

**RELATIVELY HYPERBOLIC GROUPS, RAPID DECAY ALGEBRAS
AND A GENERALIZATION OF THE BASS CONJECTURE**

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Abstract. *By deploying dense subalgebras of $\ell^1(G)$ we generalize the Bass conjecture in terms of Connes' cyclic homology theory. In particular, we propose a stronger version of the ℓ^1 -Bass Conjecture. We prove that hyperbolic groups relative to finitely many subgroups, each of which posses the polynomial conjugacy-bound property and nilpotent periodicity property, satisfy the ℓ^1 -Stronger-Bass Conjecture. Moreover, we determine the conjugacy-bound for relatively hyperbolic groups and compute the cyclic cohomology of the ℓ^1 -algebra of any discrete group.*

INTRODUCTION

Let R be a discrete ring with unit. A classical construction of Hattori-Stallings ([Ha], [St]) yields a homomorphism of abelian groups

$$K_0^a(R) \xrightarrow{Tr^{HS}} HH_0(R)$$

where $K_0^a(R)$ denotes the zeroth algebraic K -group of R , and $HH_0(R) = R/[R, R]$ its zeroth Hochschild homology group. Precisely, given a projective module P over R , one chooses a free R -module R^n containing P as a direct summand, resulting in a map

$$End_R(P) \xrightarrow{\iota} End_R(R^n) = M_{n,n}(R) \xrightarrow{Trace} HH_0(R)$$

One then verifies the equivalence class of $Tr^{HS}([P]) := Trace(\iota(Id_P))$ in $HH_0(R)$ depends only on the isomorphism class of P , is invariant under stabilization, and sends direct sums to sums. Hence it extends uniquely to a map on $K_0(R)$, as above.

Let G be a discrete group, k a commutative ring and $k[G]$ the group algebra of G with coefficients in $k \subseteq \mathbb{C}$. Let $\langle G \rangle$ denote the set of conjugacy classes of G . Taking tensor products over k , $HH_0(k[G]) = \bigoplus_{x \in \langle G \rangle} HH_0(k[G])_x = \bigoplus_{x \in \langle G \rangle} k$ decomposes as a direct sum of copies of k indexed on $\langle G \rangle$. Let $\pi_x : HH_0(k[G]) \rightarrow k$ be the projection which is the identity on the summand indexed by x and sends the other summands to zero. Given a finitely-generated projective module P over $k[G]$, the P -rank of an element $g \in G$ is given by $r_P(g) := \pi_{\langle g \rangle} Tr^{HS}([P])$. By convention, the conjugacy classes associated to elements of finite order are referred to as elliptic, and the sum of all such the elliptic summand. In [Ba], H. Bass stated the following conjecture.

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Bass Conjecture. *If P is a finitely-generated projective module over $\mathbb{Z}[G]$, then $r_P(g) = 0$ if $g \neq id$.*

Although still unknown in general, many partial results are known, beginning with the case G a linear group, verified by Bass himself in [Ba]. For the integral group algebra, Linell has shown that $r_P(g) = 0$ whenever $1 \neq g$ is of finite order. Also, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$ induces an inclusion $HH_0(\mathbb{Z}[G]) = \bigoplus_{x \in \langle G \rangle} \mathbb{Z} \hookrightarrow HH_0(\mathbb{C}[G]) = \bigoplus_{x \in \langle G \rangle} \mathbb{C}$ (where $HH_0(\mathbb{C}[G])$ is computed with tensors over \mathbb{C}) which is the obvious inclusion on each summand. Thus the Bass conjecture is implied by the

Strong Bass Conjecture - SBC. *For each non-elliptic class x , the image of the composition $\pi_x \circ Tr^{HS} : K_0^a(\mathbb{C}[G]) \rightarrow HH_0(\mathbb{C}[G]) \rightarrow HH_0(\mathbb{C}[G])_x$ is zero.*

Henceforth all algebras and tensor products are assumed to be over \mathbb{C} . If A is a Fréchet algebra and A^δ denotes A with the discrete topology, then there is a commuting diagram

$$\begin{array}{ccc} K_0^a(A^\delta) & \xrightarrow{Tr^{HS}} & HH_0(A^\delta) \\ \downarrow & & \downarrow \\ K_0^t(A) & \xrightarrow{ch_0} & HC_0^t(A) \end{array}$$

where the vertical maps are induced by the continuous map $A^\delta \rightarrow A$ which is the identity on elements, ch_0 is the Connes-Karoubi-Chern character in dimension zero [CK], and $HC_0^t(A)$ the zeroth (reduced topological) cyclic homology group of A . By [Ti], this same diagram exists for arbitrary topological algebras A when A is topologized by the fine topology, in which case one has an equality $HC_*^a(A) = HC_*^t(A)$. The advantage to working with ch_* is that it fits into a family of Chern characters

$$ch_n^m : K_n^t(A) \rightarrow HC_{n+2m}^t(A)$$

which satisfy the identity $S \circ ch_n^m = ch_n^{m-1}$ for $m \geq 1$, where $S : HC_{n+2m}^t(A) \rightarrow HC_{n+2(m-1)}^t(A)$ is the S operator. Moreover, For when A is a (locally convex) algebra topologized with the fine topology, there is an isomorphism $HC_*(A) = HC_*^t(A)$, with the left-hand side denoting algebraic cyclic homology. In particular, we can take $A = \mathbb{C}[G]$ equipped with the fine topology. Then for each conjugacy class x , there is a summand $HC_*(\mathbb{C}[G])_x$ of $HC_*(\mathbb{C}[G]) = HC_*^t(\mathbb{C}[G])$, and a corresponding projection operator $(\pi_x)_* : HC_*(\mathbb{C}[G]) \rightarrow HC_*(\mathbb{C}[G])_x$. In light of this, it is natural to consider the following stronger version of SBC.

Stronger Bass conjecture - SrBC. *For each non-elliptic class x , the image of the composition $(\pi_x)_* \circ ch_* : K_*^t(\mathbb{C}[G]) \rightarrow HC_*(\mathbb{C}[G]) \rightarrow HC_*(\mathbb{C}[G])_x$ is zero.*

It follows by [Og1] that the image of $ch_* : K_*^t(\mathbb{C}[G]) \rightarrow HC_*(\mathbb{C}[G])$ is at least the elliptic summand; the non-trivial part of the conjecture is to show it is no more than this.

For a given non-elliptic class x , we say that x satisfies the nilpotency condition if the operator $S^* : HC^*(\mathbb{C}[G])_x \rightarrow HC^{*+2}(\mathbb{C}[G])_x$ is nilpotent. We say that G satisfies the nilpotency condition if x does for each non-elliptic conjugacy class x . Combining the results of Connes-Karoubi with those of Burghelea, one has the following immediate consequence, used in [Ec1], [Ji3] and [Em]:

Observation. *Let x be a non-elliptic conjugacy class satisfying the nilpotency condition. Then the composition $K_n(\mathbb{C}[G]) \rightarrow HC_n(\mathbb{C}[G]) \xrightarrow{\pi_x} HC_n(\mathbb{C}[G])_x$ is zero for all $n \geq 0$. Consequently, if G satisfies the nilpotency condition, then the SrBC is true for G .*

Unlike SrBC, there are known examples of discrete groups which do not satisfy the nilpotency condition. However, there is a large class of groups which do. The most inclusive results are those

of Emmanouil [Em]. Following (loc. sit.), we denote by $\mathcal{E}(\mathbb{C})$ the class of groups which satisfy the nilpotency condition over \mathbb{C} (this is the same as over \mathbb{Q} , but the notation is more in keeping with the rest of our paper).

Let $\ell^1(G)$ denote the ℓ^1 algebra of G . Fixing a proper word-length function L on G , we denote by $H_L^{1,\infty}(G)$ the standard ℓ^1 rapid-decay algebra of G ; this is a Fréchet subalgebra of $\ell^1(G)$ containing $\mathbb{C}[G]$ which is smooth in $\ell^1(G)$ - that is, the inclusion $H_L^{1,\infty}(G) \hookrightarrow \ell^1(G)$ induces an isomorphism on topological K -groups in all degrees [Jo1], [Jo2]. The Connes-Karoubi-Chern character therefore produces a map

$$K_*^t(\ell^1(G)) \xleftarrow{\cong} K_*^t(H_L^{1,\infty}(G)) \rightarrow HC_*^t(H_L^{1,\infty}(G))$$

Analogous to the case of the group algebra, there is a naturally defined collection of summands $\{HH_*^t(H_L^{1,\infty}(G))_x\}$ resp. $\{HC_*^t(H_L^{1,\infty}(G))_x\}$ of $HH_*^t(H_L^{1,\infty}(G))$ resp. $HC_*^t(H_L^{1,\infty}(G))$ indexed on the set of conjugacy classes of G , with a similar result for topological Hochschild and cyclic cohomology. This leads to the following variant of SrBC, which was one of our original motivations for this paper.

ℓ^1 -SrBC. *For each non-elliptic conjugacy class x , the image of the composition $(\pi_x)_* \circ ch_* : K_*^t(\ell^1(G)) \rightarrow HC_*^t(H_L^{1,\infty}(G)) \rightarrow HC_*^t(H_L^{1,\infty}(G))_x$ is zero.*

The ℓ^1 -Bass Conjecture states that the ℓ^1 Hattori-Stallings trace: $HS^1 : K_0(\ell^1(G)) \rightarrow HH_0(\ell^1(G))$ has a range concentrated in elliptic summands. In [BCM] it is shown that the Baum-Connes Conjecture for a discrete group implies its ℓ^1 -Bass Conjecture. Thus, a-T-menable groups, in particular, amenable groups satisfy the ℓ^1 -Bass conjecture, following the work of Higson and Kasparov [HK]. The ℓ^1 -SrBC we have proposed here is stronger than the ℓ^1 -Bass Conjecture. Our method requires determining the nilpotency of the periodicity operator on cyclic cohomology and some geometric analysis on the group. However, it does not require the verification of Baum-Connes conjecture.

An outline of the paper is as follows. In section 1, following a preliminary review of Hochschild, cyclic and periodic cyclic (co)homology, we introduce in section 1.2 the notion of a bounding function. A family \mathcal{B} of such functions, together with a word-length function L on G , equips $\mathbb{C}[G]$ with a collection of semi-norms, for which the completion $H_{\mathcal{B},L}(G)$ of $\mathbb{C}[G]$ is a subalgebra of $\ell^1(G)$. In this paper the term rapid decay algebra refers to any subalgebra of $\ell^1(G)$ of this type. In fact, the rapid decay algebra $H_L^{1,\infty}(G)$ occurs as $H_{\mathcal{P},L}(G)$, where \mathcal{P} is the family of polynomial bounding functions. The category $\underline{\mathcal{B}}$ of all families of bounding functions is a poset, with partial ordering given by inclusion. This category has an initial object \mathcal{B}_{min} , a terminal object \mathcal{B}_{max} , and for all (G, L) (meaning a discrete group G equipped with a word-length function L) there are equalities $H_{\mathcal{B}_{min},L}(G) = \ell^1(G)$, $H_{\mathcal{B}_{max},L}(G) = \mathbb{C}[G]$. Moreover, for each (G, L) , the association $\mathcal{B} \mapsto H_{\mathcal{B},L}(G)$ defines a contravariant functor $F_{G,L} : \underline{\mathcal{B}} \rightarrow (top. alg.)$ such that the unique inclusion $\mathcal{B}_{min} \hookrightarrow \mathcal{B}_{max}$ maps under $F_{G,L}$ to the natural inclusion $\mathbb{C}[G] \hookrightarrow \ell^1(G)$. The result is a class of rapid decay algebras indexed by $\underline{\mathcal{B}}$, lying between the group algebra $\mathbb{C}[G]$ and the ℓ^1 algebra $\ell^1(G)$. Now there is a topological as well as a bornological approach to extending (co)homology theories (Hochschild, cyclic, periodic cyclic, bar, etc . . .) to non-discrete algebras. In general, these extensions apply to different categories and hence yield different theories. However, for Fréchet algebras these extensions are the same [Me2]. For \mathcal{B} countable, $H_{\mathcal{B},L}(G)$ is a Fréchet subalgebra of $\ell^1(G)$; consequently, on the subclass of rapid decay algebras $\{H_{\mathcal{B},L}(G) \mid \mathcal{B} \text{ countable}\}$, these two extensions agree. This is formalized in section 1.3. Consequently, we write $F_*^t(H_{\mathcal{B},L}(G))$ resp. $F_t^*(H_{\mathcal{B},L}(G))$ to denote either topological or bornological homology resp. cohomology groups, where F represents Hochschild, cyclic, periodic cyclic or bar homology. Moreover, as we further show in this section, the cohomology theories identify with the the corresponding \mathcal{B} -bounded theory computed as the cohomology of the subcomplex of \mathcal{B} -bounded cochains in the

cocomplex associated to the group algebra. This idea, which is a natural extension of both [Og2] and [M1], is most conveniently cast in the context of \mathcal{B} -bounded cohomology theories associated to weighted simplicial sets (also defined in section 1.3). To illustrate, there is an isomorphism

$$HH_t^*(H_{\mathcal{B},L}(G)) \cong HH_{\mathcal{B}}^*(\mathbb{C}[G])$$

where the groups on the left are the topological Hochschild cohomology groups of $H_{\mathcal{B},L}(G)$, while the groups on the right are the \mathcal{B} -bounded Hochschild cohomology groups of $\mathbb{C}[G]$, computed as the \mathcal{B} -bounded cohomology of the weighted simplicial set $N.^{cy}(G) =$ the cyclic bar construction on G , with weight function induced by L . Additionally, these groups contain summands indexed by the conjugacy classes of G , and which may be identified with appropriate \mathcal{B} -bounded cohomology groups of weighted simplicial summands of $N.^{cy}(G)$. This is made precise in section 1.4. To further identify the summands in the manner suggested by Burghlea's result, we need to consider various (albeit natural) restrictions on G . Under the assumption that G has a \mathcal{B} -solvable conjugacy-bound there is for each conjugacy class $x \in \langle G \rangle$ an isomorphism $HH_{\mathcal{B}}^*(\mathbb{C}[G])_x \cong H_{\mathcal{B}}^*(BG_x; \mathbb{C})$, where G_x denotes the centralizer of h in G and $\langle h \rangle = x$ (Cor. 1.4.6). In the language of group theory, \mathcal{B} -solvability of the conjugacy-bound is essentially saying that for each conjugacy class there is a bounding function $\phi \in \mathcal{B}$ for which the length of a conjugator g for two conjugate elements u and v is bounded by ϕ of the lengths of u and v . Again, for each conjugacy class x $F_*^t(H_{\mathcal{B},L}(G))$ contains a naturally defined summand $F_*^t(H_{\mathcal{B},L}(G))_x$ ($F = HH, HC, HPer$), which suggests the following extension of ℓ^1 -SrBC.

\mathcal{B} -SrBC. *For each non-elliptic class x , the image of the composition $(\pi_x)_* \circ ch_* : K_*^t(H_{\mathcal{B},L}(G)) \rightarrow HC_*^t(H_{\mathcal{B},L}(G)) \rightarrow HC_*^t(H_{\mathcal{B},L}(G))_x$ is zero.*

We say G satisfies the \mathcal{B} -nilpotency condition if the operator $S_x^* : HC^*(H_{\mathcal{B},L}(G))_x \rightarrow HC^{*+2}(H_{\mathcal{B},L}(G))_x$ is nilpotent for each non-elliptic conjugacy class x . As above, it follows by Connes-Karoubi that

Observation. *If G satisfies the \mathcal{B} -nilpotency condition, then the conjecture \mathcal{B} -SrBC is true for G .*

In section 2, we use the machinery of section 1 to verify the \mathcal{B} -nilpotency condition in a number of interesting cases. Although not a theorem, in practice it is generally the case that if G satisfies the \mathcal{B} -nilpotency condition and $\mathcal{B} \subset \mathcal{B}'$, then G satisfies the \mathcal{B}' -nilpotency condition. The class of groups satisfying the \mathcal{B} -nilpotency condition will be denoted by $\mathcal{C}_{\mathcal{B}}(\mathbb{C})$. The weakest condition is \mathcal{B}_{max} -nilpotency, which is nothing else but the nilpotency condition defined above for the group algebra $\mathbb{C}[G]$ (i.e., $\mathcal{C}_{\mathcal{B}_{max}}(\mathbb{C}) = \mathcal{C}(\mathbb{C})$). As a warm-up, and to illustrate this method, we show in section 2.1 that G satisfies \mathcal{B} -nilpotency for all $\mathcal{B} \neq \mathcal{B}_{min}$ when G is (i) f.g. nilpotent or (ii) word-hyperbolic. This includes the case $\mathcal{B} = \mathcal{P}$, which are results due to the first author [Ji1], [Ji2]. In section 2.2 we extend this by showing

Theorem A. *If $\mathcal{P} \subset \mathcal{B}$ and G is hyperbolic relative to a finite set of subgroups H_i where each H_i is in the class $\mathcal{C}_{\mathcal{B}}(\mathbb{C})$ and has a \mathcal{B} -solvable conjugacy bound, then G is in the class $\mathcal{C}_{\mathcal{B}}(\mathbb{C})$. In particular, If $\mathcal{B} = \mathcal{P}$, then G satisfies the ℓ^1 -SrBC.*

Theorem A depends heavily on the following theorem on relatively hyperbolic groups:

Theorem B. *Let G be a finitely generated group with generating set X , such that G is relatively hyperbolic with respect to $\{H_{\lambda}\}_{\lambda \in \Lambda}$, each member of which has \mathcal{P} -solvable conjugacy-bound property. Then G has a \mathcal{P} -solvable conjugacy-bound.*

The conjugacy problem for relatively hyperbolic groups was solved in [Bu]. However, the \mathcal{P} -solvable conjugacy-bound property described in Theorem B is new. Recall that the idempotent conjecture for $\ell^1(G)$ states that the only idempotent elements in $\ell^1(G)$ are 0 and 1 if G is torsion-free. By a standard argument we have

Corollary C. *If G satisfies the hypothesis of Theorem B and is torsion-free, then the idempotent conjecture holds for $\ell^1(G)$.*

In section 2.3, we consider the case when G_x is synchronously combable for non-elliptic x ; the results of this section are preliminary, and indicate a need for further study. One important family of combable groups are those which are semi-hyperbolic in the sense of [AB]. As we have already seen, \mathcal{B} -solvability is needed at the non-elliptic classes in order to properly identify the resulting summands of the topological Hochschild and cyclic groups. Our second main result is

Theorem D. *If G is semi-hyperbolic, and \mathcal{B} is any class containing \mathcal{E} (the exponential bounding class), then G is in $\mathcal{C}_{\mathcal{B}}(\mathbb{C})$ if it is in $\mathcal{C}_{\mathcal{B}_{max}}(\mathbb{C})$.*

For certain classes of semi-hyperbolic groups, this result can be strengthened to where \mathcal{E} is replaced by \mathcal{P} in the above theorem. Finally in section 2.4, we deal with the class \mathcal{B}_{min} (corresponding to the full ℓ^1 -algebra $\ell^1(G)$). In marked contrast to the structure of the topological cyclic homology groups of $H_{\mathcal{B},L}(G)$ when $\mathcal{B} \neq \mathcal{B}_{min}$, we have

Theorem E. *For each conjugacy class x , there is an isomorphism*

$$HC_t^*(\ell^1(G))_x \cong H_b^*(BG_x; \mathbb{C}) \otimes HC^*(\mathbb{C})$$

where the S_x^* map on the left identifies with $Id \otimes S^*$ on the right, and $H_b^*(-)$ denotes bounded cohomology. Thus in the case of the ℓ^1 -algebra, the map S_x^* is never nilpotent for any conjugacy class x . Consequently, the class $\mathcal{C}_{\mathcal{B}_{min}}(\mathbb{C})$ is empty.

The essential reason for this result is the vanishing of bounded cohomology for amenable groups in positive degrees. It is this theorem that motivates consideration of larger classes of bounding functions (or equivalently, the restriction to rapid decay subalgebras of $\ell^1(G)$).

Finally, it is natural to expect a connection between the conjecture \mathcal{B} -SrBC and the corresponding Baum-Connes conjecture for the rapid decay algebra $H_{\mathcal{B},L}(G)$. In fact, the rational surjectivity of the (appropriately defined) Baum-Connes assembly map for $K_*^t(H_{\mathcal{B},L}(G))$ implies \mathcal{B} -SrBC (a very special case of this result was shown in [BCM]). A proof of this result is provided in the appendix.

Some of the ideas of the paper grow out of the workshop “The property of rapid decay” in Jan. 2006 at the American Institute of Mathematics. The first and third authors wish to thank the institute for providing active and friendly atmosphere for the workshop.

1. HOCHSCHILD AND CYCLIC (CO)HOMOLOGY OF GENERAL RAPID DECAY ALGEBRAS

1.1. Preliminaries.

We recall the construction of Hochschild and cyclic homology ([Co1], [Lo]), assuming throughout that the base field is \mathbb{C} , with all algebras and tensor products being over \mathbb{C} . For an associative unital algebra A and an A -bimodule M , let $C_n(A, M) = M \otimes A^{\otimes n}$. Define $b_n : C_n(A, M) \rightarrow C_{n-1}(A, M)$ by the equation

$$b_n(m, a_1, \dots, a_n) = (ma_1, a_2, \dots, a_n) + \sum_{i=1}^{n-1} (-1)^i (m, \dots, a_i a_{i+1}, \dots, a_n) + (-1)^n (a_n m, a_1, \dots, a_{n-1})$$

This is a differential, and the resulting complex $C_*(A, M) = (C_*(A, M), b)$ is the Hochschild complex of A with coefficients on M . Its homology is denoted $HH_*(A, M)$. When $M = A$, viewed in the obvious way as an A -bimodule, we write $C_*(A, A)$ simply as $C_*(A)$, referred to as the Hochschild complex of A . The complex $C_*(A)$ admits a degree one chain map B_* given by

$$B_n(a_0, a_1, \dots, a_n) = \sum_{i=0}^n (-1)^{ni} (1, a_i, a_{i+1}, \dots, a_n, a_0, a_1, \dots, a_{i-1}) \\ - \sum_{i=0}^n (-1)^{ni} (a_i, 1, a_{i+1}, \dots, a_n, a_0, \dots, a_{i-1})$$

Let $\mathcal{B}'(A)_{**}$ denote the first-quadrant bicomplex with $\mathcal{B}'(A)_{2p,q} = \otimes^{q+1} A$ for $p \geq 0$, $\mathcal{B}'(A)_{2p,q} = 0$ for $p < 0$ and $\mathcal{B}'(A)_{2p-1,q} = 0$ for all p . The differentials are $b_q : \mathcal{B}'(A)_{2p,q} \rightarrow \mathcal{B}'(A)_{2p,q-1}$ and $B_q : \mathcal{B}'(A)_{2p,q} \rightarrow \mathcal{B}'(A)_{2p-2,q+1}$ for $p \geq 1$. The associated total complex is the cyclic complex of A , denoted $CC_*(A)$; because we are over a field of characteristic 0, this is quasi-isomorphic to the smaller complex $(C_*(A)/(1 - \tau_*) , b_*)$ (which we also denote by $CC_*(A)$ when there is no confusion), where $\tau_n(a_0, \dots, a_n) = (-1)^n (a_n, a_0, \dots, a_{n-1})$. One may also form the larger 2-periodic bicomplex $\mathcal{B}(A)_{**}$ with $\mathcal{B}(A)_{2p,q} = \otimes^{q+1} A$, $\mathcal{B}(A)_{2p-1,q} = 0$. The differentials are $b_q : \mathcal{B}(A)_{2p,q} \rightarrow \mathcal{B}(A)_{2p,q-1}$ and $B_q : \mathcal{B}(A)_{2p,q} \rightarrow \mathcal{B}(A)_{2p-2,q+1}$, as before. The total complex $T_*(A) = (\bigoplus_{2p+q=*} \mathcal{B}(A)_{2p,q}, b + B)$ of the resulting bicomplex $\mathcal{B}(A)_{**} = \{\mathcal{B}(A)_{**}; b, B\}$ is clearly $\mathbb{Z}/2$ -graded (and contractible). There is a decreasing filtration by subcomplexes $F^k T_*(A) = (b(C_k(A)) \bigoplus_n C_{n \geq k}(A), b + B)$, and the periodic cyclic complex $CC_*^{per}(A)$ is the completion of $T_*(A)$ with respect to this filtraton: $CC_*^{per}(A) = \varprojlim_n T_*(A) / F^k T_*(A)$,

with homology $HC_*^{per}(A)$. The dual constructions yield Hochschild, cyclic and periodic cyclic cohomology. Finally we recall the definition of Hochschild homology and cohomology in terms of derived functors. For a unital algebra A , A^{op} denotes A with the opposite multiplication. An A -bimodule M is naturally a right $A \otimes A^{op}$ -module via the operation $M \otimes (A \otimes A^{op}) \ni m \otimes (a \otimes b) \mapsto bma$ (in particular, A , via its canonical A -bimodule structure, is an $A \otimes A^{op}$ -module). One then has $HH_*(A, M) = Tor_*^{A \otimes A^{op}}(A, M)$, and the dual groups $HH^*(A, M) = Ext_{A \otimes A^{op}}^*(A, M)$ defined for a left $A \otimes A^{op}$ -module M .

Note 1.1.1 The Hochschild cohomology of A , denoted $HH^*(A)$, is the cohomology of the cocomplex $(Hom(C_*(A), \mathbb{C}), b^*)$. In terms of the above, this identifies with $HH^*(A, A^*)$, where A^* is the A -bimodule of linear functionals on A . Similarly, the cyclic resp. periodic cyclic cohomology of A is computed as $HC^*(A) = H^*(CC^*(A))$ resp. $HPer^*(A) = H^*(CPer^*(A))$, where $CC^*(A) := Hom(CC_*(A), \mathbb{C})$ resp. $CPer^*(A) := Hom(CPer_*(A), \mathbb{C})$.

A fundamental result and starting point for the computations in this paper is the well-known computation of the cyclic homology of $\mathbb{C}[G]$, due to Burghelea [Bur], which we summarize as:

Theorem 1.1.2 – Burghelea.

(i) *There are canonical isomorphisms*

$$HH_*(\mathbb{C}[G]) \cong \bigoplus_{x \in \langle G \rangle} HH_*(\mathbb{C}[G])_x \\ HC_*(\mathbb{C}[G]) \cong \bigoplus_{x \in \langle G \rangle} HC_*(\mathbb{C}[G])_x$$

with the maps in the Connes-Gysin long-exact preserving this decomposition.

(ii) For all x , there is a canonical isomorphism $HH_*(\mathbb{C}[G])_x = H_*(BC_g; \mathbb{C})$, where C_g denotes the centralizer of $g \in G$ ($\langle g \rangle = x$).

(iii) If x is elliptic, then $HC_*(\mathbb{C}[G])_x = H_*(BC_g; \mathbb{C}) \otimes HC_*(\mathbb{C})$, where $S_x : HC_*(\mathbb{C}[G])_x \rightarrow HC_{*-2}(\mathbb{C}[G])_x$ is given by $S_x = Id \otimes S_*^{\mathbb{C}}$, $S_*^{\mathbb{C}} : HC_*(\mathbb{C}) \rightarrow HC_{*-2}(\mathbb{C})$.

(iv) If x is non-elliptic, then $HC_*(\mathbb{C}[G])_x = H_*(BN_g; \mathbb{C})$, where $N_g = C_g/\langle g \rangle =$ the quotient of C_g by the infinite cyclic subgroup generated by g . Under this identification, the summand of the Connes-Gysin sequence indexed by x identifies with the Gysin sequence in homology (with coeff.s in \mathbb{C}) associated to the fibration $S^1 = \mathbb{Z} \rightarrow BC_g \rightarrow BN_g$.

(v) The analogues of (i) - (iv) hold for the Hochschild and cyclic cohomology of $\mathbb{C}[G]$ (with direct sum replaced by direct product in (i)).

1.2. Bounding functions and rapid decay algebras.

We begin with

Definition 1.2.1. A bounding class \mathcal{B} is a non-empty set of non-decreasing functions $\{f : \mathbb{R}_+ \rightarrow \mathbb{R}^+\}$ satisfying the following conditions

(i) It is closed under the operation of taking positive rational linear combinations.

(ii) If $f_1, f_2 \in \mathcal{B}$, there is an $f_3 \in \mathcal{B}$ such that $f_1 \circ f_2 \leq f_3$.

(iii) it contains the constant function 1.

(iv) If $f_1 \in \mathcal{B}$ and g is a non-decreasing linear function on \mathbb{R}_+ , then there is an $f_2 \in \mathcal{B}$ with $f_1 \circ g \leq f_2$.

By (1.2.1)(iii) The smallest bounding class is $\mathcal{B}_{min} = \mathbb{Q}^+ =$ the set of constant functions with value a positive rational number. There is also a largest class $\mathcal{B}_{max} = \{f : \mathbb{R}_+ \rightarrow \mathbb{R}^+ \mid f \text{ is non-decreasing}\}$. Other basic classes that will appear in this paper are $\mathcal{L} := \{a + bx \mid a, b \in \mathbb{Q}^+\}$, $\mathcal{P} :=$ the set of positive rational linear combinations of $\{(1+x)^m\}_{m \in \mathbb{N}}$, and $\mathcal{E} :=$ the closure, under composition, of the set of positive rational linear combinations of $\{C^x\}_{C \in \mathbb{Q}, C \geq 1}$ (note: if $\mathcal{L} \subseteq \mathcal{B}$, then condition (ii) implies condition (iv)). Two bounding classes \mathcal{B} and \mathcal{B}' are equivalent, written $\mathcal{B} \simeq \mathcal{B}'$, if for each f in one class, there is an f' in the other with $f \leq f'$. For two bounding classes $\mathcal{B}, \mathcal{B}'$ we write $\mathcal{B} + \mathcal{B}'$ to indicate the bounding class generated by $\mathcal{B} \cup \mathcal{B}'$; this is the smallest bounding class containing both \mathcal{B} and \mathcal{B}' . If $\mathcal{B} \simeq \mathcal{B} + \mathcal{B}'$, we write $\mathcal{B}' \preceq \mathcal{B}$, and $\mathcal{B}' \prec \mathcal{B}$ if $\mathcal{B}' \preceq \mathcal{B}$ but $\mathcal{B} \not\preceq \mathcal{B}'$. Thus $\mathcal{B}_{min} \prec \mathcal{L} \prec \mathcal{P} \prec \mathcal{E} \prec \mathcal{B}_{max}$.

A weighted set (X, w) will refer to a countable discrete set X together with a proper function $w : X \rightarrow \mathbb{R}_+$. Let $C_c(X)$ denote the space of complex-valued functions on X with compact (i.e., finite) support. A morphism $\phi : (X, w) \rightarrow (X', w')$ is \mathcal{B} -bounded if

$$\forall f \in \mathcal{B} \exists f' \in \mathcal{B} \text{ such that } f(w'(\phi(x))) \leq f'(w(x)) \quad \forall x \in X$$

Given a weighted set (X, w) and a bounding class \mathcal{B} , one may form the topological space

$$(1.2.2) \quad H_{\mathcal{B}, w}(X) := \left\{ \phi : X \rightarrow \mathbb{C} \mid \forall f \in \mathcal{B}, \phi|_f := \sum_{x \in X} |\phi(x)| f(w(x)) < \infty \right\}$$

A \mathcal{B} -isomorphism $\phi : (X, w) \rightarrow (X', w')$ is an isomorphism of sets which is \mathcal{B} -bounded, and which admits a two-sided inverse which is also \mathcal{B} -bounded. There is a natural embedding $H_{\mathcal{B}, w}(X) \hookrightarrow \ell^1(X)$, and so we will typically represent elements of $H_{\mathcal{B}, w}(X)$ in terms of their corresponding (infinite) sums $H_{\mathcal{B}, w}(X) \ni \phi \Leftrightarrow \sum_{x \in X} \phi(x)x$.

In most applications below, \mathcal{B} is countable, in which case $H_{\mathcal{B}, w}(X)$ is a Fréchet space which alternatively may be viewed as the completion of $C_c(X)$ with respect to the collection of semi-norms $\{|\cdot|_f\}_{f \in \mathcal{B}}$.

Proposition 1.2.3. *Let $(X, w), (X', w')$ be weighted sets, $\mathcal{B}, \mathcal{B}'$ bounding classes and (G, L) a discrete group with proper word-length function.*

(i) $H_{\mathcal{B}_{min}, w}(X) = \ell^1(X)$. In particular, $H_{\mathcal{B}_{min}, L}(G) = \ell^1(G)$, the ℓ^1 -algebra of G .

(ii) $H_{\mathcal{B}_{max}, w}(X) = C_c(X)$. In particular, $H_{\mathcal{B}_{max}, L}(G) = \mathbb{C}[G]$, the complex group algebra of G .

(iii) If $\mathcal{B} \simeq \mathcal{B}'$ then $H_{\mathcal{B}, w}(X) = H_{\mathcal{B}', w}(X)$.

(iv) If $\mathcal{B}' \prec \mathcal{B}$, there is a natural inclusion $H_{\mathcal{B}, w}(X) \subseteq H_{\mathcal{B}', w}(X)$.

(v) If $\phi : (X, w) \rightarrow (X', w')$ is \mathcal{B} -bounded, it induces a continuous map of topological vector spaces $\phi : H_{\mathcal{B}, w}(X) \rightarrow H_{\mathcal{B}, w'}(X')$. If ϕ is a \mathcal{B} -isomorphism, this induced map is a continuous isomorphism of topological vector spaces.

(vi) $H_{\mathcal{B}, w}(X) \cap H_{\mathcal{B}', w}(X) = H_{\mathcal{B}+\mathcal{B}', w}(X)$.

(vii) For all (G, L) , there is an equality $H_{\mathcal{P}, w}(G) = H_L^{1, \infty}(G) =$ the ℓ^1 -Schwartz algebra of G (this algebra is the completion of $\mathbb{C}[G]$ in the seminorms $\{\nu_k\}_{k \in \mathbb{N}}$ where $\nu_k(\lambda) := \sum_{g \in G} |\lambda(g)|(1 + L(g))^k$; cf. [Jo1], [Jo2]).

(viii) For all (G, L) and bounding class \mathcal{B} , $H_{\mathcal{B}, L}(G)$ is a subalgebra of $\ell^1(G)$.

Proof. The only point that is not immediate is (viii). To verify this last property, it suffices to show $H_{\mathcal{B}, L}(G)$ is closed under multiplication. Given $f \in \mathcal{B}$, we have

$$\begin{aligned} & \left| \left(\sum \lambda_{g_1} g_1 \right) \left(\sum \lambda_{g_2} g_2 \right) \right|_f \\ &= \sum_g \left| \sum_{g_1 g_2 = g} \lambda_{g_1} \lambda_{g_2} \right| f(L(g)) \\ &\leq \sum_g \left(\sum_{g_1 g_2 = g} |\lambda_{g_1} \lambda_{g_2}| f(L(g_1) + L(g_2)) \right) \\ &\leq \sum_g \left(\sum_{g_1 g_2 = g, L(g_1) \leq L(g_2)} |\lambda_{g_1} \lambda_{g_2}| f(2L(g_2)) \right) + \sum_g \left(\sum_{g_1 g_2 = g, L(g_2) \leq L(g_1)} |\lambda_{g_1} \lambda_{g_2}| f(2L(g_1)) \right) \\ &\leq \left| \sum \lambda_{g_1} g_1 \right|_1 \left| \sum \lambda_{g_2} g_2 \right|_{f_2} + \left| \sum \lambda_{g_1} g_1 \right|_{f_2} \left| \sum \lambda_{g_2} g_2 \right|_1 < \infty \end{aligned}$$

where $|\sum \lambda_g g|_1$ is the ℓ^1 -norm and f_2 denotes any element of \mathcal{B} satisfying $f(2x) \leq f_2(x)$ for all x (this element exists by (1.2.1) (iv)). //

Note The term rapid decay subalgebra of $\ell^1(G)$ conventionally refers to the Schwartz algebra $H_L^{1, \infty}(G)$. For this paper, a rapid decay subalgebra of $\ell^1(G)$ will mean any one of the subalgebras $H_{\mathcal{B}, L}(G)$ associated to a bounding class \mathcal{B} and a group with word-length (G, L) .

Suppose now that (X, w) is a weighted set, and $\lambda : X \rightarrow \mathbb{C}$. We call f a bounding function for λ if $|\lambda(x)| \leq f(w(x))$ for each $x \in X$. We set

$$Hom_{\mathcal{B}}(X, \mathbb{C}) = \{\lambda : \mathcal{B} \rightarrow \mathbb{C} \mid \lambda \text{ has a bounding function in } \mathcal{B}\}$$

Note that property (1.2.1) (i) implies $Hom_{\mathcal{B}}(X, \mathbb{C})$ is a vector space over \mathbb{C} . Moreover, given $\lambda \in Hom_{\mathcal{B}}(X, \mathbb{C})$ and $\phi \in H_{\mathcal{B}, w}(X)$, the series $\lambda^*(\phi) := \sum_{x \in X} \lambda(x)\phi(x)$ is absolutely convergent, as $\sum_{x \in X} |\lambda(x)\phi(x)| \leq |\phi|_f$ whenever f is a bounding function for λ . It is easy to see from this last inequality that λ^* so defined yields a continuous linear functional on $H_{\mathcal{B}, w}(X)$. Alternatively, the association $\lambda \mapsto \lambda^*$ yields a natural linear transformation $Hom_{\mathcal{B}}(X, \mathbb{C}) \rightarrow H_{\mathcal{B}, w}(X)$ which is clearly injective. What is not so clear is whether it is surjective. This leads us to

Lemma 1.2.4. *Let (X, w) be as above, with w a proper weight function, and \mathcal{B} a countable bounding set. Let $H_{\mathcal{B},w}(X)^*$ denote the space of continuous linear functionals on $H_{\mathcal{B},w}(X)$, and $\text{Hom}_{\mathcal{B}}(X, \mathbb{C})$ the vector space of \mathcal{B} -bounded functions $f : X \rightarrow \mathbb{C}$. Then the natural inclusion $\text{Hom}_{\mathcal{B}}(X, \mathbb{C}) \hookrightarrow H_{\mathcal{B},w}(X)^*$ is an isomorphism of vector spaces.*

Proof. As \mathcal{B} is countable, we can assume there exists a sequence $\{f_k \in \mathcal{B}\}$ satisfying the following properties: (i) for any $f \in \mathcal{B}$ there exists k with $f \leq f_k$, (ii) $2f_k \leq f_{k+1}$ for each $k \geq 1$. Now there is an obvious inclusion of vector spaces. Suppose that this inclusion is not also a surjection. Then there exists $\phi \in H_{\mathcal{B},w}(X)^*$ such that for each $k \geq 1$ there is an $x_k \in X$ with $|\phi(x_k)| \geq f_k(w(x_k))$. Because w is proper, we may additionally assume that $w(x_{k+1}) > w(x_k)$ for each k . Fix such a choice of x_k for each positive integer k . Let $u_k = \phi(x_k)/|\phi(x_k)|$. Also for each k choose α_k with $\alpha_k/|\alpha_k| = \overline{u_k}$ and $|\alpha_k| = f_k(w(x_k))^{-1}$. Then

$$\left| \sum_k \alpha_k x_k \right|_{f_n} \leq A_n + \sum_{k>n} |\alpha_k| |x_k|_{f_n} \leq A_n + \sum_{m \geq 1} 2^{-m} < \infty$$

for all $n \geq 0$, which by property (i) implies $\sum_k \alpha_k x_k \in H_{\mathcal{B},w}(X)$. On the other hand,

$$\left| \phi \left(\sum_k \alpha_k x_k \right) \right| = \sum_k |\alpha_k| |\phi(x_k)| \geq \sum_k 1 = \infty$$

which is a contradiction. Hence no such ϕ can exist. //

1.3. Topological vs. bornological algebras, and weighted simplicial sets.

Throughout this section we assume \mathcal{B} is countable. There are a priori different definitions of the topological (bornological) Hochschild and cyclic homology and cohomology groups associated to the rapid decay algebras $H_{\mathcal{B},L}(G)$. We give a brief description of each, and then show their equivalence. As we have observed above, $H_{\mathcal{B},L}(G)$ is a Fréchet algebra, hence a complete, locally convex topological algebra. To do homological algebra in this setting, we recall some basic definitions.

A topological A -module is a complete locally convex space, together with a jointly continuous A -module structure. A topological A -module is said to be topologically projective if it is a direct summand of a topological A -module of the form $A \hat{\otimes} E$, for E a complete locally convex space, and where $\hat{\otimes}$ is the complete projective topological tensor product.

Let M be a topological A -module. The proper framework for defining topological homology and cohomology groups is that of relative homological algebra, where continuous surjections of topological A -modules are required to be \mathbb{C} -split [T1, §2]. In this setting, a topological projective resolution of M is an exact sequence of topological A -modules:

$$0 \leftarrow M \xleftarrow{\epsilon} M_0 \xleftarrow{\partial_1} M_1 \xleftarrow{\partial_2} M_2 \xleftarrow{\partial_3} \dots$$

such that each M_i is topologically projective, all of the maps are continuous A -module maps, and the resolution has a continuous linear homotopy $s_i : M_i \rightarrow M_{i+1}$ such that $\partial_{i+1}s_i + s_{i-1}\partial_i = id_{M_i}$.

Given a topological projective resolution P_\bullet of M and any topological A -module N , one can form the complexes

$$(1.3.1) \quad \begin{aligned} P_\bullet \hat{\otimes}_A N &= 0 \leftarrow M_0 \hat{\otimes}_A N \xleftarrow{\partial_1 \otimes Id} M_1 \hat{\otimes}_A N \xleftarrow{\partial_2 \otimes Id} M_2 \hat{\otimes}_A N \xleftarrow{\partial_3 \otimes Id} \dots \\ \text{Hom}_A(P_\bullet, N) &= 0 \rightarrow \text{Hom}_A(M_0, N) \xrightarrow{\partial_1^*} \text{Hom}_A(M_1, N) \xrightarrow{\partial_2^*} \text{Hom}_A(M_2, N) \xrightarrow{\partial_3^*} \dots \end{aligned}$$

where $Hom_A(M', M'')$ denotes the internal Hom -functor evaluated on (M', M'') (i.e., the vector space of continuous A -linear maps from M' to M''). As in the algebraic case, one has $Tor_*^A(M, N) := H_*(P_\bullet \hat{\otimes}_A N)$, $Ext_A^*(M, N) := H^*(Hom_A(P_\bullet, N))$. These definitions are independent of the particular topological projective resolution by the usual arguments [Ta].

Note 1.3.2 Given a topological chain complex (C_*, d_*) with continuous boundary maps, the unreduced n^{th} homology group is typically meant to mean $ker(d_n)/im(d_{n+1})$, while the reduced n^{th} homology group is computed as $ker(d_n)/\overline{im(d_{n+1})}$, where $\overline{im(d_{n+1})}$ denotes the closure of $im(d_{n+1})$ in C_n . The unreduced groups are “algebraically correct”, while the reduced groups are “topologically correct”, in the sense that the induced subquotient topology is Hausdorff if it was to begin with on the chain level. In this paper, homology groups of topological chain complexes will always be unreduced, and the homology groups viewed simply as vector spaces, without any inherited topology. The same remarks apply to topological cochain complexes and cohomology. We will return to this point briefly later on.

Given a complete, locally convex topological algebra A with unit and a topological A -bimodule M , one may define the topological Hochschild homology of A with coefficients in M as $HH_*^t(A, M) := Tor_*^{A \hat{\otimes} A^{op}}(A, M)$. These derived functors may alternatively be computed as the homology of the topological Hochschild complex $(C_*^t(A, M), b_*)$ where $C_n(A, M) := M \hat{\otimes} A^{\hat{\otimes} n}$, with b_n given as above in (1.1). When $M = A$, the topological cyclic and periodic cyclic complexes are defined similarly, by taking the algebraic definition in (1.1) and replacing each occurrence of $(C_*(A, A), b_*)$ with $(C_*^t(A, A), b_*)$. For cohomology, one defines $HH_t^*(A, M)$ as $Ext_{A \hat{\otimes} A^{op}}^*(A, M)$. Again, by convention, the topological Hochschild cohomology of A - $HH_t^*(A)$ - which is dual to $HH_*^t(A)$, may be computed as the cohomology of $Hom(C_*^t(A), \mathbb{C}) =$ the cocomplex of continuous functionals on $C_*^t(A)$, which in turn identifies with the cocomplex $C_t^*(A, A^*)$ (A^* = the continuous dual of A ; cf. (1.1.1), [Co1]). Similarly, $HC_t^*(A)$ resp. $HPer_t^*(A)$ is computed as the cohomology of the cocomplex $Hom(CC_*^t(A), \mathbb{C})$ resp. $Hom(CPer_*^t(A), \mathbb{C})$.

An alternative (and sometimes preferable) setting for doing homological algebra is the category of bornological algebras and vector spaces, as shown by Meyer in [Me2]. Briefly, a bornology on a vector space consists of a collection of bounded subsets. The bornology is convex if every bounded subset is contained in an absolutely convex bounded subset, and complete if every bounded subset is contained in a complete disk. Bornologies are assumed to be complete and convex (the original reference for these definitions are [HN1], [HN2], but [M2, §2] will be sufficient for our purposes). As in the topological case, resolutions in this category are assumed to be \mathbb{C} -split; moreover the (projective) complete tensor product of two bornological vector spaces V and W exists and is also denoted $V \hat{\otimes} W$ when there is no confusion. Working in the category of bornological unital algebras and bornological bi-modules over those algebras, one may proceed as above and define bornological Hochschild, cyclic and periodic cyclic (co)homology in terms of the appropriate derived functors. The corresponding groups will be indicated with a subscript or superscript “bor” in place of “t”. As these functors are originating in different categories, they are different functors. However, our primary interest will be the rapid decay algebras defined in (1.1). These are Fréchet algebras, which can be viewed either topologically or bornologically.

Lemma 1.3.3. *For all (G, L) and countable \mathcal{B} , there are natural isomorphisms*

$$\begin{aligned} F_*^t(H_{\mathcal{B}, L}(G)) &\cong F_*^{bor}(H_{\mathcal{B}, L}(G)) \\ F_t^*(H_{\mathcal{B}, L}(G)) &\cong F_{bor}^*(H_{\mathcal{B}, L}(G)) \end{aligned}$$

for $F = HH, HC$ or $HPer$.

Proof. Consider two weighted sets (X, w) and (X', w') . As above, we may associate to these respectively the Fréchet spaces $H_{\mathcal{B}, w}(X)$ and $H_{\mathcal{B}, w'}(X')$. It is then easy to see that in both the topological and

bornological categories there is an isomorphism of topological resp. bornological vector spaces

$$H_{\mathcal{B},w}(X) \hat{\otimes} H_{\mathcal{B},w'}(X') \cong H_{\mathcal{B},w \times w'}(X \times X')$$

where $(w \times w')(x, x') = w(x) + w'(x')$. Iterating this isomorphism yields isomorphisms

$$H_{\mathcal{B},w}(X) \hat{\otimes}^m \cong H_{\mathcal{B},w^m}(X^m)$$

which apply in either category (where $w^m = w \times w \times \dots \times w$ m times). Specializing to the case $(X, w) = (G, L)$, we see that for both projective complete tensor products there is an isomorphism

$$H_{\mathcal{B},L}(G) \hat{\otimes}^m \cong H_{\mathcal{B},L^m}(G^m)$$

by which we may identify (up to isomorphism) both the topological and bornological Hochschild complexes with the complex $(\{H_{\mathcal{B},L^{n+1}}(G^{n+1})\}_{n \geq 0}, b_*)$, where b_n is the unique additive extension of the Hochschild boundary map defined on elements (g_0, g_1, \dots, g_n) by the usual formula (1.1.1). With this isomorphism established for the Hochschild complex, the isomorphism for the cyclic and periodic cyclic complexes follows. It also dualizes to prove the corresponding result in cohomology. //

In the above situation there is a third, simpler description of the cochain complex. Again, consider $H_{\mathcal{B},L}(G)$. By Lemma 1.2.4, there is an isomorphism $H_{\mathcal{B},L}(G)^* \cong \text{Hom}_{\mathcal{B}}(G, \mathbb{C})$. Passing to completed tensor powers and taking continuous duals results in an identification of both $C_t^*(H_{\mathcal{B},L}(G))$ and $C_b^*(H_{\mathcal{B},L}(G))$ with the cocomplex

$$(1.3.4) \quad C_{\mathcal{B}}^*(\mathbb{C}[G]) := \left\{ f \in C^m(\mathbb{C}[G]) \mid \exists F \in \mathcal{B} \text{ with } |f(g_0, g_1, \dots, g_n)| \leq F \left(\sum_{i=0}^n L(g_i) \right) \right\}_{n \geq 0}$$

The condition above on the right is simply the statement that f is \mathcal{B} -bounded (with respect to the norm on \mathbb{C}). Thus this cocomplex is the subcocomplex of $C^*(\mathbb{C}[G])$ consisting of \mathcal{B} -bounded cochains; we denote its cohomology by $HH_{\mathcal{B}}^*(\mathbb{C}[G])$. We note that the B_* map of section 1.1 is \mathcal{B} -bounded for all \mathcal{B} , so one can define the \mathcal{B} cyclic cocomplex as

$$CC_{\mathcal{B}}^*(\mathbb{C}[G]) := \{f \in CC^m(\mathbb{C}[G]) \mid f \text{ is } \mathcal{B} \text{ - bounded}\}_{n \geq 0}$$

with corresponding cohomology groups denoted $HC_{\mathcal{B}}^*(\mathbb{C}[G])$. And similarly with periodic cyclic cohomology. For any inclusion $\mathcal{B} \hookrightarrow \mathcal{B}'$ of bounding sets there is an evident transformation of cohomology theories $F_{\mathcal{B}}^*(\mathbb{C}[G]) \rightarrow F_{\mathcal{B}'}^*(\mathbb{C}[G])$ which preserves all of the standard structure maps relating the different groups; in particular, it induces a map of Connes-Gysin sequences. To further understand these cohomology groups, we introduce some additional terminology.

A \mathcal{B} -simplicial set is a simplicial set X together with weight functions $w_n : X_n \rightarrow \mathbb{R}_+$ such that the face and degeneracy maps are \mathcal{B} -bounded. As before, let (G, L) be a discrete group G equipped with a proper word-length function L . A \mathcal{B} - G -set consists of a weighted set (S, w_S) together with a specified left (resp. right) action of G on S such that there exists an $h \in \mathcal{B}$ with $w_S(gs)$ (resp. $w_S(sg)$) $\leq w_S(s) + h(L(g))$ for all $s \in S$ and $g \in G$ (in other words, the action is uniformly \mathcal{B} -bounded). A \mathcal{B} - G -map between two \mathcal{B} -(left resp. right) G -sets is a \mathcal{B} -bounded map which is G -equivariant; it is a \mathcal{B} - G -isomorphism if it admits a two-sided inverse which is also a \mathcal{B} - G -map. Finally, a \mathcal{B} - G -bi-set is a weighted set equipped with both a left and right action of G which is \mathcal{B} -bounded in the above sense, an isomorphism of two such having the obvious properties. The following are two basic examples relevant to this paper.

Example 1.3.5 (i) The cyclic bar construction with coefficients Let (G, L) be as above and let (S, w_S) be a \mathcal{B} - G -bi-set. Then $N.^{cy}(G, S) = \{n \mapsto S \times G^n\}_{n \geq 0}$ with

$$\begin{aligned} \partial_0(s, g_1, \dots, g_n) &= (sg_1, g_2, \dots, g_n), \\ \partial_i(s, g_1, \dots, g_n) &= (s, g_1, \dots, g_i g_{i+1}, \dots, g_n), \quad 1 \leq i \leq n-1, \\ \partial_n(s, g_1, \dots, g_n) &= (g_n s, g_1, g_2, \dots, g_{n-1}), \\ s_j(s, g_1, \dots, g_n) &= (s, \dots, g_j, 1, g_{j+1}, \dots, g_n) \end{aligned}$$

The simplicial weight is given by $w_n(s, g_0, \dots, g_n) = w_S(s) + \sum_{i=1}^n L(g_i)$. The condition on (S, w_S) implies $N.^{cy}(G, S)$ is a \mathcal{B} -simplicial set. When $(S, w_S) = (G, L)$, we denote this \mathcal{B} -simplicial set simply by $N.^{cy}(G)$

Example 1.3.5 (ii) The bar resolution with coefficients Let (S, w_S) be a right \mathcal{B} - G -set. Recall that the non-homogeneous bar resolution of G is $EG. = \{n \mapsto G^{n+1}\}_{n \geq 0}$ with

$$\begin{aligned} \partial_i[g_0, \dots, g_n] &= [g_0, \dots, g_i g_{i+1}, \dots, g_n], \quad 0 \leq i \leq n-1, \\ \partial_n[g_0, \dots, g_n] &= [g_0, \dots, g_{n-1}], \\ s_j[g_0, \dots, g_n] &= [g_0, \dots, g_j, 1, g_{j+1}, \dots, g_n] \end{aligned}$$

The simplicial weight function on $EG.$ is given by $w([g_0, \dots, g_n]) = \sum_{i=0}^n L(g_i)$. The left G -action is given, as usual, by $g[g_0, g_1, \dots, g_n] = [gg_0, g_1, \dots, g_n]$. Note that with respect to the given weight function and action of G , $EG.$ is a \mathcal{B} -simplicial G -set for any \mathcal{B} . Viewing S as a simplicial set with trivial simplicial structure, we can form the simplicial set $S \times EG.$ with diagonal simplicial structure. Then $S \times_G EG.$ is the quotient of $S \times EG.$ by the relation $(sg, [g_0, \dots, g_n]) \simeq (s, [gg_0, \dots, g_n])$. This is a \mathcal{B} -simplicial set where the simplicial weight is given by

$$w([s, [g_0, \dots, g_n]]) := \inf_{g \in G} \left(w_S(sg^{-1}) + L(gg_0) + \sum_{i=1}^n L(g_i) \right)$$

To a given \mathcal{B} -simplicial set $(X., w.)$ one can associate the cochain cocomplex $C_{\mathcal{B}}^*(X, \mathbb{C})$ of \mathcal{B} -bounded singular cocains, where

$$C_{\mathcal{B}, w.}^n(X., \mathbb{C}) = \{f : X_n \rightarrow \mathbb{C} \mid f \text{ is } \mathcal{B}\text{-bounded with respect to the norm on } \mathbb{C}\}$$

This is naturally a subcocomplex of the usual cocomplex of singular cochains on $X.$ with coefficients in \mathbb{C} . The \mathcal{B} -cohomology of $X.$ (with respect to the weight function $w.$) is $H_{\mathcal{B}, w.}^*(X.) := H^*(C_{\mathcal{B}, w.}^*(X., \mathbb{C}))$. The following is now evident.

Lemma 1.3.6. *Under the natural isomorphism $C^*(\mathbb{C}[G]) \cong C^*(N.^{cy}(G), \mathbb{C})$, we have a natural identification*

$$C_t^*(H_{\mathcal{B}, L}(G)) \cong C_b^*(H_{\mathcal{B}, L}(G)) \cong C_{\mathcal{B}, w.}^*(N.^{cy}(G), \mathbb{C})$$

for all (G, L) and countable \mathcal{B} .

For convenience, we will denote the groups $C_{\mathcal{B}, w.}^*(N.^{cy}(G), \mathbb{C})$ simply by $C_{\mathcal{B}}^*(N.^{cy}(G))$.

1.4. The decomposition.

As in the case of the group algebra, the Hochschild , cyclic and periodic cyclic (co)homologies of the afore-mentioned rapid decay algebras admit a decomposition indexed on $\langle G \rangle =$ the set of conjugacy classes of G . The starting point is the following simple

Proposition 1.4.1. *For any class of bounding functions \mathcal{B} , there is an isomorphism of \mathcal{B} -simplicial sets*

$$N.^{cy}(G) \cong \coprod_{x \in \langle G \rangle} (N.^{cy}(G)_x) \cong \coprod_{x \in \langle G \rangle} S_x \times_G EG.$$

(cf. (1.3.5 (i), (ii) above), where $(N.^{cy}(G)_x)$ denotes the \mathcal{B} -simplicial set of $N.^{cy}(G)$ with $(N.^{cy}(G)_x)_n = \{(g_0, \dots, g_n) \mid \langle \prod_{i=0}^n g_i \rangle = x\}$, and $S_x = \{g \in G \mid \langle g \rangle = x\}$ with weight function induced by L and the inclusion into G . The right G -action on S_x is given by $h(g) := g^{-1}hg$ (conjugation by g^{-1}).

Proof. This equivalence is well-known, and the first decomposition is clearly a \mathcal{B} -simplicial isomorphism. It remains to note that the maps

$$(N.^{cy}(G)_x)_n \ni (g_0, g_1, \dots, g_n) \mapsto [(\prod_{i=0}^n g_i), [g_0, \dots, g_n]] = [(\prod_{i=0}^n g_i)^{g_0^{-1}}, [1, \dots, g_n]] \in S_x \times_G EG_n$$

$$S_x \times_G EG_n \ni [g, [1, \dots, g_n]] \mapsto ((\prod_{i=1}^n g_i)^{-1}g, g_1, \dots, g_n)$$

are \mathcal{B} -bounded for any \mathcal{B} . //

Together with the results of the previous section, we have the following immediate consequence.

Corollary 1.4.2. *For any conjugacy class x the decomposition of (1.4.1) induces factorizations in homology and cohomology*

$$\bigoplus_{x \in \langle G \rangle} F_*^t(H_{\mathcal{B},L}(G))_x \rightarrow F_*^t(H_{\mathcal{B},L}(G)) \rightarrow \prod_{x \in \langle G \rangle} F_*^t(H_{\mathcal{B},L}(G))_x$$

$$\bigoplus_{x \in \langle G \rangle} F_t^*(H_{\mathcal{B},L}(G))_x \rightarrow F_t^*(H_{\mathcal{B},L}(G)) \rightarrow \prod_{x \in \langle G \rangle} F_t^*(H_{\mathcal{B},L}(G))_x$$

for $F = HH, HC, HPer$, and with the maps in the Connes-Gysin sequence preserving this decomposition. These summands are preserved by homomorphisms coming from an inclusion of algebras $H_{\mathcal{B},L}(G) \hookrightarrow H_{\mathcal{B}',L}(G)$ (induced by the relation $\mathcal{B}' \prec \mathcal{B}$). Moreover there exists, for each x , isomorphisms

$$HH_t^*(H_{\mathcal{B},L}(G))_x \cong H_{\mathcal{B}}^*(S_x \times_G EG.)$$

We note that the the homomorphism $F_*^t(H_{\mathcal{B},L}(G)) \rightarrow \prod_{x \in \langle G \rangle} F_*^t(H_{\mathcal{B},L}(G))_x$ may not be injective (with the same remark holding for the cohomology groups). In this paper, however, we will only be concerned with the summands indicated in the sums on the left.

Fix a conjugacy class x and group element h with $\langle h \rangle = x$, let G_h denote the centralizer of h in G , and $G_h \backslash G$ the corresponding right coset space. This is a weighted set, with weight $\bar{L}(G_h g) := \min_{g' \in G_h} L(g'g)$. The right G -action on $G_h \backslash G$ is the obvious one.

Proposition 1.4.3. *The natural map $\pi_x : G_h \backslash G \rightarrow S_x$ given by $G_h g \mapsto g^{-1} h g$ is a G -equivariant isomorphism of right G -sets, and \mathcal{B} -bounded.*

Proof. It is obviously a G -equivariant isomorphism. It is also linearly bounded in terms of the respective weight functions. //

Unfortunately, and here is the key point, this map is not a \mathcal{B} - G -isomorphism of right \mathcal{B} - G -sets in general. In order to proceed beyond Cor. 1.4.2 and get an identification analogous to Theorem 1.1.2 (ii), we will need to impose conditions on (G, L) . We say that (G, L) has a \mathcal{B} -solvable conjugacy-bound at the conjugacy class $x = \langle h \rangle \in \langle G \rangle$ if there exists $f \in \mathcal{B}$ with the property: for each $y \in S_x$, there exists $g = g_y \in G$ with $y = h^{g_y}$ and $L(g_y) \leq f(L(y))$. In other words, there exists a map $\psi_x : S_x \ni y \mapsto [g_y] \in G_h \backslash G$ which is an inverse to π_x and is \mathcal{B} -bounded. As π_x is G -equivariant, so is ψ_x . The pair (G, L) has a \mathcal{B} -solvable conjugacy-bound if it does so at each non-elliptic conjugacy class x .

Lemma 1.4.4. *The pair (G, L) has a \mathcal{B} -solvable conjugacy-bound at x iff π_x is a \mathcal{B} - G -isomorphism of right \mathcal{B} - G -sets. In this case, there is a natural isomorphism of \mathcal{B} -simplicial sets*

$$(\pi_x)_\cdot : G_h \backslash G \times_G EG \xrightarrow[\mathcal{B}]{\cong} S_x \times_G EG.$$

which induces an isomorphism on \mathcal{B} -bounded cohomology.

Proof. Given the definitions of the respective simplicial weight functions, it suffices to check that $(\pi_x)_1$ is a \mathcal{B} -isomorphism, which is implied by the fact π_x is a \mathcal{B} -isomorphism. //

Proposition 1.4.5. *For each h the inclusion $G_h \hookrightarrow G$ yields an inclusion of \mathcal{B} -simplicial sets*

$$BG_h = (G_h \backslash G) \times_{G_h} E(G_h) \hookrightarrow G_h \backslash G \times_G EG.$$

which induces an isomorphism on \mathcal{B} -bounded cohomology groups.

Proof. This result is shown in [Og2] for $\mathcal{B} = \mathcal{P}$, but the same argument works in this more general case (this point is also noted in [Me1] for \mathcal{P} and \mathcal{E}). //

Combining 1.4.2 and 1.4.5 we have

Corollary 1.4.6. *If the pair (G, L) has a \mathcal{B} -solvable conjugacy-bound, then for each non-elliptic conjugacy class $x = \langle h \rangle$, there is an isomorphism of cohomology groups*

$$HH_t^*(H_{\mathcal{B}, L}(G))_x \cong H_{\mathcal{B}}^*(S_x \times_G EG) \cong H_{\mathcal{B}}^*(BG_h; \mathbb{C})$$

We observe that having a \mathcal{B} -solvable conjugacy-bound is a local condition, and strictly weaker than requiring that the G -map

$$\coprod_{x = \langle h \rangle \in \langle G \rangle} (G_h \backslash G) \xrightarrow{x = \langle h \rangle \in \langle G \rangle \pi_x} \coprod_{x = \langle h \rangle \in \langle G \rangle} (S_x)$$

is a \mathcal{B} - G -isomorphism. Given G , it is also possible that certain conjugacy classes have a \mathcal{B} -solvable conjugacy-bound, while others are not. To illustrate, $\langle 1 \rangle$ has a \mathcal{B} -solvable conjugacy-bound for any G and bounding class \mathcal{B} , for trivial reasons. This statement is false in general for conjugacy classes $x \neq \langle 1 \rangle$.

Borrowing some terminology from [Me2], we say that (G, L) is \mathcal{B} -isocohomological if the transformation $H_{\mathcal{B}}^*(BG; \mathbb{C}) \rightarrow H^*(BG; \mathbb{C})$ (induced by the inclusion $\mathcal{B} \hookrightarrow \mathcal{B}_{max}$) is an isomorphism. Putting this all together, we have the following result.

Theorem 1.4.7. *Let (G, L) be a discrete group equipped with proper word-length function L . Suppose there exists a countable bounding class \mathcal{B} such that*

- (i) *Each non-elliptic conjugacy class x has a \mathcal{B} -solvable conjugacy-bound, and*
- (ii) *For each non-elliptic $x = \langle h \rangle \in \langle G \rangle$, G_h is \mathcal{B} -isocohomological.*

Then for each such x there are isomorphisms

$$\begin{aligned} HH_t^*(H_{\mathcal{B},L}(G))_x &= HH_{\mathcal{B}}^*(\mathbb{C}[G])_x \cong HH^*(\mathbb{C}[G])_x = H^*(BG_h; \mathbb{C}) \\ HC_t^*(H_{\mathcal{B},L}(G))_x &= HC_{\mathcal{B}}^*(\mathbb{C}[G])_x \cong HC^*(\mathbb{C}[G])_x \end{aligned}$$

for each $x = \langle h \rangle \in \langle G \rangle$.

Proof. As we have seen, $HH_{\mathcal{B}}^*(\mathbb{C}[G])_x \cong H_{\mathcal{B}}^*(BG_h; \mathbb{C})$ when x has a \mathcal{B} -solvable conjugacy-bound. Condition (ii) then implies $H_{\mathcal{B}}^*(BG_h; \mathbb{C}) \cong H^*(BG_h; \mathbb{C})$. The statement for cyclic cohomology follows by a comparison of Connes-Gysin sequences. //

As with non-bounded Hochschild and cyclic cohomology groups, it is relatively easy to show that when x has a \mathcal{B} -solvable conjugacy-bound and is elliptic, there is an isomorphism

$$HC_{\mathcal{B}}^*(\mathbb{C}[G])_x \cong H_{\mathcal{B}}^*(BG_h; \mathbb{C}) \otimes HC^*(\mathbb{C})$$

where the S^* -map on the right is given by $Id \otimes S_{\mathbb{C}}^*$. For x non-elliptic, the situation is more delicate. The next lemma follows by a straightforward adoption of the method of Nistor [Ni1] (cf. [Ji2]). As it is not needed for the rest of the paper, we state it without proof.

Lemma 1.4.8. *Given (G, L) satisfying the conditions of Theorem 1.4.7, and $x = \langle h \rangle \in \langle G \rangle$ a non-elliptic class. Let $C_h \subset G_h$ denote the infinite cyclic subgroup generated by h , and let $L_h = L|_{C_h}$. Let \mathbb{Z} denote the infinite (multiplicative) cyclic group generated by t , with $L_{\mathbb{Z}}(t^n) = n$. If the isomorphism $(\mathbb{Z}, L_{\mathbb{Z}}) \rightarrow (C_h, L_h); t \mapsto h$ is a \mathcal{B} -isomorphism, then $HC_t^*(H_{\mathcal{B},L}(G))_x \cong H_{\mathcal{B}}^*(N_h; \mathbb{C})$, where $N_h = G_h/C_h$ equipped with the quotient word-length function coming from G_h .*

Our main application of Theorem 1.4.7 is

Corollary 1.4.9. *Suppose G is in the class $\mathcal{C}(\mathbb{C})$, and (G, L) satisfies the conditions of Theorem 1.4.7. Then (G, L) is in the class $\mathcal{C}_{\mathcal{B}}(\mathbb{C})$. Consequently the conjecture \mathcal{B} -SrBC holds for G .*

In the next section we exhibit certain classes of groups to which this Corollary applies.

2. GEOMETRIC GROUPS SATISFYING \mathcal{B} -SRBC

2.1. Nilpotent and hyperbolic groups.

In previous work, the first author has shown (G, L) satisfies the \mathcal{P} -nilpotency condition for any proper word-length function L on G when G is of polynomial growth [Ji1] or word-hyperbolic [Ji3]. We show that for word hyperbolic groups or finitely-generated torsion free two-step nilpotent groups, these results can be efficiently recovered (and extended) using the machinery of section 1.

Proposition 2.1.1. *Let G be a finitely-generated torsion free two-step nilpotent group, and L a proper word-length function on G . Then for all bounding families $\mathcal{B} \supseteq \mathcal{P}$*

- (i) *Each conjugacy class x has a \mathcal{B} -solvable conjugacy-bound,*

(ii) The centralizer G_h for $h \in x$ is \mathcal{B} -isocohomological for each conjugacy class x , and

(iii) For each non-elliptic $x = \langle h \rangle$, $G_x/(h)$ has finite rational cohomological dimension ($r.c.d. < \infty$).

Hence G satisfies \mathcal{B} -nilpotency for all $\mathcal{B} \supseteq \mathcal{P}$.

Proof. For (i), we note that polynomial solvability of the conjugacy problem implies the existence of a \mathcal{B} -solvable conjugacy-bound for all $\mathcal{B} \supseteq \mathcal{P}$. Note that by [Bl] the conjugacy problem for a finitely-generated nilpotent group is solvable. We must then show that there is a polynomial solution. Consider the torsion-free nilpotent group G given by the central extension $C \rightarrow G \rightarrow N$, with C and N free abelian groups, with bases $\{e'_1, \dots, e'_n\}$ and $\{f'_1, \dots, f'_m\}$ respectively. Let g and h be conjugate elements in G , with $ug = hu$. We can write these elements as $g = f_1^{x_1} f_2^{x_2} \dots f_m^{x_m} c$, $h = f_1^{y_1} f_2^{y_2} \dots f_m^{y_m} c'$, and $u = f_1^{z_1} f_2^{z_2} \dots f_m^{z_m}$ with x_i, y_i , and z_i integers, c and c' elements of C . We can also write $c = e_1^{a_1} \dots e_n^{a_n}$, and $c' = e_1^{b_1} \dots e_n^{b_n}$, for integers a_i and b_j . (Here we have identified C with its image in G , and f_i is the image of f'_i in G under a fixed cross-section $N \rightarrow G$. $f'_i f'_j = f'_j f'_i$ so that $f_i f_j = f_j f_i C_{i,j}$ for some $C_{i,j} \in C$.) From $ug = hu$ we have $f_1^{z_1} f_2^{z_2} \dots f_m^{z_m} f_1^{x_1} f_2^{x_2} \dots f_m^{x_m} c = f_1^{y_1} f_2^{y_2} \dots f_m^{y_m} f_1^{z_1} f_2^{z_2} \dots f_m^{z_m} c'$. Applying the conjugation properties we obtain $f_1^{x_1+z_1} f_2^{x_2+z_2} \dots f_m^{x_m+z_m} C_{m,1}^{x_m z_1} \dots C_{1,1}^{x_2 z_1} C_{m,2}^{x_m z_2} \dots C_{m,m-1}^{x_m z_{m-1}} c = f_1^{z_1+y_1} f_2^{z_2+y_2} \dots f_m^{z_m+y_m} C_{m,1}^{z_m y_1} \dots C_{1,1}^{z_2 y_1} C_{m,2}^{z_m y_2} \dots C_{m,m-1}^{z_m y_{m-1}} c'$. Moving all of the f_i 's to one side we obtain $f_1^{x_1-y_1} f_2^{x_2-y_2} \dots f_m^{x_m-y_m} C_{m,1}^{x_m z_1 - y_1 z_m} \dots C_{1,1}^{x_2 z_1 - y_1 z_2} C_{m,2}^{x_m z_2 - y_2 z_m} \dots C_{m,m-1}^{x_m z_{m-1} - y_{m-1} z_m} c c'^{-1} = e$. As $\{f'_1, \dots, f'_m\}$ are free abelian generators for N , we see that $x_i = y_i$ for all i . The last equation reduces to $C_{m,1}^{x_m z_1 - x_1 z_m} \dots C_{1,1}^{x_2 z_1 - x_1 z_2} C_{m,2}^{x_m z_2 - x_2 z_m} \dots C_{m,m-1}^{x_m z_{m-1} - x_{m-1} z_m} c c'^{-1} = e$. From this we obtain a linear system to determine the z_i in terms of the x_j s, a_j s, and b_j s. As g and h are conjugate, this system has a solution. Thus there is a solution which is bounded by a polynomial of the absolute values of the entries in the coefficient matrix of this system.

As G is nilpotent, so is G_x for all x . Choose an element $z \in C$ with $ord(z) = \infty$. Let C_z denote the infinite cyclic subgroup generated by z . Associated to the short-exact sequence of groups $\mathbb{Z} = C_z \rightarrow G \rightarrow \bar{G}$ is a Serre spectral sequence in \mathcal{B} -cohomology (the case $\mathcal{B} = \mathcal{P}$ is done in detail in [Og2]. The more general case follows by similar reasoning, with similar restrictions - cf. [Og3]). By a spectral sequence comparison together with induction on n , we see that nilpotent groups are \mathcal{B} -isocohomological for all $\mathcal{B} \supseteq \mathcal{P}$. Finally, any f.g. nilpotent group G' satisfies $r.c.d.(G') < \infty$; as subquotients of f.g. nilpotent groups are again f.g. nilpotent, this implies (iii). //

Proposition 2.1.2. *Let G be a word-hyperbolic group, and L a proper word-length function on G . Then for all bounding families $\mathcal{B} \supseteq \mathcal{L}$*

(i) Each conjugacy class x has a \mathcal{B} -solvable conjugacy-bound,

(ii) The centralizer G_h is \mathcal{B} -isocohomological for each element h in the non-elliptic conjugacy class x , and

(iii) For each non-elliptic class $x = \langle h \rangle$, $G_h/(h)$ has $r.c.d. < \infty$.

Hence G satisfies \mathcal{B} -nilpotency for all $\mathcal{B} \supseteq \mathcal{L}$.

Proof. In the case of word-hyperbolic groups, the situation is even nicer. If x is a non-elliptic conjugacy class, then the conjugacy-bound in this case is linear [Gr], [Bu]. Moreover, G_h is virtually cyclic and isometrically embedded in G for any $h \in x$, implying it is \mathcal{B} -isocohomological [Ji1], [Og2]. It also implies the quotient $G_h/(h)$ is finite, so $r.c.d. = 0$. //

2.2. Relatively hyperbolic groups.

We begin by recalling the basic setup for relatively hyperbolic groups. Thus, let G be a discrete group, and $\{H_\lambda\}_{\lambda \in \Lambda}$ a family of subgroups of G . Set $\mathcal{H} = \bigcup_{\lambda \in \Lambda} H_\lambda \setminus \{1_G\}$. A subset $X \subset G$ is a relative generating set of G , with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$, if $G = \langle X \cup \mathcal{H} \rangle$. In this case, G can be considered as the quotient of a free product $F = F(X) * (*_{\lambda \in \Lambda} H_\lambda)$, where $F(X)$ is the free group with basis X .

Let $\mathcal{R} \subset F$ be such that the normal closure of \mathcal{R} is the kernel of the quotient map. In this case G has a relative presentation with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$ given by $\langle X, \mathcal{H} | R = 1, R \in \mathcal{R} \rangle$. G is finitely relatively presented if \mathcal{R} is finite.

Let W be a path in F representing $1_G \in G$. Then W can be represented in the following form: $W = \prod_{i=1}^k f_i R_i^{\pm 1} f_i^{-1}$, where the $f_i \in F$ and $R_i \in \mathcal{R}$. We can then define the relative area of such a path: $area^{rel}(W) = \min\{k \in \mathcal{N} | W \text{ can be written as } \prod_{i=1}^k f_i R_i^{\pm 1} f_i^{-1} \text{ where } f_i \in F \text{ and } R_i \in \mathcal{R}\}$.

G is relatively hyperbolic with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$ if G is finitely relatively presented with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$ and there is a constant C such that for any word $W \in F$ representing $1_G \in G$ we have that $area^{rel}(W) \leq C \|W\|$, where $\|W\|$ is the length of W in F [Os]. In particular if G is relatively hyperbolic with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$, then the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ is a hyperbolic metric space, in the graph metric.

An alternate description is given by Farb in [Fa]. Let G be a finitely generated group, and let $\{H_\lambda\}_{\lambda \in \Lambda}$ be a family of finitely generated subgroups of G . Fix X , a finite symmetric set of generators of G . To the corresponding Cayley graph $\Gamma(G, X)$ add a vertex $v(gH_\lambda)$ for each left coset of each H_λ in the family, and connect $v(gH_\lambda)$ with the vertex corresponding to each element $x \in gH_\lambda$ by an edge of length $1/2$. The resulting graph, $\hat{\Gamma}(G, X; \{H_\lambda\}_{\lambda \in \Lambda})$ is called the coned-off Cayley graph of G relative to $\{H_\lambda\}_{\lambda \in \Lambda}$. In Farb's notation, the group G is weakly relative hyperbolic with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$ if $\hat{\Gamma}(G, X; \{H_\lambda\}_{\lambda \in \Lambda})$ is a hyperbolic metric space. As $\hat{\Gamma}(G, X; \{H_\lambda\}_{\lambda \in \Lambda})$ is quasi-isometric to $\Gamma(G, X \cup \mathcal{H})$, this property is weaker than that defined above. However, when augmented with the bounded coset penetration property (defined below), this notion is equivalent to relative hyperbolicity.

Definition. Let a group G be weakly relatively hyperbolic to a family $\{H_\lambda\}_{\lambda \in \Lambda}$ of finitely generated subgroups. $(G, \{H_\lambda\}_{\lambda \in \Lambda})$ is said to satisfy the Bounded Coset Penetration (BCP) property if for each constant k there exists a constant $c = c(k)$ such that for every pair of k -quasi-geodesics p and q in the coned-off Cayley graph with the same endpoints and without backtracking, satisfy

(i) If p penetrates a coset gH_λ and q does not penetrate gH_λ , then the point at which p enters the coset is at most a $\Gamma(G, X)$ -distance of $c(k)$ from the point at which p leaves the coset;

(ii) If p and q both penetrate a coset gH_λ , then the points at which p and q enter gH_λ are at most a $\Gamma(G, X)$ -distance of $c(k)$ from each other. Similarly the points at which p and q exit the coset are within a $\Gamma(G, X)$ -distance of $c(k)$ from one-another.

Definition. Let G be a finitely generated group which is relatively hyperbolic with respect to the collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. An element of G is parabolic if it is conjugate to an element in one of the H_λ . Otherwise the element is said to be hyperbolic.

Proposition 2.2.1. Let G be a finitely generated group, $\{H_\lambda\}_{\lambda \in \Lambda}$ a collection of nontrivial subgroups of G . Suppose that G is finitely presented with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$. Then the collection $\{H_\lambda\}_{\lambda \in \Lambda}$ is finite. In particular, if G is relatively hyperbolic with respect to a collection $\{H_\lambda\}_{\lambda \in \Lambda}$ of nontrivial subgroups, then $\text{card } \Lambda < \infty$.

Proof. This is Corollary 2.48 of [Os]. //

Proposition 2.2.2. Let G be a finitely generated group with generating set X , such that G is relatively hyperbolic with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$, and let $h \in G$ be a nontorsion element which does not conjugate into

any of the H_λ (such an element is said to be hyperbolic). Then the centralizer G_h is finitely generated, virtually cyclic, and the embedding $G_h \rightarrow G$ is a quasi-isometric embedding.

Proof. By Theorem 4.19 of [Os], G_h is a strongly relatively quasi-convex subgroup of G . Theorem 4.13 of [Os] shows that G_h is finitely generated and that the inclusion $G_h \rightarrow G$ is a quasi-isometric embedding. Theorem 4.16 of [Os] gives that G_h is a word-hyperbolic group. As the infinite cyclic subgroup generated by h lies in the center of the hyperbolic group G_h , we have that G_h is virtually cyclic. //

Proposition 2.2.3. *Let G be a finitely generated group with generating set X , such that G is relatively hyperbolic with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$, and let $h \in G$ be a nontorsion parabolic element such that $h \in xH_\lambda x^{-1}$. Then $G_h = xH_{\lambda, x^{-1}hx}x^{-1}$, where $H_{\lambda, x^{-1}hx}$ is the centralizer of $x^{-1}hx$ in H_λ .*

Proof. This is due to the fact that H_λ and $xH_\lambda x^{-1}$ are conjugated by x . //

Proposition 2.2.4. *Let G be a finitely generated group with generating set X , such that G is relatively hyperbolic with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$, and let $\mathcal{H} = \bigcup_{\lambda \in \Lambda} H_\lambda \setminus \{1_G\}$. Then for every k there is an $\epsilon = \epsilon(k)$ such that given any two k -quasi-geodesics without backtracking in $\Gamma(G, X \cup \mathcal{H})$, say p and q , the following conditions hold:*

(1) *If p penetrates a coset gH_λ and q does not penetrate gH_λ , then the point at which p enters the coset is at most a $\Gamma(G, X)$ -distance of $\epsilon(k)$ from the point at which p leaves the coset;*

(2) *If p and q both penetrate a coset gH_λ , then the points at which p and q enter gH_λ are at most a $\Gamma(G, X)$ -distance of $\epsilon(k)$ from each other. Similarly the points at which p and q exit the coset are within a $\Gamma(G, X)$ -distance of $\epsilon(k)$ from one-another.*

Thus ϵ works for the constant c in the definition of the BCP property.

Proof. This is Theorem 3.23 of [Os]. //

Lemma 2.2.5. *Let (H, d) be a δ -hyperbolic geodesic metric space. For every k there is a constant $\tilde{N} = \tilde{N}(k)$ such that if p is a k -quasigeodesic with endpoints x and y , then p lies within the \tilde{N} -neighborhood of every geodesic $[x, y]$ joining x to y in H . Moreover \tilde{N} can be bounded by a polynomial in k .*

Proof. This is theorem 1.7 of Part III.H of [BH]. [BH] actually claims only the first statement above, but the existence of the polynomial follows from the proof. In particular,

$$\tilde{N}(k) \leq (\delta \log_2(2k^3 + 6k^2 + 3k + 2) + \delta \log_2[\delta \log_2(2k^3 + 6k^2 + 3k + 2)] + 1)(k^2 + 1) + \frac{1}{2}(2k^3 + 3k)$$

which is clearly bounded by a polynomial in k . //

Corollary 2.2.6. *Let (H, d) be a δ -hyperbolic geodesic metric space. For every k and R there is a constant $N = N(k, R)$ such that if p and q are k -quasigeodesics whose starting points are within R from each other, and whose ending points are within R from each other, then p and q lie within the N -neighborhood of one-another. Moreover $N(k, R)$ can be bounded by a polynomial in k and R .*

Proof. Denote the starting and ending points of p (resp. q) by x and y (resp. x' and y'). By Lemma 2.2.5, p and $[x, y]$ lie within an $\tilde{N}(k)$ -neighborhood of one-another. Similarly for q and $[x', y']$. Consider the geodesic quadrilateral x, y, y', x' . The sides $[x, x']$ and $[y, y']$ each have length less than R by hypothesis. For any point on $[x, y]$ which is at least $R + 2\delta$ from either endpoint, hyperbolicity gives a point on $[x', y']$ which is within a distance of 2δ . Similarly for points on $[x', y']$ at least $R + 2\delta$ from either endpoint. As every point of $[x, y]$ is within $R + 2\delta$ from such a point, it follows that $[x, y]$ and $[x', y']$ lie in $R + 4\delta$ -neighborhoods of one-another. It follows that $N(k, R) = \tilde{N}(k) + R + 2\delta$ works. Moreover, as $\tilde{N}(k)$ is bounded by a polynomial in k , we have $N(k, R)$ is bounded by a polynomial in k and R . //

Proposition 2.2.7. *Let G , \mathcal{H} and ϵ be as in Proposition 2.2.4. Then $\epsilon(k)$ is bounded by a polynomial in k .*

Proof. In [Os] the proof of theorem 3.23 bounds $\epsilon(k)$ by the maximum of three terms, each of which can be expressed as a polynomial in $N = N(k)$. By the above lemma, this is bounded by a polynomial in k .

In particular, $\epsilon(k) = \max\{\epsilon'(k), C'(k), D(k)\}$ where $C'(k) = LM(1 + k(2\epsilon'(k) + 1) + 2\epsilon'(k))$, $D(k) = LM(1 + k(2\epsilon'(k) + 1) + 2\epsilon'(k) + k + 1)$, $\epsilon'(k) = 2K(k)^2LM(4k + 1)$, for L and M positive constants depending only on the finite relative presentation $\langle X, \mathcal{H} | R = 1, R \in \mathcal{R} \rangle$. Let $K_0 = N(k, 0)$, where $N(k, R)$ is the constant from Corollary 2.2.6. Then $K(k) = N(k, K_0) + \frac{1}{2}$ is the $K(k)$ showing up in the definition of ϵ' . It is clear that $\epsilon(k)$ is bounded by a polynomial in k . //

The following is, at heart, the Pseudo-Anosov case of Theorem 4.1 of [Ma].

Lemma 2.2.8. *Let u and v be conjugate nontorsion hyperbolic elements of G . Then there exists a constant K_h and a $g \in G$ with $u = g^{-1}vg$, and $\ell_{\hat{\Gamma}}(\hat{g}) \leq K_h(\ell_{\Gamma}(u) + \ell_{\Gamma}(v))$, where \hat{g} is the “shortened path” in the relative graph $\hat{\Gamma}$ corresponding to the element $g \in G$. Moreover, K_h is independant of the choice of u and v .*

Proof. By the work in [Os] on translation length of hyperbolic elements, there exists a $d > 0$ such that for any hyperbolic element $h \in G$ one has $\ell_{\hat{\Gamma}}(\hat{h}^n) \geq |n|d$. Moreover, for any $x \in G$ we have $d_{\hat{\Gamma}}(x, h^n x) \geq |n|d$.

Let α be a quasi-axis for the hyperbolic element h , acting on $\hat{\Gamma}$ as a hyperbolic isometry. That is, α is a bi-infinite geodesic such that for every n , $h^n \alpha$ lies in a 2δ neighborhood of α , where δ is the hyperbolicity constant of $\hat{\Gamma}$.

If $x \in G$ then there exists constant K such that $d_{\hat{\Gamma}}(x, \alpha) \leq Kd_{\hat{\Gamma}}(x, hx)$, for K independant of x and h . This is Lemma 4.3 of [Ma], which relies only on $\hat{\Gamma}$ being a hyperbolic metric space, on which h acts as a hyperbolic isometry.

Thus if u and v are two conjugate hyperbolic elements of G , let g be some conjugator with $u = g^{-1}vg$. Let α and β be the axes of u and v , respectively, as hyperbolic isometries. Let a be the closest point on α to the identity element, e , and let b be the closest point on β to e . Applying our above inequality yields $d_{\hat{\Gamma}}(e, a) \leq Kd_{\hat{\Gamma}}(e, u) = K\ell_{\hat{\Gamma}}(\hat{u})$ and $d_{\hat{\Gamma}}(e, b) \leq K\ell_{\hat{\Gamma}}(\hat{v})$. The conjugator g takes the axis β to the axis α , so $gb \in \alpha$. The translation length of u is at most $\ell_{\hat{\Gamma}}(\hat{u})$, so there is an n such that $d_{\hat{\Gamma}}(a, u^n gb) \leq \ell_{\hat{\Gamma}}(\hat{u})$, as an orbit of u is $\ell_{\hat{\Gamma}}(\hat{u})$ -dense on α . Let $g' = u^n g$. Then g' serves to conjugate u to v . Moreover, $d_{\hat{\Gamma}}(e, g') \leq d_{\hat{\Gamma}}(e, a) + d_{\hat{\Gamma}}(a, g'b) + d_{\hat{\Gamma}}(g'b, g')$. Thus $\ell_{\hat{\Gamma}}(\hat{g}') = d_{\hat{\Gamma}}(e, g') \leq K\ell_{\hat{\Gamma}}(\hat{u}) + \ell_{\hat{\Gamma}}(\hat{u}) + K\ell_{\hat{\Gamma}}(\hat{v})$. The lemma follows by observing that $\ell_{\Gamma}(u) \geq \ell_{\hat{\Gamma}}(\hat{u})$ and $\ell_{\Gamma}(v) \geq \ell_{\hat{\Gamma}}(\hat{v})$. //

Lemma 2.2.9. *Let u be a nontorsion parabolic elements of G , lying in $\gamma H \gamma^{-1}$. Then there exists a universal polynomial K_p , of degree one, such that $\ell_{\hat{\Gamma}}(\hat{\gamma}) \leq K_p(\ell_{\Gamma}(u))$.*

Proof. For a coset γH , let $\ell_{\Gamma}^*(\gamma H) = \min\{\ell_{\Gamma}(xh) \text{ for } h \in H\}$. Let x be a point of γH such that $\ell_{\Gamma}(x) = \ell_{\Gamma}^*(\gamma H)$. By Lemma 4.24 of [DS] there exists a constant C , independant of u , γ and H , such that either $d_{\Gamma}(x, ux) \leq C$ or $\ell_{\Gamma}(u) \geq \ell_{\Gamma}^*(\gamma H) + \frac{1}{2}d_{\Gamma}(x, ux) - C$.

Thus if $d_{\Gamma}(x, ux) > C$ then $\ell_{\Gamma}^*(\gamma H) \leq \ell_{\Gamma}(u) - \frac{1}{2}d_{\Gamma}(x, ux) + C$. As $\ell_{\hat{\Gamma}}(\hat{\gamma}) \leq \ell_{\hat{\Gamma}}^*(\gamma H) + 1 \leq \ell_{\Gamma}^*(\gamma H) + 1$, it follows that $\ell_{\hat{\Gamma}}(\hat{\gamma}) \leq \ell_{\Gamma}(u) + C + 1$.

Otherwise $\ell_{\Gamma}(x^{-1}ux) = d_{\Gamma}(x, ux) \leq C$. Consider the sequence of elements $x^{-1}u^n x \in G$. As u is nontorsion, this is an infinite family of elements. As ℓ_{Γ} is a proper length function, there can be at most finitely many elements of G with length not exceeding C . Let M be the cardinality of the ball of radius C centered at the identity. Then there is an $N \leq M + 1$ such that $\ell_{\Gamma}(x^{-1}u^N x) > C$. As u^N is a nontorsion parabolic element lying in $\gamma H \gamma^{-1}$, one has $\ell_{\Gamma}(u^N) \geq \ell_{\Gamma}^*(\gamma H) + \frac{1}{2}d_{\Gamma}(x, u^N x) - C$ so that $\ell_{\hat{\Gamma}}(\hat{\gamma}) \leq \ell_{\Gamma}(u^N) + C + 1$. As $\ell_{\Gamma}(u^N) \leq N\ell_{\Gamma}(u) \leq (M + 1)\ell_{\Gamma}(u)$, we have $\ell_{\hat{\Gamma}}(\hat{\gamma}) \leq (M + 1)\ell_{\Gamma}(u) + C + 1$.

//

Theorem 2.2.10. *Let G be a finitely generated group with generating set X , such that G is relatively hyperbolic with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$, each member of which has a \mathcal{P} -solvable conjugacy-bound. Then G has a \mathcal{P} -solvable conjugacy-bound.*

Proof. Let δ denote the hyperbolicity constant of the coned-off Cayley graph $\hat{\Gamma}$. As each H_λ has a \mathcal{P} -solvable conjugacy-bound, there are polynomials P_λ such that if $u, v \in H_\lambda$ are conjugate in H_λ , then there is a $g \in H_\lambda$ with $u^g = v$ and $\ell_{H_\lambda}(g) \leq P_\lambda(\ell_{H_\lambda}(u) + \ell_{H_\lambda}(v))$. As $(H_\lambda, \ell_{H_\lambda}) \rightarrow (G, \ell_G)$ is a quasi-isometric embedding, there exists a polynomial Q_λ such that $\ell_\Gamma(g) \leq Q_\lambda(\ell_\Gamma(u) + \ell_\Gamma(v))$. If u and v are two elements of H_λ , with $g \in G$ with $u^g = v$, then $v \in H_\lambda^g \cap H_\lambda$. If $g \in G \setminus H_\lambda$, then Proposition 2.36 of [Os] gives $v = 1$. Thus if u and v are nontrivial elements of H_λ which have a conjugator in G , then they have a conjugator in H_λ . The $\Gamma(G, X)$ -length of the minimal conjugator is thus bounded by the polynomial Q_λ . Moreover, as Λ must be finite by Proposition 2.2.1 above, $Q = \sum_{\lambda \in \Lambda} Q_\lambda$ is a polynomial which works to bound the conjugator length for every H_λ .

Let u and v be conjugate hyperbolic elements of G , and let $L = \ell_\Gamma(u) + \ell_\Gamma(v)$. According to [Bu] Lemma 5.11 we have that the Γ -distance which g travels in a coset it penetrates is bounded by $2(L) + 10c(8L)$, where $c(k)$ is the function given by the BCP property. As shown above this is bounded by a polynomial in L . By Lemma 2.2.8 above, $\ell_{\hat{\Gamma}}(\hat{g}) \leq K_h(L)$. As each step of this shortened path corresponds to a Γ -distance bounded by a polynomial in L , it follows that there is an element $g' \in G$ conjugating u to v with $\ell_\Gamma(g') \leq (2(\ell_\Gamma(u) + \ell_\Gamma(v)) + 10c(8\ell_\Gamma(u) + 8\ell_\Gamma(v)))K_h(\ell_\Gamma(u) + \ell_\Gamma(v))$. As $c(k)$ can be bounded by a polynomial in k , we see that this expression is bounded by some universal polynomial $Q_h(\ell_\Gamma(u) + \ell_\Gamma(v))$.

Let u be a nontorsion parabolic element of $G \setminus H_\lambda$, and v an element of H_λ with u conjugate to v . Let g be a conjugator of minimal relative length with $g^{-1}ug = v$. Then $u \in gH_\lambda g^{-1}$. By Lemma 2.2.9 we have that $\ell_{\hat{\Gamma}}(\hat{g}) \leq K_p(\ell_\Gamma(u))$.

Let g' be an element with minimal relative length which conjugates u into H , and let $h = g'^{-1}ug'$. By Theorem 5.13 of [Bu] we have the Γ -length is bounded by $c(7\ell_\Gamma(u) + 7\ell_\Gamma(v))$. In this case, from our above argument, we have that the element of minimal length conjugating h to v has length bounded by a polynomial in $\ell_\Gamma(u) + \ell_\Gamma(v)$. Then to bound the length of g , we may assume that g has minimal relative length amongst all elements conjugating u into H .

Consider the geodesic quadrilateral in $\hat{\Gamma}$ with sides $[e, u]$, $[u, ug]$, $[ug, g]$ and $[e, g]$, where the paths $[e, g]$ and $[ug, g]$ correspond to the shortened path \hat{g} . Denote by \hat{p} the path $[u, ug]$ and by \hat{q} the path $[e, g]$. Let \hat{p} travel through the coset fH along the word k . By [Bu] Corollary 5.4, if \hat{q} do not penetrate fH , then Γ -distance of k is at most $2(\ell_\Gamma(u) + \ell_\Gamma(v)) + c(2\ell_\Gamma(u) + 2\ell_\Gamma(v) + 1) + 2c(2)$. Otherwise \hat{q} also travels through fH . Assume that \hat{q} travels through fH along k' .

If $k = k'$ so that p and q travel synchronously through fH , let γ_1 be an H -geodesic connecting the first point of p in fH to the first point of q in fH . Let γ_2 be an H -geodesic connecting the last point of p in fH to the last point of q in fH . Then u and v are conjugate to γ_1 and γ_2 , each of which are in H . By the argument in the proof of Theorem 5.12 of [Bu], the relative lengths of γ_1 and γ_2 are each bounded by

$$\left(7\ell_\Gamma(u) + 7\ell_\Gamma(v) + 2\delta + 2\tilde{N}(2\ell_\Gamma(u) + 2\ell_\Gamma(v) + 1)\right)$$

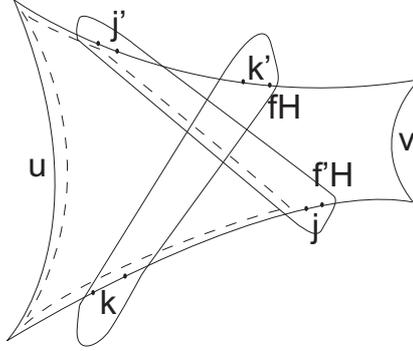
Moreover, in any coset they penetrate they can travel a Γ -distance at most

$$\left(3c(56\ell_\Gamma(u) + 56\ell_\Gamma(v) + 18\delta + 16\tilde{N}(2\ell_\Gamma(u) + 2\ell_\Gamma(v) + 1)) + 4\ell_\Gamma(u) + 4\ell_\Gamma(v) + 10c(8\ell_\Gamma(u) + 8\ell_\Gamma(v))\right)$$

Thus their Γ -lengths are bounded by the product of these two quantities, a polynomial in $\ell_\Gamma(u) + \ell_\Gamma(v)$. The subpath of g conjugating γ_1 to γ_2 can then be thought of as an element of H conjugating γ_1 to γ_2 .

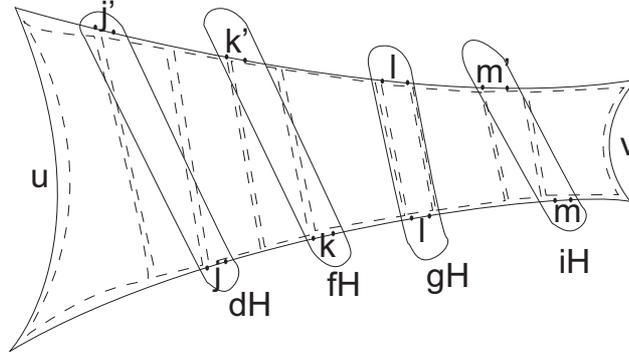
It's Γ -length can then be bounded by a polynomial in $\ell_\Gamma(\gamma_1) + \ell_\Gamma(\gamma_2)$. Thus the Γ -distance traveled by p and q through fH is bounded by a universal polynomial in $\ell_\Gamma(u) + \ell_\Gamma(v)$.

If this is not the case, then k and k' are not synchronously in fH . Assume that there is a coset $f'H$ penetrated by both \hat{p} and \hat{q} , such that \hat{p} travels through $f'H$ along j , \hat{q} travels through $f'H$ along j' with \hat{p} traveling along k before j and \hat{q} traveling along j' before k' .



Let a be the first point of q in $f'H$, and let b be the first point of p in $f'H$, and consider the paths w_1 and w_2 where w_1 is the indicated paths from e to u then to a , and w_2 is the path from e to b then to a , such that the path from b to a is an H -geodesic. As in the 'skew'-coset case of Lemma 5.5 of [Bu], \hat{w}_1 and \hat{w}_2 are two $\ell_\Gamma(u) + \ell_\Gamma(v) + 2\ell_{\hat{f}}(\hat{g})$ -quasigeodesics, and the Γ -length of k is bounded by $c(\ell_\Gamma(u) + \ell_\Gamma(v) + 2\ell_{\hat{f}}(\hat{g}))$. As $\ell_{\hat{f}}(\hat{g})$ is bounded by a polynomial in $\ell_\Gamma(u) + \ell_\Gamma(v)$, we have that the Γ -length of k itself is bounded by a polynomial in $\ell_\Gamma(u) + \ell_\Gamma(v)$.

If this is not the case, then we perform a series of surgeries on the paths p and q as follows. Let I denote the set of i such that $p(i)$ is a point where p enters or exits a coset $f'H$ which both p and q penetrate, as well as i such that $q(i)$ is a point where q enters or exits a coset $f'H$ which both p and q penetrate. Connect $p(i)$ to $q(i)$ by a Γ -geodesic, denoted by γ_i , for each $i \in I$. The Γ -length of each γ_i is bounded in the same way as γ_1 and γ_2 above. This splits the paths p and q into a series of subpaths which conjugate one γ_i to the next. We will denote by γ_0 and γ_∞ the paths u and v respectively, and consider 0 and ∞ as elements of I .



A priori, for each successive γ_i, γ_j conjugacy, we have one of the following:

- (1) γ_i and γ_j are in the same coset.
- (2) γ_i and γ_j do not lie in a single common coset, and the subpaths p' and q' connecting γ_i to γ_j do not penetrate a common coset.
- (3) γ_i and γ_j do not lie in a single common coset, and the subpaths p' and q' connecting γ_i to γ_j do penetrate a common coset.

In each of these cases, the subword of g which conjugates γ_i to γ_j can have its length bounded by that of the minimal length conjugator of γ_i and γ_j . In case (1) the conjugator is a word in H which we have already seen has length bounded by a polynomial in the Γ -lengths of γ_i and γ_j . Thus this case

is solvable. In case (2) the argument above where the paths p and q do not penetrate a common coset bounds the length of the minimal conjugator by a polynomial in the Γ -lengths of γ_i and γ_j . Thus this case is also solvable. In case (3), the subpaths p' and q' of p and q connecting γ_i to γ_j both travel a positive distance through a common coset. Denote this coset by fH . By the definition of I , $p(i)$ is the point where p enters fH and $p(j)$ is the point where p exits fH , and similarly for $q(i)$ and $q(j)$. Then γ_i and γ_j are elements of H , and the above argument bounds the length of their conjugator.

As the number of such pairs is bounded by the $\hat{\Gamma}$ -length of \hat{g} , which is itself bounded by a polynomial in $\ell_\Gamma(u) + \ell_\Gamma(v)$, and each γ_i has length bounded by a polynomial in $\ell_\Gamma(u) + \ell_\Gamma(v)$. Putting this together with the above arguments, we see that, through any coset which they penetrate, the Γ -distance which p and q can travel is bounded by a polynomial in $\ell_\Gamma(u) + \ell_\Gamma(v)$. Thus there is a universal polynomial U such that the element of minimal length conjugating u to v has length bounded by $U(\ell_\Gamma(u) + \ell_\Gamma(v))$.

Let u and v be two conjugate parabolic elements of $G \setminus H_\lambda$, which can each be conjugated into H_λ . Assume that $u^g = v$, for g a conjugator of minimal length. There is $h \in H_\lambda$ such that u is conjugate to h , and v is conjugate to h , and by Lemma 5.13 of [Bu] we have $\ell_\Gamma(h) \leq c(7\ell_\Gamma(u) + 7\ell_\Gamma(v))$. By the argument above, there is an element conjugating u to h whose length is bounded by a polynomial in $\ell_\Gamma(u) + \ell_\Gamma(h)$ and an element conjugating v to h whose length is bounded by a polynomial in $\ell_\Gamma(v) + \ell_\Gamma(h)$. The product of these two conjugators gives an element of G which conjugates u to v , and whose length is bounded by a universal polynomial in $\ell_\Gamma(u) + \ell_\Gamma(v)$. //

Recall that if L_1 and L_2 are two length functions on a group G , then L_1 dominates L_2 if there are constants A and $B > 0$ such that for all $g \in G$, $L_2(g) \leq A(L_1(g) + 1)^B$. L_1 and L_2 are equivalent if L_1 dominates L_2 and L_2 dominates L_1 . Viewed as weight functions on G , we see that this classical notion of equivalence corresponds to \mathcal{P} -equivalence as defined in section 1.2 above. Thus equivalent length functions on a discrete group yield the same algebra of polynomially bounded functions.

Theorem 2.2.11. *Let G be a finitely generated group, such that G is relatively hyperbolic with respect to H_1, H_2, \dots, H_n , where each H_i is in the class $\mathcal{C}_\mathcal{P}(\mathbb{C})$ and has a \mathcal{P} -solvable conjugacy-bound. Then G is in the class $\mathcal{C}_\mathcal{P}(\mathbb{C})$. In particular, the ℓ^1 -SrBC holds for G .*

Proof. By Theorem 2.2.10 and the methods of section 1.4 above,

$$\begin{aligned} HH_t^*(H_L^{1,\infty}(G))_x &\cong HH_t^*(H_L^{1,\infty}(G_h)) \\ HC_t^*(H_L^{1,\infty}(G))_x &\cong HC_t^*(H_L^{1,\infty}(G_h)) \end{aligned}$$

It remains to show that G_h belongs to the class $\mathcal{C}_\mathcal{P}(\mathbb{C})$, for each non-torsion element h .

Let $h \in G$ be a nontorsion element.

If h is hyperbolic, then by Prop. 2.2.2, the embedding $G_h \rightarrow G$ is a quasi-isometric embedding. This is precisely the statement that $L_G|_{G_h}$ is equivalent to L_{G_h} , where L_G denotes the standard word-length function on G (determined by the generating set). We also have G_h is virtually cyclic. Thus, $G_h \in \mathcal{C}_\mathcal{P}(\mathbb{C})$.

Suppose that h is parabolic. Then by Prop. 2.2.3, $G_h = xH_{i,x^{-1}hx}x^{-1}$, for some $x \in G$. Thus, $L_{G_h} = L_{x(H_{i,x^{-1}hx})x^{-1}}$. As all of the H_i are quasi-convex in G , the restriction $L_G|_{xH_ix^{-1}}$ is equivalent to $L_{xH_ix^{-1}}$. Then $L_G|_{xH_{i,x^{-1}hx}x^{-1}}$ is equivalent to $L_{xH_ix^{-1}}|_{xH_{i,x^{-1}hx}x^{-1}}$. That is, $L_G|_{G_h}$ is equivalent to $L_{xH_ix^{-1}}|_{x(H_{i,x^{-1}hx})x^{-1}}$, which is by hypothesis equivalent to $L_{xH_{i,h}x^{-1}}$. Conjugating where appropriate we obtain $L_G|_{G_h}$ is equivalent to L_{G_h} . Since $G_h \cong H_{i,x^{-1}hx}$, while H_i are in the class $\mathcal{C}_\mathcal{P}(\mathbb{C})$ the theorem is proved. //

This verifies Theorem A in the case $\mathcal{B} = \mathcal{P}$. However, in the more general case when $\mathcal{P} \prec \mathcal{B}$, the argument is the same. In fact, the \mathcal{B} -analogue of Theorem 2.2.10 clearly holds whenever $\mathcal{P} \prec \mathcal{B}$, at which point the argument for proving Theorem 2.2.11 carries over with \mathcal{B} replacing \mathcal{P} throughout.

2.3. Combable groups.

We begin with the definition of a combing for a discrete group given in [Me1] (this notion is originally due to Thurston [CEHLPT]).

Definition 2.3.1. *If $(X, *)$ is a basepointed discrete metric space with distance function d , a combing on X consists of a sequence of maps $f_n : X \rightarrow X$ satisfying*

- (i) $f_0 = *$ and for all $x \in X$ there is an N such that $f_n(x) = x \forall n \geq N$;
- (ii) there exists a constant $C > 0$ such that $d(f_n(x), f_n(y)) \leq Cd(x, y) + 1$;
- (iii) there exists a constant $S \in \mathbb{N}$ such that $d(f_n(x), f_{n+1}(x)) \leq S$ for all $x \in X$ and $n \in \mathbb{N}$.

For a given x , let $J(x)$ be the number of integers n for which $f_n(x) \neq f_{n+1}(x)$. The combing is polynomial resp. exponential if $J(x) \leq g(l(x))$ for all $x \in X$, where $g(t) = C'(1+t)^m$ resp. $C'e^{kt}$ for some $C', k \in \mathbb{R}_+$ and $m \in \mathbb{N}$ (here $l(x) := d(*, x)$). This applies in particular to discrete groups with word-length (G, L) , viewed as a basepointed metric space with metric $d(g_1, g_2) := L(g_1^{-1}g_2)$.

That $HH_t^*(H_{\mathcal{B}, L}(G))$ and $HC_t^*(H_{\mathcal{B}, L}(G))$ should be computable in terms of the Hochschild and cyclic cohomology of $\mathbb{C}[G]$ for combable groups is strongly suggested by the following two results.

Theorem 2.3.2 - [Me1], [Me2], [Og2]. *Let (G, L) be a synchronously combable group with polynomial combing function, L the standard word-length function on G . Then there is an isomorphism*

$$H_{\mathcal{P}, G}^*(EG.; A) \xrightarrow{\cong} H_G^*(EG.; A)$$

for any p -bounded semi-normed (p.s.) G -module A .

Actually the result of [Me1] only deals with the case $A =$ the trivial G -module \mathbb{C} , but also proves the same result when G has an exponential combing function and \mathcal{P} is replaced by \mathcal{E} ; for more general coefficient modules one needs to use either [Me2] or [Og2] (the definition of a p.s. G -module is as in [Og2]).

Theorem 2.3.3 - [Me2]. *Let (G, L) be as above, and let P_* be a bimodule resolution of $\mathbb{C}[G]$ by free $\mathbb{C}[G]$ -bimodules. Then $H_{\mathcal{B}, L}(G) \otimes_{\mathbb{C}[G]} P_* \otimes_{\mathbb{C}[G]} H_{\mathcal{B}, L}(G)$ is a (bornological) bimodule resolution of $H_{\mathcal{B}, L}(G)$ by free (bornological) $H_{\mathcal{B}, L}(G)$ -bimodules ($\mathcal{B} = \mathcal{P}$ or \mathcal{E}).*

[Note: What is done in [Me2] is much more general; Theorem 2.3.2 only states a special case relevant to our setting].

Proposition 2.3.4. *Assume that (G, L) is a group equipped with a polynomial combing in the above sense. Then for each conjugacy class x there are isomorphisms*

$$HH_t^*(H_{\mathcal{P}, L}(G))_x = HH_{\mathcal{P}}^*(\mathbb{C}[G])_x = H_{\mathcal{P}, G}^*(EG.; Hom_{\mathcal{P}}(S_x, \mathbb{C})) \cong H_G^*(EG.; Hom_{\mathcal{P}}(S_x, \mathbb{C}))$$

where (i) S_x is a weighted G -set with weight function induced by L and the inclusion into G , (ii) $Hom_{\mathcal{P}}(S_x, \mathbb{C})$ denotes the subspace of $Hom(S_x, \mathbb{C})$ consisting of those maps which are polynomially bounded with respect to the weight function on S_x , and (iii) $H_{\mathcal{P}, G}^*(-)$ denotes the cohomology of the cocomplex of G -equivariant, polynomially bounded cochains.

Proof. The first equality was verified above. Corollary 1.4.2 gives the identification of $HH_{\mathcal{P}}^*(\mathbb{C}[G])_x$ with $H_{\mathcal{P}}^*(S_x \times EG) = H_{\mathcal{P},G}^*(S_x \times EG)$. These last groups are computed as the cohomology of $\{C_{\mathcal{P},G}^n(S_x \times EG, \mathbb{C})\}_{n \geq 0} = \{Hom_{\mathcal{P},G}(S_x \times EG_n, \mathbb{C})\}_{n \geq 0}$. For each n taking partial adjoints yields an isomorphism $Hom_{\mathcal{P},G}(S_x \times EG_n, \mathbb{C}) \cong Hom_{\mathcal{P},G}(EG_n, Hom_{\mathcal{P}}(S_x, \mathbb{C}))$ - this is just the usual isomorphism on cochains, restricted to polynomially bounded cochains (cf. [Og2]); consequently, it commutes with the coboundary maps and extends to an isomorphism of cocomplexes

$$\{Hom_{\mathcal{P},G}(S_x \times EG_n, \mathbb{C})\}_{n \geq 0} \cong C_{\mathcal{P},G}^*(EG, Hom_{\mathcal{P}}(S_x, \mathbb{C})) := \{Hom_{\mathcal{P},G}(EG_n, Hom_{\mathcal{P}}(S_x, \mathbb{C}))\}_{n \geq 0}$$

which produces the second isomorphism in cohomology. The third follows from Theorem 2.3.2. //

Note that $Hom_{\mathcal{P}}(S_x, \mathbb{C})$ is dense in $Hom(S_x, \mathbb{C})$, and it is reasonable to ask (or even conjecture)

Question/Conjecture 2.3.5. *For (G, L) as above, does the inclusion of coefficients $Hom_{\mathcal{P}}(S_x, \mathbb{C}) \hookrightarrow Hom(S_x, \mathbb{C})$ induce an isomorphism $H_G^*(EG; Hom_{\mathcal{P}}(S_x, \mathbb{C})) \xrightarrow{\cong} H_G^*(EG; Hom(S_x, \mathbb{C}))$?*

If this is the case, then for each x there would be an isomorphism $HH_t^*(H_{\mathcal{P},L}(G))_x \cong HH^*(\mathbb{C}[G])_x$, and consequently isomorphisms $HC_t^*(H_{\mathcal{P},L}(G))_x \cong HC^*(\mathbb{C}[G])_x$. However, we have not been able to verify these isomorphisms except in certain special cases in which the general theory of section 1.4 can be applied. The following is an immediate consequence of Proposition 2.3.4.

Corollary 2.3.6. *If \mathbb{Q} admits a projective resolution of length N over $\mathbb{Q}[G]$, then $HH_t^*(H_{\mathcal{P},L}(G))_x = HH^*(\mathbb{C}[G])_x = 0$ for all $* > N$.*

An important subclass of combable groups are the semi-hyperbolic groups defined by Alonso and Bridson in their paper [AB]. We have

Theorem 2.3.7. *Let (G, L) be semi-hyperbolic in the sense of [AB]. Then for each conjugacy class $x = \langle h \rangle$, there are isomorphisms $HH_t^*(H_{\mathcal{E},L}(G))_x \cong H^*(BG_h; \mathbb{C}) = HH^*(\mathbb{C}[G])_x$, $HC_t^*(H_{\mathcal{E},L}(G))_x \cong HC^*(\mathbb{C}[G])_x$.*

Proof. By [AB], we know that quasi-convex subgroups of semi-hyperbolic are again semi-hyperbolic. This applies in particular to the centralizer subgroups G_x . Consequently, each G_x admits a synchronous (linear) combing. By [Me1], G_x is \mathcal{E} -isocohomological. Again, by [AB], we have that G has an \mathcal{E} -solvable conjugacy-bound. Thus the hypothesis of Theorem 1.4.7 is satisfied for any \mathcal{B} containing \mathcal{E} , and the result follows. //

Corollary 2.3.8. *If (G, L) is semi-hyperbolic and G satisfies the nilpotency condition, then it satisfies the \mathcal{E} -nilpotency condition.*

In the absence of an affirmative answer to (2.3.5) above, it would be quite desirable to strengthen Corollary 2.3.8 by replacing “ \mathcal{E} -nilpotency” with “ \mathcal{P} -nilpotency”. Certainly the groups G_x are \mathcal{P} -isocohomological (their combings are even linear). The obstruction lies with the time needed to solve the conjugacy problem. Currently it is not known if the exponential bound on the solution time given in [AB] is actually a sharp bound in the general case, or whether it can be improved to polynomial time. However, for certain semi-hyperbolic groups, this improved bound is known for certain conjugacy classes. In particular, if G is a mapping class group, then the conjugacy-bound is polynomial [MM] when the conjugacy class is pseudo-Anosov. In general the \mathcal{P} -solvable conjugacy-bound problem is still solvable but we will address this issue in a future publication.

Finally we remark that the Bass Conjecture for semi-hyperbolic groups was proved by Eckman [Ec2]

2.4. Hochschild and cyclic homology of $\ell^1(G)$.

As we have observed, for $\mathcal{B} = \mathcal{B}_{min}$, the corresponding rapid decay algebra $H_{\mathcal{B}_{min},L}(G)$ is equal to $\ell^1(G)$, the full ℓ^1 -algebra of G . More generally, one has an identification of Banach spaces $H_{\mathcal{B}_{min},w}(X) = \ell^1(X)$ for any weighted space (X, w) . Since the class \mathcal{B}_{min} consists of constant functions, the space $H_{\mathcal{B}_{min},w}(X)$ is independent of w . For this reason, we will drop w and simply write $\ell^1(X)$ for the corresponding space. As in the proof of Lemma 1.3.3, there is, for any two sets X, X' , a natural isomorphism of Banach spaces

$$(2.4.1) \quad \ell^1(X) \hat{\otimes} \ell^1(X') \cong \ell^1(X \times X')$$

and by Lemma 1.2.4 a natural identification

$$(2.4.2) \quad \ell^1(X)^* \cong Hom_b(X, \mathbb{C})$$

between the continuous dual of $\ell^1(X)$ and the space of bounded, complex-valued maps on X . Consequently, as in Lemma 1.3.6, we have an isomorphism

$$C_t^*(\ell^1(G)) \cong C_b^*(N.^{cy}(G))$$

between the topological Hochschild cocomplex for $\ell^1(G)$ and the cocomplex of bounded cochains on $N.^{cy}(G)$. Again, this isomorphism preserves the respective decompositions by conjugacy classes, and commutes with the maps in the Connes-Gysin sequence. Thus for any conjugacy class x we have an identification

$$HH_t^*(\ell^1(G))_x \cong H_b^*(N.^{cy}(G))_x$$

Unlike the case $\mathcal{B} \neq \mathcal{B}_{min}$, we see that the isomorphism $G_h \backslash G \cong S_x$ of right G -sets (Proposition 1.4.3, Lemma 1.4.4) producing the isomorphism of simplicial sets $(\pi_x) : G_h \backslash G \times_G EG. \xrightarrow{\cong} S_x \times_G EG.$, will always yield an isomorphism on bounded cochains: $C_b^*(S_x \times_G EG.) \xrightarrow{\cong} C_b^*(G_h \backslash G \times_G EG.)$. Given that Proposition 1.4.5 holds without restriction, we conclude

Corollary 2.4.3. *For all discrete groups G and conjugacy classes $x = \langle h \rangle$, there is a natural isomorphism $HH_t^*(\ell^1(G))_x \cong H_b^*(BG_h)$.*

We will write $S_{\mathbb{C}}^*$ for the S -map $S^* : HC^*(\mathbb{C}) \rightarrow HC^{*+2}(\mathbb{C})$. The main result of this section is

Theorem 2.4.4. *For all discrete groups G and conjugacy classes $x = \langle h \rangle$ there is an isomorphism*

$$HC_t^*(\ell^1(G))_x \cong H_b^*(BG_h) \otimes HC^*(\mathbb{C})$$

where the map S_x^* on the left identifies with $Id \otimes S_{\mathbb{C}}^*$ on the right.

Proof. The proof is an adaption of the method due to Nistor [Ni1](see also [Ji2]). For $h \in G$, let $\tilde{L}(G, h) = (EG)$. Writing x for $\langle h \rangle$, define a projection map $P_h : \tilde{L}(G, h) \rightarrow N.^{cy}(G)_x$ by $P_h(g_0, \dots, g_n) = (g_n^{-1}hg_0 \otimes g_0^{-1}g_1 \otimes \dots \otimes g_{n-1}^{-1}g_n)$. It is easy to see that P_h induces an isomorphism of simplicial sets $\overline{P}_h : * \times_{G_h} \tilde{L}(G, h) \xrightarrow{\cong} N.^{cy}(G)_x$, and hence an isomorphism of bounded cochain cocomplexes

$$C_b^*(N.^{cy}(G))_x \xrightarrow{\cong} C_b^*(* \times_{G_h} \tilde{L}(G, h)) \cong C_{b, G_h}^*(\tilde{L}(G, h))$$

where $C_{b, G_h}^*(\tilde{L}(G, h))$ denotes the subcocomplex of $C_b^*(\tilde{L}(G, h))$ consisting of G_h -equivariant cochains (with coefficients in the trivial G_h -module \mathbb{C}). Now consider the map T defined on $\tilde{L}(G, h)$ by

$T_{n+1}(g_0, \dots, g_n) = (hg_n, g_0, \dots, g_{n-1})$. If H is any subgroup of G_h containing h , then we see that T descends to a map \overline{T} on $* \times_H \tilde{L}(G, h)$, giving $* \times_H \tilde{L}(G, h)$ the structure of a cyclic simplicial set, and making \overline{P}_h an isomorphism of cyclic simplicial sets. Denote the singular resp. cyclic complex of a cyclic simplicial set $X.^c$ (with coeff.s in \mathbb{C}) by $C_*(X.^c)$ resp. $CC_*(X.^c)$, and the corresponding dual cocomplexes by $C^*(X.^c)$ resp. $CC^*(X.^c)$. Clearly an isomorphism $\phi : X.^c \xrightarrow{\cong} Y.^c$ of cyclic simplicial sets induces an isomorphism on both singular and cyclic (co)complexes. It also induces an isomorphism of bounded cyclic cocomplexes $CC_b^*(\phi) : CC_b^*(X.^c) \xrightarrow{\cong} CC_b^*(Y.^c)$ (as we are in char. 0, $CC_b^*(Z.^c)$ is quasi-isomorphic to the subcocomplex of $C_b^*(Z.^c)$ consisting of cochains invariant under the cyclic operator). Applying these observations to our situation, we get for each conjugacy class $x = \langle h \rangle$ an isomorphism

$$CC_t^*(\ell^1(G))_x \cong CC_{b, G_h}^*(\tilde{L}(G, h).)$$

Let $C_h \subset G_h$ denote the cyclic subgroup generated by h , and N_h the quotient of G_h by C_h . Denote the cohomology groups of $CC_{b, C_h}^*(\tilde{L}(G, h).)$ by $HC_{b, C_h}^*(\tilde{L}(G, h).)$. We claim these groups are Hausdorff, or alternatively, that the reduced and unreduced cohomology groups agree (see Note (1.3.2)). In fact, as C_h is amenable for all h (it is abelian), the augmentation map $\tilde{L}(G, h). \rightarrow *$ induces isomorphisms on bounded Hochschild cohomology, and hence also bounded cyclic cohomology. The Hausdorff condition guarantees the existence of a first-quadrant spectral sequence strongly converging to $HC_{b, G_h}^*(\tilde{L}(G, h).) := H^*(CC_{b, G_h}^*(\tilde{L}(G, h).))$ with E_2^{**} -term

$$E_2^{**} = H_b^*(N_h; HC_{b, C_h}^*(\tilde{L}(G, h).))$$

(compare [No], [Ji1, (3.7)]). As we have just observed, the augmentation map on $\tilde{L}(G, h).$ yields an isomorphism $HC_{b, C_h}^*(\tilde{L}(G, h).) \xrightarrow{\cong} HC^*(\mathbb{C})$, and hence an isomorphism

$$E_2^{**} = H_b^*(N_h; HC_{b, C_h}^*(\tilde{L}(G, h).)) \xrightarrow{\cong} H_b^*(N_h; HC^*(\mathbb{C}))$$

of E_2^{**} -terms. The spectral sequence on the right converges to $H_b^*(N_h) \otimes HC^*(\mathbb{C})$; however by the Serre spectral sequence in bounded cohomology associated to the exact sequence $C_h \hookrightarrow G_h \xrightarrow{proj_h} N_h$ (loc. sit.), we have $H_b^*(N_h) \xrightarrow[\cong]{proj_h} H_b^*(G_h)$, completing the proof. //

Thus, although computable, the non-elliptic summands of the topological cyclic cohomology groups of $\ell^1(G)$ will never have a nilpotent S -operator, and so are not useful in verifying the ℓ^1 -Bass conjecture by the methods used in this paper. Note that if G itself is amenable, then each G_x is as well, and Theorem 2.4.4 produces an isomorphism between $HC_t^*(\ell^1(G))$ and the ℓ^1 -completion of the vector space $\bigoplus_{x \in \langle G \rangle} HC^*(\mathbb{C})$.

Appendix - The Baum-Connes assembly map and the generalized Bass Conjecture

C. Ogle (OSU)

In the early 1980's, P. Baum and A. Connes defined an assembly map

$$(A.0.1) \quad \mathcal{A}_*^{G,a} : KK_*^G(C(\underline{EG}), \mathbb{C}) \rightarrow K_*^t(C_r^*(G))$$

where G denotes a locally compact group, \underline{EG} the classifying space for proper G -actions, $C(\underline{EG})$ the G -algebra of complex-valued functions on \underline{EG} vanishing at infinity, and $KK_*^G(C(\underline{EG}), \mathbb{C})$ the G -equivariant KK -groups of (\underline{EG}) with coefficients in \mathbb{C} , while $K_*^t(C_r^*(G))$ represents the topological K -groups of the reduced C^* -algebra of G^\dagger . The original details of this map appeared (a few years later) in [BC1] and [BC2], with further elaborations in [BCH]. As shown in [BC3], when G is discrete the left-hand side admits a Chern character which may be represented as

$$ch_*^{BC}(G) : KK_*^G(C(\underline{EG}), \mathbb{C}) \rightarrow \bigoplus_{x \in \text{fin}(\langle G \rangle)} H_*(BG_x; \mathbb{C}) \otimes HPer_*(\mathbb{C})$$

where $\text{fin}(\langle G \rangle)$ is the set of conjugacy classes of G corresponding to elements of finite order, G_x the centralizer of g in G where $x = \langle g \rangle$, and $HPer_*(\mathbb{C})$ the periodic cyclic homology of \mathbb{C} . Note that $H_*(BH; \mathbb{C}) \otimes HPer_*(\mathbb{C})$ are simply the 2-periodized complex homology groups of BH , and (via the classical Atiyah-Hirzebruch Chern character) can be alternatively viewed as the complexified K -homology groups of BH . Upon complexification, the map $ch_*^{BC}(G)$ is an isomorphism. The original construction of Baum and Connes $\mathcal{A}_*^{G,a}$ was analytical. Motivated by the need to construct a homotopical analogue to their map, we constructed an assembly map in [Ogl] which we will denote here as

$$\mathcal{A}_*^{G,h} \otimes \mathbb{C} : H_*\left(\prod_{x \in \text{fin}(\langle G \rangle)} BG_x; \mathbf{K}(\mathbb{C})\right) \otimes \mathbb{C} \rightarrow K_*^t(C_r^*(G)) \otimes \mathbb{C}$$

where $\mathbf{K}(\mathbb{C})$ denotes the 2-periodic topological K -theory spectrum of \mathbb{C} . The construction of this map amounted to an extension of the classical assembly map constructed in [L] which was designed to take into account the contribution coming from the conjugacy classes of finite order.

As before, let $\mathbb{C}[G]$ denote the complex group algebra topologized with the fine topology. To this algebra we associate the space

$$K^f(\mathbb{C}[G]) := |[q] \mapsto K_0(S_q(\mathbb{C}[G])) \times BGL^+(S_q(\mathbb{C}[G]))|$$

where $\{S_q(X)\}$ denotes the total singular complex of the topological space X , and $|[q] \mapsto Y_q|$ the geometric realization of the simplicial set Y . This space has a natural infinite-loop structure coming from the standard (non-connective) Quillen-Grayson-Wagoner delooping of the spaces on the right, and the associated spectrum is a module over the spectrum delooping of $K^f(\mathbb{C})$. In particular, there is a

†For non-cocompact \underline{EG} , the equivariant KK -groups $KK_*^G(C(\underline{EG}), \mathbb{C})$ are defined as a colimit over the directed set $\text{Co}_G(\underline{EG})$ of G -cocompact subsets of \underline{EG} , with the relation given by inclusion:

$$KK_*^G(C(\underline{EG}), \mathbb{C}) := \text{colim}_{\text{Co}_G(\underline{EG})} KK_*^G(C(X), \mathbb{C})$$

Bott element $\beta \in \pi_2(K^f(\mathbb{C}))$ (corresponding to the usual Bott element in $K_2^t(\mathbb{C})$), and multiplication with this element induces a map

$$\beta_* : \pi_*(K^f(\mathbb{C}[G])) \rightarrow \pi_{*+2}(K^f(\mathbb{C}[G]))$$

For the purpose of this appendix, we define the topological K -theory of $\mathbb{C}[G]$ as

$$K_*^t(\mathbb{C}[G]) := \pi_*(K^f(\mathbb{C}[G]))[\beta^{-1}]$$

The two essential features of $\mathcal{A}_*^{G,h} \otimes \mathbb{C}$, shown in [Og1], were (i) it factors through $K_*^t(\mathbb{C}[G]) \otimes \mathbb{C}$, and (ii) the composition of $\mathcal{A}_*^{G,h} \otimes \mathbb{C}$ with the complexified Chern-Connes-Karoubi-Tillmann character $ch_*^{CK} : K_*(\mathbb{C}[G]) \otimes \mathbb{C} \rightarrow HC_*(\mathbb{C}[G])$ was effectively computable (see below). What we did not do in [Og1] was show that $\mathcal{A}_*^{G,a} \otimes \mathbb{C}$ and $\mathcal{A}_*^{G,h} \otimes \mathbb{C}$ agree. Since this initial work, there have been numerous extensions and reformulations of the Baum-Connes assembly map, as well as of the original Baum-Connes conjecture, which states that the map in (0.1) is an isomorphism. These extensions typically are included under the umbrella term ‘‘Isomorphism Conjecture’’, (formulated for both algebraic and topological K -theory; cf. [DL], [FJ], [LR]). Thanks to [HP], we now know that the different formulations of these assembly maps (e.g., homotopy-theoretic vs. analytical) agree.

Abbreviating $KK_*^G(C(\underline{EG}), \mathbb{C})$ as $K_*^G(\underline{EG})$ (read: the equivariant K -homology of the proper G -space \underline{EG}), our main result is

Theorem A.1. *There is a commuting diagram*

$$\begin{array}{ccc} K_*^G(\underline{EG}) & \xrightarrow{\mathcal{A}_*^{G,DL}} & K_*^t(\mathbb{C}[G]) \\ \downarrow ch_*^? & & \downarrow ch_*^{CK} \\ HC_*^{fin}(\mathbb{C}[G]) & \longrightarrow & HC_*(\mathbb{C}[G]) \end{array}$$

where $^{fin}HC_*(\mathbb{C}[G]) := \bigoplus_{x \in \text{fin}(\langle G \rangle)} HC_*(\mathbb{C}[G])_x \cong \bigoplus_{x \in \text{fin}(\langle G \rangle)} H(BG_x; \mathbb{C}) \otimes HC_*(\mathbb{C})$ is the elliptic summand of $HC_*(\mathbb{C}[G])$, $\mathcal{A}_*^{G,DL}$ is the homotopically defined assembly map of [DL], the lower horizontal map is the obvious inclusion, and the Chern character $ch_*^?$ becomes an isomorphism upon complexification for $* \geq 0$.

Let β denote a bounding class, (G, L) a discrete group equipped with a word-length, and $H_{\beta,L}(G)$ the rapid decay algebra associated with this data. We write $K_*^t(H_{\beta,L}(G))$ for the Bott-periodic topological K -theory of the topological algebra $H_{\beta,L}(G)$. The Baum-Connes assembly map for $H_{\beta,L}(G)$ is defined to be the composition

$$(BC) \quad \mathcal{A}_*^{G,\beta} : K_*^G(\underline{EG}) \xrightarrow{\mathcal{A}_*^{G,DL}} K_*^t(\mathbb{C}[G]) \rightarrow K_*^t(H_{\beta,L}(G))$$

where the second map is induced by the natural inclusion $\mathbb{C}[G] \hookrightarrow H_{\beta,L}(G)$. In the introduction, we conjectured that the image of $ch_* : K^t(H_{\beta,L}(G)) \rightarrow HC_*^t(H_{\beta,L}(G))$ lies in the elliptic summand $^{fin}HC_*^t(H_{\beta,L}(G))$ (conjecture β -SrBC). As the inclusion $\mathbb{C}[G] \hookrightarrow H_{\beta,L}(G)$ sends $^{fin}HC_*(\mathbb{C}[G])$ to $^{fin}HC_*^t(H_{\beta,L}(G))$, naturality of the Chern character ch_*^{CK} and Theorem 1 implies

Corollary A.2. *If $\mathcal{A}_*^{G,\beta}$ is rationally surjective, then β -SrBC is true.*

Since going down and then across is rationally injective, we also have (compare [O1])

Corollary A.3. *The assembly map $\mathcal{A}_*^{G,DL} \otimes \mathbb{Q}$ is injective for all discrete groups G .*

We do not claim any great originality in this paper. In fact, Theorem A.1, although not officially appearing in print before this time, has been a “folk-theorem” known to experts for many years. The connection between the Baum-Connes Conjecture (more precisely a then-hypothetical Baum-Connes-type Conjecture for $\mathbb{C}[G]$) and the stronger Bass Conjecture for $\mathbb{C}[G]$ discussed in the introduction was noted by the author in [O4].

There is some overlap of this paper with the results presented in [Ji]. A special case of Theorem 1 (for $*$ = 0 and $\mathbb{C}[G]$ replaced by the ℓ^1 -algebra $\ell^1(G)$) appeared as the main result of [BCM].

PROOF OF THEOREM A.1

We use the notation $F_*^{fin}(\mathbb{C}[G])$ to denote the elliptic summand $\bigoplus_{x \in fin(\langle G \rangle)} F_*(\mathbb{C}[G])_x$ of $F_*(\mathbb{C}[G])$ where $F_*(-) = HH_*(-), HN_*(-), HC_*(-)$ or $HPer(-)$. To maximize consistency with [LR], we write \mathbf{S} for the (unreduced) suspension spectrum of the zero-sphere S^0 , $\mathbf{HN}(R)$ resp. $\mathbf{HH}(R)$ the Eilenberg-MacLane spectrum whose homotopy groups are the negative cyclic resp. Hochschild homology groups of the discrete ring R , and $\mathbf{K}^a(R)$ the non-connective algebraic K -theory spectrum of R , with $K_*^a(R)$ representing its homotopy groups. By [LR, diag. 1.6] there is a commuting diagram

$$(A.1.1) \quad \begin{array}{ccccc} H_*^G(\underline{EG}; \mathbf{S}) & \longrightarrow & & \longrightarrow & K_*^a(\mathbb{Z}[G]) \\ \downarrow & & & & \downarrow NTr_* \\ H_*^G(\underline{EG}; \mathbf{HN}(\mathbb{Z})) & \xrightarrow{\cong} & HN_*^{fin}(\mathbb{Z}[G]) & \longrightarrow & HN_*(\mathbb{Z}[G]) \\ \downarrow & & \downarrow & & \downarrow h_* \\ H_*^G(\underline{EG}; \mathbf{HH}(\mathbb{Z})) & \xrightarrow{\cong} & HH_*^{fin}(\mathbb{Z}[G]) & \longrightarrow & HH_*(\mathbb{Z}[G]) \end{array}$$

where the top horizontal map is the composition

$$H_*^G(\underline{EG}; \mathbf{S}) \rightarrow H_*^G(\underline{EG}; \mathbf{K}^a(\mathbb{Z})) \xrightarrow{\mathcal{A}_*^{G,DL}} K_*^a(\mathbb{Z}[G])$$

referred to as the the restricted assembly map for the algebraic K -groups of $\mathbb{Z}[G]$. The other two horizontal maps are the assembly maps for negative cyclic and Hochschild homology respectively. The upper left-hand map is induced by the map from the sphere spectrum to the Eilenberg-MacLane spectrum \mathbf{HN} , which may be expressed as the composition of spectra $\mathbf{S} \rightarrow \mathbf{K}^a(\mathbb{Z}) \rightarrow \mathbf{HN}$. By [LR], the composition on the left is a rational equivalence.

Let \mathbb{C}^δ denote the complex numbers \mathbb{C} equipped with the discrete topology. Tensoring with \mathbb{C} and combined with the inclusion of group algebras $\mathbb{Z}[G] \hookrightarrow \mathbb{C}^\delta[G]$, (A.1.1) yields the commuting diagram

$$(A.1.2) \quad \begin{array}{ccc} H_*^G(\underline{EG}; \mathbb{Q}) \otimes \mathbb{C} & \longrightarrow & K_*^a(\mathbb{C}^\delta[G]) \otimes \mathbb{C} \\ \downarrow \cong & & \downarrow NTr_* \\ HN_*^{fin}(\mathbb{C}[G]) & \longrightarrow & HN_*(\mathbb{C}[G]) \end{array}$$

Next, we consider the transformation from algebraic to topological K -theory, induced by the map of group algebras $\mathbb{C}^\delta[G] \rightarrow \mathbb{C}[G]$ which is the identity on elements. By the results of [CK], [W] and [T],

there is a commuting diagram

$$(A.1.3) \quad \begin{array}{ccc} K_*^a(\mathbb{C}^\delta[G]) \otimes \mathbb{C} & \longrightarrow & K_*^t(\mathbb{C}[G]) \otimes \mathbb{C} \\ \downarrow NTr_* & & \downarrow ch_*(\mathbb{C}[G]) \\ HN_*(\mathbb{C}[G]) & \longrightarrow & HPer_*(\mathbb{C}[G]) \end{array}$$

where $ch_*(\mathbb{C}[G])$ is the Connes-Karoubi Chern character for the fine topological algebra $\mathbb{C}[G]$, and the bottom map is the transformation from negative cyclic to periodic cyclic homology.

We can now consider our main diagram

$$(A.1.4) \quad \begin{array}{ccccccc} H_*^G(\underline{EG}; \mathbb{C}) \otimes K_*(\mathbb{C}) & \longrightarrow & K_*^a(\mathbb{C}^\delta[G]) \otimes \mathbb{C} \otimes K_*(\mathbb{C}) & \longrightarrow & K_*^t(\mathbb{C}[G]) \otimes \mathbb{C} \otimes K_*(\mathbb{C}) & \longrightarrow & K_*^t(\mathbb{C}[G]) \otimes \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow ch_*(\mathbb{C}[G]) \otimes ch_*(\mathbb{C}\{id\}) & & \downarrow ch_*(\mathbb{C}[G]) \\ HN_*^{fin}(\mathbb{C}[G]) \otimes K_*(\mathbb{C}) & \longrightarrow & HN_*(\mathbb{C}[G]) \otimes K_*(\mathbb{C}) & \longrightarrow & HPer_*(\mathbb{C}[G]) \otimes HPer_*(\mathbb{C}) & \longrightarrow & HPer_*(\mathbb{C}[G]) \\ \downarrow & & \downarrow & & \uparrow & & \uparrow \\ HPer_*^{fin}(\mathbb{C}[G]) \otimes K_*(\mathbb{C}) & \xrightarrow{\cong} & HPer_*^{fin}(\mathbb{C}[G]) \otimes HPer_*(\mathbb{C}) & \longrightarrow & HPer_*^{fin}(\mathbb{C}[G]) & & HPer_*^{fin}(\mathbb{C}[G]) \end{array}$$

The top left square commutes by (A.1.2), and the middle top square commutes by (A.1.3). The upper right square commutes by virtue of the fact that the Connes-Karoubi-Chern character is a homomorphism of graded modules, which maps the $K_*^t(\mathbb{C})$ -module $K_*^t(\mathbb{C}[G])$ to the $HPer_*(\mathbb{C})$ -module $HPer_*(\mathbb{C}[G])$, with the map of base rings induced by isomorphism $ch_*(\mathbb{C}\{id\}) : K_*^t(\mathbb{C}) \otimes \mathbb{C} \xrightarrow{\cong} HPer_*(\mathbb{C})$. The lower left square commutes trivially, while the lower right commutes by the naturality of the inclusion $HPer_*^{fin}(\mathbb{C}[G]) \hookrightarrow HPer_*(\mathbb{C}[G])$ with respect to the module structure over $HPer_*(\mathbb{C})$. Summarizing, we get a commuting diagram

$$(1.5) \quad \begin{array}{ccc} H_*^G(\underline{EG}; \mathbb{C}) \otimes K_*(\mathbb{C}) & \longrightarrow & K_*^t(\mathbb{C}[G]) \otimes \mathbb{C} \\ \downarrow \cong & & \downarrow ch_*(\mathbb{C}[G]) \\ HPer_*^{fin}(\mathbb{C}[G]) & \longrightarrow & HPer_*(\mathbb{C}[G]) \\ \downarrow & & \downarrow \\ HC_*^{fin}(\mathbb{C}[G]) & \longrightarrow & HC_*(\mathbb{C}[G]) \end{array}$$

where the top map is the complexified Davis-Lück assembly map for topological group algebra $\mathbb{C}[G]$, and the bottom square is induced by the transformation $HPer_*(-) \rightarrow HC_*(-)$ (which respects the summand decomposition indexed on conjugacy classes). Restricted the elliptic summand yields the map $HPer_*^{fin}(\mathbb{C}[G]) \rightarrow HC_*^{fin}(\mathbb{C}[G])$ which is an isomorphism for $* \geq 0$, implying the result stated in Theorem A.1.

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