BOUNDED COHOMOLOGY AND AMENABLE GROUPS

BOBBY RAMSEY

Abstract. We recall the definition of bounded cohomology for a discrete group and give an elementary proof that for amenable groups this bounded cohomology is trivial.

1. Bounded Cohomology

Let $G$ be a discrete group, and let $C^n(G) = \{\phi : G^n \to \mathbb{R}\}$ with $\delta_n : C^n(G) \to C^{n+1}(G)$ defined by

$$\delta_n \phi(x_1, ..., x_{n+1}) = \phi(x_2, ..., x_{n+1}) - \phi(x_1x_2, x_3, ..., x_{n+1})$$

$$+ ... + (-1)^i \phi(x_1, ..., x_ix_{i+1}, ..., x_{n+1}) + ...$$

$$+ (-1)^{n+1} \phi(x_1, ..., x_n)$$

This yields a cochain complex:

$$C^0(G) \xrightarrow{\delta_0} C^1(G) \xrightarrow{\delta_1} C^2(G) \xrightarrow{\delta_2} ...$$

The cohomology of this complex, denoted by $H^*(G)$, is the standard cohomology of the group $G$ (with $\mathbb{R}$ coefficients).

Let us now restrict our attention to bounded functions [Ger]. Denote by $C^n_b(G)$ those $\phi \in C^n(G)$ with

$$\|\phi\|_\infty = \sup_{(x_1, ..., x_n) \in G^n} |\phi(x_1, ..., x_n)| < \infty$$

If $\phi \in C^n_b(G)$, then consider $\delta_n \phi \in C^{n+1}(G)$

$$\delta_n \phi(x_1, ..., x_{n+1}) = \phi(x_2, ..., x_{n+1}) - \phi(x_1x_2, x_3, ..., x_{n+1})$$

$$+ ... + (-1)^i \phi(x_1, ..., x_ix_{i+1}, ..., x_{n+1}) + ...$$

$$+ (-1)^{n+1} \phi(x_1, ..., x_n)$$

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So that when considering boundedness:

$$
\|\delta_n \phi \|_\infty \leq \|\phi\|_\infty + \|\phi\|_\infty \\
+ \cdots + \|\phi\|_\infty + \cdots \\
+ \|\phi\|_\infty
$$

Thus the differentials preserve boundedness. That is

$$
C^0_b(G) \xrightarrow{\delta_0} C^1_b(G) \xrightarrow{\delta_1} C^2_b(G) \xrightarrow{\delta_2} \ldots
$$

is a sub-cochain complex of the above complex. The cohomology of this complex, denoted by $H^*_b(G)$, is the bounded cohomology of the group $G$. As an example we will start by showing the following:

**Proposition 1.1.** For any discrete group $G$, $H^1_b(G) = 0$.

**Proof.** Let $\phi \in C^1_b(G)$ with $\delta_1 \phi = 0$. Then for all $a, b \in G$ we have $(\delta_1 \phi)(a, b) = \phi(b) - \phi(ab) + \phi(a) = 0$. That is, $\phi(ab) = \phi(a) + \phi(b)$ so that $\phi$ is a homomorphism from $G$ to $\mathbb{R}$.

Assume that there is a $g \in G$ with $\phi(g) \neq 0$. Then $\phi(g^n) = n\phi(g)$ so that $\lim_{n \to \infty} |\phi(g^n)| = \infty$, contradicting that $f \in C^1_b(G)$. Thus $\ker \delta_1 = 0$, so $H^1_b(G) = 0$. □

Now $C^0_b(G) = \{ \phi : G^0 \to \mathbb{R} \|\phi\|_\infty < \infty \}$, which we can identify with $\mathbb{R}$, with $\delta_0 = 0$. Thus we also have $H^0_b(G) = \mathbb{R}$, but this is not surprising as we are dealing with $\mathbb{R}$-coefficients.

## 2. Amenable Groups

**Definition 2.1.** [Pat] A discrete group $G$ is amenable if there is a left-invariant mean $m : \ell^\infty G \to \mathbb{C}$.

This means that $m$ is a positive linear functional on $\ell^\infty G$, with $\|m\| = m(1) = 1$. Moreover, for each $\phi \in \ell^\infty G$ and $g \in G$, $m(\phi) = m(\phi_g)$ where $\phi_g(x) = \phi(g \cdot x)$.

The following result is attributed in [Gro] to Hirsch and Thurston [HT]. We give an elementary proof.

**Theorem 2.1.** Let $G$ be discrete amenable group. Then for $n \geq 1$ we have $H^n_b(G) = 0$.

**Proof.** Denote a left-invariant mean by $m$. We will first note that, as $m$ is positive, $m(\psi) \in \mathbb{R}$ for any real-valued $\psi \in \ell^\infty G$. We have already seen that $H^1_b(G) = 0$. Let us start by examining $H^2_b(G)$.

Let $\phi \in C^2_b(G)$ with $\delta_2 \phi = 0$. That is, for all $a, b, c \in G$ we have

$$(\delta_2 \phi)(a, b, c) = \phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b) = 0$$
So that $\phi(a, b) = \phi(b, c) - \phi(ab, c) + \phi(a, bc)$. Let $f : G \to \mathbb{R}$ be defined by $f(a) = m(\phi(a, x))$, where by $\phi(a, x)$ we mean the bounded function from $G$ to $\mathbb{R}$ obtained by fixing the first argument of $\phi$. As $f(a) = m(\phi(a, x)) \leq ||m||\|\phi\|_{\infty} < \infty$, we have $f \in C^1_b(G)$.

$$
(\delta_1 f)(a, b) = f(b) - f(ab) + f(a)
= m(\phi(b, x)) - m(\phi(ab, x)) + m(\phi(a, x))
= m(\phi(b, x)) - m(\phi(ab, x)) + m(\phi(a, bx))
= m(\phi(b, x)) - \phi(ab, x) + \phi(a, bx))
= m(\phi(a, b))
= \phi(a, b) \cdot m(1)
= \phi(a, b)
$$

Thus $\phi$ is in the image of $\delta_1$, so $H^2_b(G) = 0$.

The case for general $n > 1$ is similar in technique to this case. Let $\phi \in C^n_b(G)$ with $\delta_n \phi = 0$. For all $x_1, \ldots, x_{n+1} \in G$

$$
(\delta_n \phi)(x_1, \ldots, x_{n+1}) = \phi(x_2, \ldots, x_{n+1}) - \phi(x_1, x_2, x_3, \ldots, x_{n+1})
+ \cdots + (-1)^i \phi(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) + \cdots + (-1)^{n+1} \phi(x_1, \ldots, x_n)
= 0
$$

So that

$$
(-1)^n \phi(x_1, \ldots, x_n) = \phi(x_2, \ldots, x_{n+1}) - \phi(x_1, x_2, x_3, \ldots, x_{n+1})
+ \cdots + (-1)^i \phi(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) + \cdots + (-1)^n \phi(x_1, \ldots, x_n x_{n+1})
$$

Let $f : G^{n-1} \to \mathbb{R}$ be defined by

$$
f(x_1, \ldots, x_{n-1}) = m(\phi(x_1, \ldots, x_{n-1}, x))
$$

where, as before, $\phi(x_1, \ldots, x_{n-1}, x)$ is treated as a bounded function of one variable obtained by fixing the first $n - 1$ arguments of $\phi$. 
Applying $\delta_{n-1}$ to $f$, we obtain the following:

\[
(\delta_{n-1}f)(x_1, \ldots, x_n) = f(x_2, \ldots, x_n) - f(x_1x_2, x_3, \ldots, x_n)
+ \ldots + \sum_{i=1}^{n-1} (-1)^i f(x_1, \ldots, x_ix_{i+1}, \ldots, x_n) + (-1)^n f(x_1, \ldots, x_{n-1})
= m(\phi(x_2, \ldots, x_n, x)) - m(\phi(x_1x_2, x_3, \ldots, x_n, x))
+ \ldots + (-1)^i m(\phi(x_1, \ldots, x_ix_{i+1}, \ldots, x_n, x)) + \ldots
+ (-1)^n m(\phi(x_1, \ldots, x_{n-1}, x_n))
= m(\phi(x_2, \ldots, x_n, x)) - m(\phi(x_1x_2, x_3, \ldots, x_n, x))
+ \ldots + (-1)^i m(\phi(x_1, \ldots, x_ix_{i+1}, \ldots, x_n, x)) + \ldots
+ (-1)^n m(\phi(x_1, \ldots, x_{n-1}, x_n))
= \sum_{i=1}^{n-1} (-1)^i \phi(x_1, \ldots, x_ix_{i+1}, \ldots, x_n, x) + (-1)^n \phi(x_1, \ldots, x_{n-1}, x_n)
= (-1)^n \phi(x_1, \ldots, x_n) \cdot m(1)
= (-1)^n \phi(x_1, \ldots, x_n)
\]

Thus $\phi = \delta_{n-1}(-1)^n f$, so the kernel of $\delta_n$ lies in the image of $\delta_{n-1}$, resulting in $H^*_b(G) = 0$. \qed

References


