mathfrak{D}$.

By Remark 4.2, we have $\mathfrak{D}^1 \subset \mathfrak{D}^2 \subset \mathfrak{D}^3 \subset \cdots \subset \mathrm{w}\mathfrak{D}$.

In [5], Dranishnikov and Zarichnyi give the following generalization of FDC, whose applications to algebraic K–theory have been studied by Goldfarb [9].

Definition 4.6. *A metric family* $\mathcal{X}$ *has* straight finite decomposition complexity (*sFDC*) *if, for every sequence* $R_1 < R_2 < \dots$ *of positive numbers, there exists an* $n \in \mathbb{N}$ *and metric families* $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ *such that* $\mathcal{X} = \mathcal{X}_0$*, the family* $\mathcal{X}_i$ *is* R_{i+1}*–decomposable over* $\mathcal{X}_{i+1}$*, and the family* $\mathcal{X}_n$ *is uniformly bounded. The collection of metric families with sFDC is denoted by* s$\mathfrak{D}$.

We say that a metric space X *has sFDC if the single-element family* $\{X\}$ *has sFDC.*

Proposition 4.7. *Every metric family with weak FDC also has straight FDC. In fact, the class of metric families* s$\mathfrak{D}$ *is stable under weak decomposition.*

We need the following lemma, which was pointed out to us by Daniel Kasprowski (personal communication). A similar idea appears in Dydak–Virk [6].

Lemma 4.8. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be metric families such that* $\mathcal{X}$ *admits a* (k, s) *decomposition over* $\mathcal{Y}$*. There there exists a sequence of decompositions*

$$\mathcal{X} = \mathcal{X}_0 \xrightarrow{S} \mathcal{X}_1 \xrightarrow{S} \cdots \xrightarrow{S} \mathcal{X}_{k-1} \xrightarrow{S} \mathcal{X}_k = \mathcal{Y}.$$

Proof. For each $X \in \mathcal{X}$, there exists a k–fold decomposition $X = X_1 \cup \cdots \cup X_k$, with each X_i an s–disjoint union

$$X_i = \bigsqcup_{j \in J_i}^{s-\text{disjoint}} X_{ij},$$

such that $X_{ij} \in \mathcal{Y}$ for each i, j. For $l = 1, 2, \dots, k$, we define $\mathcal{X}_l$ to be the metric family consisting of all the spaces X_{ij} with $1 \leqslant i \leqslant l$ and $j \in J_i$, together with the space $X_{l+1} \cup \cdots \cup X_k$. In other words,

$$\mathcal{X}_l = \bigcup_{X \in \mathcal{X}} \left(\bigcup_{i=1}^{l} \{X_{ij} : j \in J_i\} \right) \cup \{X_{l+1} \cup \cdots \cup X_k\}.$$

The desired decompositions $\mathcal{X}_l \xrightarrow{S} \mathcal{X}_{l+1}$ are obtained by decomposing each X_{ij} with $i \leqslant l$ trivially, and decomposing $X_{l+1} \cup \cdots \cup X_k$ as the union of X_{l+1} and $X_{l+2} \cup \cdots \cup X_k$; we can then further decompose

$$X_{l+1} = \bigsqcup_{j \in J_{l+1}}^{s-\text{disjoint}} X_{(l+1)j},$$

and we can decompose $X_{l+2} \cup \cdots \cup X_k$ trivially. □

Proof of Proposition 4.7. We will show that s$\mathfrak{D}$ is stable under weak decomposition. Since s$\mathfrak{D}$ contains all uniformly bounded families and w$\mathfrak{D}$ is the smallest collection of metric families that is stable under weak decomposition and contains all uniformly bounded families, this will imply that w$\mathfrak{D} \subset$ s$\mathfrak{D}$.

Say $\mathcal{X}$ weakly decomposes over s$\mathfrak{D}$. Then there exists $k \geqslant 1$ such that for each $r > 0$, there exists $\mathcal{Y}(r) \in$ s$\mathfrak{D}$ such that

$$\mathcal{X} \xrightarrow{(k,r)} \mathcal{Y}(r).$$

By Lemma 4.8, there exist metric families $\mathcal{X}_1(r), \ldots, \mathcal{X}_{k-1}(r)$ such that

$$\mathcal{X} \xrightarrow{r} \mathcal{X}_1(r) \xrightarrow{r} \mathcal{X}_2(r) \xrightarrow{r} \cdots \xrightarrow{r} \mathcal{X}_{k-1}(r) \xrightarrow{r} \mathcal{Y}(r).$$

Now consider a sequence $R_1 < R_2 < \cdots$. Setting $r = R_k$, we have

$$\mathcal{X} \xrightarrow{R_k} \mathcal{X}_1(R_k) \xrightarrow{R_k} \mathcal{X}_2(R_k) \xrightarrow{R_k} \cdots \xrightarrow{R_k} \mathcal{X}_{k-1}(R_k) \xrightarrow{R_k} \mathcal{Y}(R_k),$$

and since $R_1, R_2, \ldots, R_{k-1} < R_k$, we in fact have

$$\mathcal{X} \xrightarrow{R_1} \mathcal{X}_1(R_k) \xrightarrow{R_2} \mathcal{X}_2(R_k) \xrightarrow{R_3} \cdots \xrightarrow{R_{k-1}} \mathcal{X}_{k-1}(R_k) \xrightarrow{R_k} \mathcal{Y}(R_k). \tag{2}$$

Since $\mathcal{Y}(R_k) \in$ s$\mathfrak{D}$, applying the definition of sFDC to the sequence

$$R_{k+1} < R_{k+2} < \cdots$$

yields a finite sequence of decompositions of $\mathcal{Y}(R_k)$ ending with a bounded family; that is, for some $n \in \mathbb{N}$ we have

$$\mathcal{Y}(R_k) \xrightarrow{R_{k+1}} \mathcal{Z}_{k+1} \xrightarrow{R_{k+2}} \mathcal{Z}_{k+2} \xrightarrow{R_{k+2}} \cdots \xrightarrow{R_{k+n}} \mathcal{Z}_{k+n} \tag{3}$$

with $\mathcal{Z}_{k+n}$ uniformly bounded. Stringing together (2) and (3) shows that $\mathcal{X}$ has sFDC. □

The notion of sFDC can be weakened in a manner analogous to the definition of weak FDC.

Definition 4.9. *A metric family $\mathcal{X}$ has* weak straight finite decomposition complexity *if there exists a sequence $\mathbf{k} = (k_1, k_2, \ldots)$ $(k_i \in \mathbb{N})$ such that for every sequence $R_1 < R_2 < \ldots$ of positive numbers, there exists an $n \in \mathbb{N}$ and metric families $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ such that $\mathcal{X} = \mathcal{X}_0$, the family $\mathcal{X}_i$ is (k_{i+1}, R_{i+1})–decomposable over $\mathcal{X}_{i+1}$, and the family $\mathcal{X}_n$ is uniformly bounded. We say that $\mathcal{X}$ has weak sFDC* with respect to the sequence $\mathbf{k} = (k_1, k_2, \ldots)$.

Dydak and Virk call this notion *countable asymptotic dimension*. In an earlier version of this article, we asked whether metric spaces with weak sFDC have Property A. In fact, Dydak and Virk show that countable asymptotic dimension is *equivalent* to sFDC [6, Theorem 8.4], and sFDC spaces have Property A by a result of Dranishnikov and Zarichnyi [5].

5. Extendability of straight finite decomposition complexity

We now consider basic extendability properties for straight finite decomposition complexity.

The usual argument for coarse inheritance of FDC also proves the following result.

Lemma 5.1. *If X has sFDC and there exists a coarse embedding $Y \to X$, then Y also has sFDC.*

For the next result, the following notion for metric families will be useful.

Definition 5.2. *Let $\mathcal{X}$ be a metric family. The* subspace closure *of $\mathcal{X}$, denoted by $\mathcal{X}'$ is the metric family $\mathcal{X}' = \{X : \text{ there exists } Y \in \mathcal{X} \text{ with } X \subset Y\}$.*

Theorem 5.3. *Let $f\colon E \to B$ be a uniformly expansive, homogeneous map. Assume that B has sFDC and assume that there exists $b_0 \in B$ such that for each $r > 0$, the space $f^{-1}(B_r(b_0))$ has sFDC. Then E has sFDC.*

In particular, sFDC satisfies the Homogeneous Fibering Theorem.

Proof. Take ρ to be the function from the definition of uniform expansion for f, and let $R_1 < R_2 < \ldots$ be given. Since B has sFDC there is an $n \in \mathbb{N}$ and a sequence of metric families $\mathcal{Y}_0 = \{B\}$, $\mathcal{Y}_1$, ..., $\mathcal{Y}_n$ such that $\mathcal{Y}_{i-1}$ is $\rho(R_i)$–decomposable over $\mathcal{Y}_i$ and $\mathcal{Y}_n$ is a uniformly bounded family. Let

$$f^{-1}(\mathcal{Y}_i) = \left\{ f^{-1}(Y) : Y \in \mathcal{Y}_i \right\}.$$

Then $f^{-1}(\mathcal{Y}_0) = \{E\}$, and $f^{-1}(\mathcal{Y}_i)$ can be R_{i+1}–decomposed over $f^{-1}(\mathcal{Y}_{i+1})$, since inverse images of $\rho(R_{i+1})$–disjoint sets in B are R_{i+1}–disjoint in E.

This yields a sequence of decompositions of E that ends with the family $f^{-1}(\mathcal{Y}_n)$, and by assumption there exists $r > 0$ such that each $Y \in \mathcal{Y}_n$ has diameter at most r. Each $f^{-1}(Y)$ is isometric, via one of the isometries $\bar{\phi}$ guaranteed by the definition of homogeneity, to a subspace of $f^{-1}(B_r(b_0))$, so by Lemma 5.1 we conclude that each space $f^{-1}(Y)$ has sFDC.

Applying the definition of sFDC to the space $f^{-1}(B_r(b_0))$ and the sequence of numbers $R_{n+1} < R_{n+2} < \cdots$ shows that there exists $N \geq 0$ and metric families

$$\mathcal{Z}_n(b_0) = \left\{ f^{-1}(B_r(b_0)) \right\}, \mathcal{Z}_{n+1}(b_0), \mathcal{Z}_{n+2}(b_0), \ldots, \mathcal{Z}_{n+N}(b_0)$$

such that $\mathcal{Z}_{n+N}(b_0)$ is uniformly bounded and for $i = 0, \ldots N-1$, $\mathcal{Z}_{n+i}(b_0)$ admits an R_{n+i+1}–decomposition over $\mathcal{Z}_{n+i+1}(b_0)$.

For $i = 0, \ldots, N$, let $\mathcal{Z}_{n+i}$ be the union over $b \in B$ of all translates of spaces in $\mathcal{Z}_{n+i}(b_0)$ under the isometries $\bar{\phi}$. Since decomposability is defined element-wise over elements in a metric family, we see that $\mathcal{Z}_{n+i}$ admits an R_{n+i+1}–decomposition over $\mathcal{Z}_{n+i+1}$. Let $\mathcal{Z}'_{n+i}$ be the subspace closure of $\mathcal{Z}_{n+i}$, and note that $\mathcal{Z}'_{n+N}$ is still uniformly bounded. If $Z' \subset Z$ are metric spaces, then each decomposition of Z can be intersected with Z' to obtain a decomposition of Z'. Hence $\mathcal{Z}'_{n+i}$ admits an R_{n+i+1}–decomposition over $\mathcal{Z}'_{n+i+1}$, and the same idea shows that $f^{-1}(\mathcal{Y}_n)$ admits an R_{n+1}–decomposition over $\mathcal{Z}'_{n+1}$.

The sequence of decompositions

$$\{E\} = f^{-1}(\mathcal{Y}_0) \xrightarrow{R_1} f^{-1}(\mathcal{Y}_1) \xrightarrow{R_2} f^{-1}(\mathcal{Y}_2) \xrightarrow{R_3} \cdots \xrightarrow{R_n} f^{-1}(\mathcal{Y}_n)$$
$$\xrightarrow{R_{n+1}} \mathcal{Z}'_{n+1} \xrightarrow{R_{n+2}} \ldots \xrightarrow{R_{n+N}} \mathcal{Z}'_{n+N}$$

shows that E has sFDC. □

The Finite Union Theorem and the Union Theorem for sFDC were established in [5, Theorems 3.5 and 3.6]. We now have the following consequence of Theorem 3.9.

Corollary 5.4. *Straight finite decomposition complexity axiomatically extendable. In particular, if G is relatively hyperbolic with respect to $H_1, \ldots, H_n$ and each H_i has sFDC, then G has sFDC.*

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Department of Mathematical Sciences, Indiana University-Purdue University, Indianapolis, 402 N. Blackford Street, Indianapolis, IN 46202,
E-mail address: dramras@iupui.edu

Department of Mathematics, The Ohio State University, Columbus, OH 43210,
E-mail address: ramsey.313@math.osu.edu